Introduction to Marine Hydrodynamics (NA235)

(2014-2015, 2nd Semester)

Assignment No.7

(8 problems, given on May 18th, submitted on May 28th, 2015)

Problem 1: Given a deep water wave with period, $\tau = 5s$, wave height, $H = 1.2m$. Calculate wavelength, phase velocity (celerity), group velocity and energy transmission rate of the wave.

Solution: From the given wave period, wave frequency is directly evaluated according to the definition

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{5} = 1.26\left(s^{-1}\right)$$

The corresponding wave number is obtained from dispersion relation for deep water wave

$$k = \frac{\omega^2}{g} = \frac{(1.26)^2}{9.81} = 0.161\left(m^{-1}\right)$$

and wave length is determined from wave number as follows

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.161} = 39.03(m)$$

Then wave velocity, i.e., celerity, equals

$$c = \frac{\lambda}{\tau} = \frac{39.03}{5} = 7.81(m/s)$$

and group velocity is

$$c_g = \frac{1}{2}c = 3.903(m/s)$$

Energy transmission rate of the wave is evaluated as

$$\begin{aligned} W &= \frac{E}{\lambda}c_g = \frac{1}{8}\rho g H^2 c_g \\ &= \frac{1}{8}\times 1000 \times 9.8 \times (1.2)^2 \times 3.903 \\ &= 6886.6 \quad (N\cdot m/s) \end{aligned}$$

Problem 2: Given a deep water wave of wavelength $6.28m$. How deep from the free surface, wave height will be reduced to half of the one at the free surface?

Solution: For deep water wave, trace of a water particle is a circle, of which radius, r, decreases with the average distance, d, of the particle from the mean water surface, that is,

$$r = a\exp(-kd)$$

where a is the wave amplitude. Then from

$$\frac{r}{a} = \exp(-kd) = \frac{1}{2}$$

it gives the answer

$$d = \frac{\ln 2}{k} = \frac{\ln 2}{2\pi/\lambda} = \frac{\ln 2}{2\pi/6.28} = 0.693(m)$$

That is, at depth 0.693 metres beneath the mean water surface, wave amplitude will be half of the one at free surface. Also wave height there is half of the one at free surface.

Problem 3: In a water field of depth $H = 10m$, there is a surface wave of amplitude, $a = 1m$, and wave number, $k = 0.2m^{-1}$. ① Calculate wavelength, phase velocity and period of the wave. ② Give out the equation of the wave elevation. ③ Write down equation of the path of a water particle at $x_0 = 0$ and $z_0 = -5m$.

Solution:

① Wave length is directly evaluated from wave number

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2} = 31.42(m)$$

Since ratio of water depth to wave length

$$\frac{H}{\lambda} = \frac{10}{31.42} = 0.318$$

is less than half, it should take account for the effect of water depth. According to dispersion relation of finite water depth, wave velocity, i.e., celerity, is evaluated as follows

$$c=\sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi H}{\lambda}}=\sqrt{\frac{9.81\times 31.42}{2\times 3.1416}\tanh\frac{2\times 3.1416\times 10}{31.4}}$$
$$=\sqrt{49\times 0.964}=6.87\left(m/s\right)$$

Then, wave period is immediately obtained

$$T=\frac{\lambda}{c}=\frac{31.4}{6.87}=4.57\left(s\right)$$

② Equation of wave elevation

Wave frequency is calculated from wave period

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{4.57}=1.37\left(s^{-1}\right)$$

Then equation of the wave elevation can be written

$$\varsigma=a\cos(kx-\omega t)=\cos(0.2x-1.37t)$$

where it is assumed that origin of the coordinate system is taken at a point on the mean water surface at the initial time $t=0$.

