Introduction to Marine Hydrodynamics (NA235)

(2014-2015, 2nd Semester)

Assignment No.6

(9 problems, given on May 4, submitted on May 14th, 2015)

Problem 1: Inside a large sphere of radius R fills with incompressible perfect fluid. A small ball of radius a is moving in it at speed $V(t)$. At initial instant t_0, the small ball is concentric with the large sphere. Please write down governing equations and boundary conditions that velocity potential of the flow between them obeys.

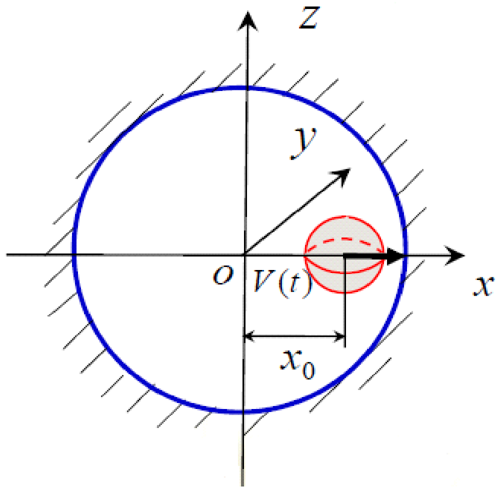

Figure 6-1

Solution: Surface of the small ball is a sphere. Its center varies with time as follows

$$x_c(t) = \int_{t_0}^{t} V(\tau)\,d\tau \qquad y_c(t) = z_c(t) = 0$$

and the surface of the small ball is expressed as

$$|\mathbf{r} - \mathbf{r}_c| = a \qquad \text{or} \qquad (x - x_c)^2 + y^2 + z^2 = a^2$$

The large sphere can be expressed

$$|\mathbf{r}| = R \qquad \text{or} \qquad x^2 + y^2 + z^2 = R^2$$

Therefore potential flow inside the large sphere and outside the small ball is governed by Laplace equation and impermeable condition on the large sphere and the small sphere. Mathematically they are written in the form

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0 & \text{in } \Omega = \{\mathbf{r} | (|\mathbf{r}| < R) \cap (|\mathbf{r} - \mathbf{r}_c| > a)\} \\ x\dfrac{\partial \phi}{\partial x} + y\dfrac{\partial \phi}{\partial y} + z\dfrac{\partial \phi}{\partial z} = 0 & \text{on } |\mathbf{r}| = R \\ (x - x_c)\dfrac{\partial \phi}{\partial x} + y\dfrac{\partial \phi}{\partial y} + z\dfrac{\partial \phi}{\partial z} = (x - x_c)V(t) & \text{on } |\mathbf{r} - \mathbf{r}_c| = a \end{cases}.$$

Problem 2: Given a planar potential flow, in which there are a point source of intensity $Q_1=20m^3/s$ at point (-1, 0), and a point sink of intensity $Q_2=40m^3/s$ at point (2, 0). Density of the fluid is $\rho = 1.8\, kg/m^3$. If pressure at origin is 0, please calculate velocities and pressures at point (0, 1) and at point (1, 1) respectively.

Solution: Velocity potential of the flow is expressed as

$$\phi = \frac{Q_1}{2\pi}\ln\sqrt{(x+1)^2 + y^2} - \frac{Q_2}{2\pi}\ln\sqrt{(x-2)^2 + y^2}$$

and from it velocity components are derived

$$u = \frac{\partial \phi}{\partial x} = \frac{Q_1}{2\pi}\frac{x+1}{(x+1)^2 + y^2} - \frac{Q_2}{2\pi}\frac{x-2}{(x-2)^2 + y^2}$$

$$v = \frac{\partial \phi}{\partial y} = \frac{Q_1}{2\pi}\frac{y}{(x+1)^2 + y^2} - \frac{Q_2}{2\pi}\frac{y}{(x-2)^2 + y^2}$$

