Introduction to Marine Hydrodynamics (NA235)

(2014-2015, 2nd Semester)

Assignment No.9

(6 problems, given on June 11th, submit on June 18th, 2015)

Problem 1: Given velocity profile of a laminar boundary layer

$$u = U\frac{y}{\delta}, \qquad 0 \le y \le \delta$$

where U is the outer velocity, and y the distance to the wall. Please ① calculate ratio of displacement thickness, δ^*, to the boundary layer thickness, δ, and ② calculate ratio of momentum thickness, θ, to the boundary layer thickness, δ.

Solution: From the given velocity profile, introducing a normalized velocity

$$f = \frac{u}{U} = \frac{y}{\delta} = \eta$$

that is, the normalized velocity is identical to the normalized distance. According to the definition of displacement thickness and momentum thickness, they can be immediately evaluated as follows

$$\delta^* = \int_0^\delta (1-\frac{u}{U})dy = \delta\int_0^1 (1-f)d\eta = \delta\int_0^1 (1-\eta)d\eta = \frac{1}{2}\delta$$

and

$$\theta = \int_0^\delta \frac{u}{U}\cdot(1-\frac{u}{U})dy = \delta\int_0^1 f\cdot(1-f)d\eta = \delta\int_0^1 \eta\cdot(1-\eta)d\eta = \frac{1}{6}\delta$$

Answer: Ratio of the displacement thickness to boundary layer thickness is 1/2, and the ratio of momentum thickness to boundary layer thickness equals 1/6.

Problem 2: In a laminar boundary layer near a flat plate, velocity gradient on the wall is expressed as $k\left(=\left.\frac{\partial u}{\partial y}\right|_{y=0}\right)$. If the flow is steady and driven by a favorable pressure gradient, $\frac{\partial P}{\partial x}$, prove that velocity profile near the wall can be expressed as $u=\frac{1}{2\mu}\frac{\partial P}{\partial x}y^2+ky$, where y is the distance to the wall.

Solution: In boundary layer equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

since the flow is steady, we have $\frac{\partial u}{\partial t}=0$. Also because the plate is long, velocity won't vary with x, that is, $\frac{\partial u}{\partial x}=0$. Due to continuity equation, it is further derived that $\frac{\partial v}{\partial y}=0$. In addition, due to non-slip condition, normal velocity component on the wall vanishes, i.e., $v=0$, which is correct in the whole boundary layer due to $\frac{\partial v}{\partial y}=0$. Therefore, for the present problem, the boundary layer equation is simplified as follows

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\rho\nu}\frac{\partial P}{\partial x}=\frac{1}{\mu}\frac{\partial P}{\partial x}$$

Integral twice to the above equation, it results

$$u=\frac{1}{2\mu}\frac{\partial P}{\partial x}y^2+Cy+C_1$$

In order to satisfying non-slip condition on the wall and the wall condition $\left.\frac{\partial u}{\partial y}\right|_{y=0}=k$, it should be

$$u=\frac{1}{2\mu}\frac{\partial P}{\partial x}y^2+ky$$

It is the same as the expression given.

Problem 3: Given velocity profile in a 2-dimensional laminar boundary layer near flat plate

$$\frac{u}{U}=\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)$$

where y is the distance to the wall. Please write down expressions of $\delta(x)$, $\delta^*(x)$, $\theta(x)$ and frictional drag R_t for a plate with length L.

Solution: From given velocity profile, we can obtain momentum thickness and shear stress on the wall as follows

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\left[1-\sin\left(\frac{\pi}{2}\cdot\frac{y}{\delta}\right)\right]dy = \frac{4-\pi}{2\pi}\delta$$

$$\tau_0 = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U\frac{\pi}{2\delta}$$

Substituting them in boundary layer momentum equation

$$\frac{d\theta}{dx} = \frac{\tau_0}{\rho U^2}$$

it gives

$$\frac{4-\pi}{2\pi}\frac{d\delta}{dx} = \frac{\mu}{\rho U}\frac{\pi}{2}\frac{1}{\delta}$$

Considering condition on the leading edge of the plate, $\delta|_{x=0} = 0$, solution of the above equation is

$$\delta = \sqrt{\frac{2\pi^2}{4-\pi}}\sqrt{\frac{\nu x}{U}} = 4.788\sqrt{\frac{\nu x}{U}}$$

The corresponding momentum thickness, displacement thickness and wall shear stress are thus obtained

$$\theta = \frac{4-\pi}{2\pi}\delta = 0.654\sqrt{\frac{\nu x}{U}}$$

$$\delta^* = \int_0^\delta\left(1-\frac{u}{U}\right)dy = \left(1-\frac{2}{\pi}\right)\delta = 1.738\sqrt{\frac{\nu x}{U}}$$

$$\tau_0 = \mu U\frac{\pi}{2\delta} = \mu U\frac{\pi}{2}\frac{1}{4.788}\sqrt{\frac{U}{\nu x}} = 0.328\mu U\sqrt{\frac{U}{\nu x}}$$

Also frictional drag on the plate of length L with unit width is evaluated as follows

$$R_f = \int_0^L \tau_0 dx = 0.328\mu U\int_0^L\sqrt{\frac{U}{\nu x}}dx = 0.656\mu U\sqrt{\frac{UL}{\nu}}.$$

Problem 4: Given velocity profile of a laminar boundary layer near a flat plate

$$u = A\sin(By) + C$$

where y is the distance to the wall. Please list boundary conditions, then determine coefficients A, B and C.

Solution: Following three conditions can be used to determine the constants A, B, C in the velocity expression.

①Non-slip wall condition: $u\big|_{y=0} = 0$

②Outer boundary condition: $u\big|_{y=\delta} = U$

③Outer boundary condition: $\left.\dfrac{\partial u}{\partial y}\right|_{y=\delta} = 0$

They uniquely determine these constants, and it finally gives out the velocity profile below

$$u = U\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right).$$

Problem 5: A flat plate is placed in a uniform flow under favorable pressure gradient. If double the length of the plate, how many times the plate drag will become? State of the boundary layer near the plate is assumed to be laminar.

Solution: For laminar boundary layer, Blasius solution has been derived. It says that drag force is parallel to the square root of the plate length, that is, $D_f \propto \sqrt{L}$. Therefore double the plate length will cause the frictional drag square root of 2 times large.

Problem 6: A flat plate is placed in a uniform flow under favorable pressure gradient. If velocity profile, $u(y)$, in the boundary layer near the wall can be expressed as a polynomial up to cubic term, where y is the distance to the wall, prove the velocity profile will be expressed as

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

where δ stands for the boundary layer thickness.

Solution: Let $\eta = \dfrac{y}{\delta}$ be the normalized distance to the wall. From the problem, velocity profile is assumed to be expressed as a cubic polynomial

$$\frac{u}{U} = a + b\eta + c\eta^2 + d\eta^3 .$$

To determine four coefficients in above expression, following conditions can be used.

①Non-slip condition on wall: $u\big|_{y=0} = 0$

②Definition of boundary layer thickness: $u\big|_{y=\delta} = U$

③Outer potential flow condition: $\left.\dfrac{\partial u}{\partial y}\right|_{y=\delta} = 0$

④Flat plate condition: $\dfrac{dU}{dx} = 0$, then $\left.\dfrac{\partial^2 u}{\partial y^2}\right|_{y=0} = -\dfrac{U}{\nu}\dfrac{dU}{dx} = 0$

These conditions can uniquely determine the four coefficients. It gives the result

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 .$$