③ Generally, trace of a water particle in a finite water depth wave field is an ellipse of form

$$\frac{\left(x-x_0\right)^2}{A^2}+\frac{\left(z-z_0\right)^2}{B^2}=1$$

at depth $d=5\left(m\right)$, i.e., $z=-d=-5\left(m\right)$

$$A=a\frac{\cosh(z_0+H)}{\sinh kH}=1\times\frac{\cosh\left[0.2\times(-5+10)\right]}{\sinh(0.2\times 10)}=\frac{\cosh 1}{\sinh 2}=\frac{1.543}{3.627}=0.43\left(m\right)$$
$$B=a\frac{\sinh(z_0+H)}{\sinh kH}=1\times\frac{\sinh\left[0.2\times(-5+10)\right]}{\sinh(0.2\times 10)}=\frac{\sinh 1}{\sinh 2}=\frac{1.175}{3.627}=0.32\left(m\right)$$

that is, the trace of a particle beneath the mean water surface is an ellipse, and it can be written

$$\frac{x^2}{0.43^2}+\frac{\left(z+5\right)^2}{0.32^2}=1$$

where point $C\left(0,-5\right)$ is the mean position of the particle.

Problem 4: Given two deep water waves with wavelength $15m$ and $150m$ respectively. ①Evaluate their wave velocities (*i.e.* phase velocities) and periods. ② Discuss variations when they propagates from deep water into a shore of $10m$ deep.

Solution:

① For deep water wave, both wave velocity and wave period can be evaluated from given wave length in accordance with the dispersion relation as follows.

$$c=\sqrt{\frac{g\lambda}{2\pi}} \quad \text{and} \quad T=\sqrt{\frac{2\pi\lambda}{g}}$$

●When $\lambda=15(m)$, it corresponds

$$c=\sqrt{\frac{g\times 15}{2\pi}}=4.8369(m/s) \quad \text{and} \quad T=\sqrt{\frac{2\pi\times 15}{g}}=3.1011(s)$$

●When $\lambda=150(m)$, it corresponds

$$c=\sqrt{\frac{g\times 150}{2\pi}}=15.3(m/s) \quad \text{and} \quad T=\sqrt{\frac{2\pi\times 150}{g}}=9.81(s)$$

② When the two waves propagate into a 10 metres deep shore, their performances will be greatly affected by the ratio of water depth to wave length.

●When $\lambda=15(m)$, the water depth to wave length ratio

$$\frac{h}{\lambda}=\frac{10}{15}=0.67$$

is greater than $1/2$ and can be reasonably considered as deep water wave. Parameters, such as wave velocity, wave period and so on, almost do not change.

●When $\lambda=150(m)$, the water depth to wave length ratio

$$\frac{h}{\lambda}=\frac{10}{150}=0.067$$

is very small. Water depth will remarkably affect parameters, while its wave period does not change, $T=9.81(s)$. The corresponding wave frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{9.81} = 0.64\left(s^{-1}\right)$$

where wave number is evaluated from dispersion relation

$$k = \frac{\omega^2}{g \tanh kh}$$

Setting initial value $k_1 = 2\pi/150 = 0.04189$, after several iteration

n	2	3	4	5	6	7	8
k_n	0.0984	0.1876	0.2371	0.2442	0.2448	0.2448	0.2448

we get wave number $k = 0.2448\left(m^{-1}\right)$. Then wave velocity takes value

$$c = \frac{\omega}{k} = \frac{0.64}{0.2448} = 2.616\left(m/s\right)$$

and wave length

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2448} = 25.667\left(m\right)$$

is much shorter than its original length 150 metres in deep water, but still far greater than the water depth, 10 metres.

Problem 5: Set a buoy on a sea of depth $H = 6.2m$. Under the excitation of a water wave, the buoy is moving up and down (*i.e.* heaving) periodically at a rate of 12 times a minute. The wave height is measured of $h = 1.2m$. Calculate the wave length and amplitudes of the velocity and dynamic pressure of a particle at the seabed.

Solution:

Since the buoy is heaving at a rate of 12 times a minute, the wave propagates at the same rate, that is, wave frequency equals

$$f = \frac{12}{60} = 0.2\left(Hz\right)$$

Supposing it is a deep water wave, wave length will be

$$\lambda = \frac{g}{2\pi f^2} = \frac{9.81}{2\pi \times 0.2^2} = 39.03\left(m\right)$$

It is greater than 6 times of the water depth. Water depth effect is remarkable, and it should be treated as a finite water depth wave. According to dispersion relation of finite depth water wave, wave number should satisfy

$$k = \frac{\left(2\pi f\right)^2}{g \tanh kh}$$

If we take $k_1 = 2\pi/\lambda = 2\pi/39.03 = 0.1610$ as an initial value, wave number can be determined by iteration

n	2	3	4	5	6	7	8	9	10
k_n	0.2116	0.1862	0.1965	0.1918	0.1939	0.1930	0.1935	0.1932	0.1932

That is, wave number is believed to be 0.1932 m^{-1}, accordingly wave length is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.1932} = 32.522(m)$$

Since velocity potential of the wave is written as

$$\varphi = \frac{ag}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\sin(kx-\omega t)$$

In the case at seabed, $z=-H$, velocity only has horizontal component

$$\begin{aligned} u &= \frac{\partial\varphi}{\partial x} = \frac{agk}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\cos(kx-\omega t) \\ &= \frac{(1.2/2)\times 9.81\times 0.1932}{2\pi\times 0.2}\frac{\cosh[0.1932\times(-6.2+6.2)]}{\cosh(0.1932\times 6.2)}\cos(0.1932x-2\pi\times 0.2t) \\ &= 0.5001\cos(0.1932x-0.4\pi t)(m/s) \end{aligned}$$

and the corresponding dynamic pressure

$$\begin{aligned} P &= -\rho\frac{\partial\varphi}{\partial t} = \rho ag\frac{\cosh k(z+H)}{\cosh kH}\cos(kx-\omega t) \\ &= 1025\times(1.2/2)\times 9.81\frac{\cosh[0.1932\times(-6.2+6.2)]}{\cosh(0.1932\times 6.2)}\cos(0.1932x-2\pi\times 0.2t) \\ &= 3338.0\cos(0.1932x-0.4\pi t)(Pa). \end{aligned}$$

Problem 6: Two kinds of fluid are separated by a horizontal plane. Their thicknesses are assumed great enough. The upper layer fluid is known of density ρ', and the lower layer of density ρ. ① Show that a surface wave of wave length λ on the separation plane will propagate at the velocity

$$c = \sqrt{\frac{g\lambda}{2\pi}\frac{\rho-\rho'}{\rho+\rho'}}$$

② Show that for any group waves, group velocity is just equal to half of the wave velocity.

Proof:

① Take a Cartesian coordinate system: origin is set on the mean separation surface, oy is vertical and upward, and ox coincides with the direction wave propagates. In this way, wave fields in upper fluid domain and the lower fluid domain have velocity potentials as follows respectively

$$\phi' = C\frac{gA}{\omega}e^{-ky}\sin\left(kx-\omega t\right)$$

and

$$\phi = D\frac{gA}{\omega}e^{ky}\sin\left(kx-\omega t\right).$$

Due to impermeable condition on wave surface, $\left.\frac{\partial\varphi'}{\partial y}\right|_{y=0} = \left.\frac{\partial\varphi}{\partial y}\right|_{y=0}$, it gives $C=-D$. Bernoulli's equations in the two fluid domains are expressed as

$$\frac{p'}{\rho'}+gy'+\frac{\partial\phi'}{\partial t}=0$$

and

$$\frac{p}{\rho}+gy+\frac{\partial\phi}{\partial t}=0.$$

On mean separation surface, $y=0$, pressures are identical, $p'=p$, and wave elevation, $y'=y=\eta\left(x,t\right)$, is derived

$$\eta = \frac{1}{g\left(\rho-\rho'\right)}\left.\left(\rho'\frac{\partial\phi'}{\partial t}-\rho\frac{\partial\phi}{\partial t}\right)\right|_{y=0} = A\frac{D\rho-C\rho'}{\rho-\rho'}\cos(kx-\omega t)$$
$$= AD\frac{\rho+\rho'}{\rho-\rho'}\cos(kx-\omega t)$$

On wave surface, the kinematic condition is written as

$$\left.\frac{\partial\phi'}{\partial y}\right|_{y=0} = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} = \left.\frac{\partial\eta}{\partial t}\right|_{y=0}$$

It is equivalent to

$$D\frac{gA}{\omega}k\sin\left(kx-\omega t\right) = A\omega\frac{D(\rho+\rho')}{\rho-\rho'}\sin(kx-\omega t)$$

that is,

$$gk = \omega^2 \frac{\rho+\rho'}{\rho-\rho'}$$

That is the dispersion relation of the wave system. Phase velocity of it is immediately derived from the dispersion relation as follows.

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}} = \sqrt{\frac{g\lambda}{2\pi}\frac{\rho-\rho'}{\rho+\rho'}}.$$

② Wave group velocity is also derived from the dispersion relation.

$$c_g = \frac{d\omega}{dk} = \frac{g}{2\omega}\frac{\rho-\rho'}{\rho+\rho'} = \frac{\omega}{2k} = \frac{1}{2}c.$$

Problem 7: Show that for deep water waves, hydrodynamic pressure of any fluid particle just equals the hydrostatic pressure of the particle at the equilibrium position in calm water, that is,

$$\frac{P}{\rho} + gz_0 = const\,.$$

Proof:

Deep water waves is generally expressed in velocity potential

$$\varphi = \frac{ag}{\omega} e^{kz} \sin(kx - \omega t + \delta)$$

Substituting it in Bernoulli's equation

$$\frac{P}{\rho} + gz + \frac{\partial\varphi}{\partial t} = const$$

we have

$$\begin{aligned} &\frac{P}{\rho} + gz + -age^{kz}\cos(kx-\omega t+\delta) \\ &= \frac{P}{\rho} + g\left[z - ae^{kz}\cos(kx-\omega t+\delta)\right] \\ &= \frac{P}{\rho} + gz_0 \\ &= const \end{aligned}$$

where $z = z_0 + \eta(z_0)$, and $\eta(z_0)$ is the wave elevation at the mean level $z = z_0$

$$\eta(z_0) = ae^{kz_0}\cos(kx-\omega t+\delta) \approx ae^{kz}\cos(kx-\omega t+\delta)$$

are is employed.

Problem 8: A standing wave is formed near a vertical breakwater. Show that a water particle with equilibrium position (x_0, z_0) will trace along the flowing straight line

$$\frac{z-z_0}{x-x_0} = -\tanh k(H+z_0)\cot kx_0$$

where H is the water depth. Oxz is a Cartesian coordinate system with Ox horizontal and perpendicular to the breakwater, and Oz vertical, upward positive.

Proof:

Take a Cartesian coordinate system: origin is at the intersection of the mean water surface and the vertical wall, oz is vertical and upward, and ox is perpendicular to the wall and parallel to wave propagating direction. In this way, the incident wave, propagating along ox-axis, is of velocity potential

$$\varphi_1 = \frac{ag}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\sin(kx-\omega t+\delta)$$

and the wave reflected from the wall is of velocity potential

$$\varphi_1 = -\frac{ag}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\sin(kx+\omega t-\delta)$$

where reflection rate is assumed to be 100%. Then, superposition of these two waves form a standing wave with velocity potential

$$\varphi = \varphi_1+\varphi_2 = -\frac{2ag}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\cos kx\sin(\omega t-\delta)$$

Velocity components of a fluid particle with mean position (x_0, z_0) are expressed as

$$u = \frac{\partial\varphi}{\partial x} = \frac{2agk}{\omega}\frac{\cosh k(z+H)}{\cosh kH}\sin kx\sin(\omega t-\delta)$$

$$w = \frac{\partial \varphi}{\partial z} = -\frac{2agk}{\omega} \frac{\sinh k(z+H)}{\cosh kH} \cos kx \sin(\omega t - \delta)$$

Denote $\left(x(t), z(t)\right)$ the position of a particle at time t, we have

$$\frac{dx}{dt} = u = \frac{\partial \varphi}{\partial x} = \frac{2agk}{\omega} \frac{\cosh k(z_0+H)}{\cosh kH} \sin kx_0 \sin(\omega t - \delta)$$
$$\frac{dz}{dt} = w = \frac{\partial \varphi}{\partial z} = -\frac{2agk}{\omega} \frac{\sinh k(z_0+H)}{\cosh kH} \cos kx_0 \sin(\omega t - \delta)$$

Above differential equations give solution

$$x - x_0 = -\frac{2agk}{\omega^2} \frac{\cosh k(z_0+H)}{\cosh kH} \sin kx_0 \cos(\omega t - \delta)$$

and

$$z - z_0 = \frac{2agk}{\omega^2} \frac{\sinh k(z_0+H)}{\cosh kH} \cos kx_0 \cos(\omega t - \delta).$$

They ratio leads to

$$\frac{z - z_0}{x - x_0} = -\tanh k(H + z_0) \cot kx_0 .$$

That is the expression to be proven.