Specifically, at point $(0,1)$

$$u_1 = \frac{20}{2\pi}\frac{0+1}{(0+1)^2 + 1^2} - \frac{40}{2\pi}\frac{0-2}{(0-2)^2 + 1^2} = 4.14(m/s)$$

$$v_1 = \frac{20}{2\pi}\frac{1}{(0+1)^2 + 1^2} - \frac{40}{2\pi}\frac{1}{(0-2)^2 + 1^2} = 0.318(m/s)$$

and at point $(1,1)$

$$u_2 = \frac{20}{2\pi}\frac{1+1}{(1+1)^2+1^2} - \frac{40}{2\pi}\frac{1-2}{(1-2)^2+1^2} = 4.46(m/s)$$

$$v_2 = \frac{20}{2\pi}\frac{1}{(1+1)^2+1^2} - \frac{40}{2\pi}\frac{1}{(1-2)^2+1^2} = -2.55(m/s)$$

and at the origin

$$u_0 = \frac{20}{2\pi}\frac{0+1}{(0+1)^2+0^2} - \frac{40}{2\pi}\frac{0-2}{(0-2)^2+0^2} = 6.37(m/s)$$

$$v_0 = \frac{20}{2\pi}\frac{0}{(0+1)^2+0^2} - \frac{40}{2\pi}\frac{0}{(0-2)^2+0^2} = 0(m/s)$$

then, the Bernoulli constant is evaluated from the given pressure and the evaluated velocity at the origin, that is,

$$H = \frac{P_0}{\rho} + \frac{V_0^2}{2} = 0 + \frac{6.37^2}{2} = 20.26(m/s)$$

Therefore, pressures at point $(0,1)$ and point $(1,1)$ are evaluated from Bernoulli's equation as

$$P_1 = \rho\left[H - \frac{1}{2}\left(u_1^2 + v_1^2\right)\right] = 1.8\times\left[20.26 - \frac{1}{2}\times\left(4.14^2 + 0.318^2\right)\right] = 20.97(Pa)$$

and

$$P_2 = \rho\left[H - \frac{1}{2}\left(u_2^2 + v_2^2\right)\right] = 1.8\times\left[20.26 - \frac{1}{2}\times\left(4.46^2 + (-2.55)^2\right)\right] = 12.78(Pa)$$

respectively.

Problem 3: Velocity field of a flow is given as follows,

$$u = y + 2z, \quad v = z + 2x, \quad w = x + 2y$$

① Determine the vorticity field and write down the equation of vortex lines; ② Calculate the flux of vorticity in a cross section of area $dS = 0.0001m^2$ on the plane $x+y+z=1$.

Solution: ① Vorticity is directly evaluated from the velocity distribution

$$\boldsymbol{\omega} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+2z & z+2x & x+2y \end{vmatrix} = (1,1,1)$$

According to definition, vortex line is determined from equation

$$\boldsymbol{\omega} \times d\mathbf{r} = 0$$

or

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$$

and finally vortex line passing through point (x_0, y_0, z_0) is expressed as

$$\begin{cases} x - x_0 = y - y_0 \\ y - y_0 = z - z_0 \end{cases}$$

or

$$\begin{cases} x - x_0 = t \\ y - y_0 = t \\ z - z_0 = t \end{cases} \qquad t \in (-\infty, +\infty).$$

② Vorticity flux across area S is expressed as

$$I = \int_S \boldsymbol{\omega} \cdot \mathbf{n} dS$$

Here unit normal vector is a constant

$$\mathbf{n} = \frac{(1,1,1)}{|(1,1,1)|} = \frac{(1,1,1)}{\sqrt{3}}$$

and $\boldsymbol{\omega} = (1,1,1)$ is also a constant, thus

$$I = \int_S \boldsymbol{\omega} \cdot \mathbf{n} dS = \boldsymbol{\omega} \cdot \mathbf{n} dS = (1,1,1) \cdot \frac{(1,1,1)}{\sqrt{3}} \times 0.0001 = 0.0001\sqrt{3} = 0.00017 \left(m^2/s\right).$$

Problem 4: A flow is a superposition of a uniform flow of speed $u_0 = 10 m/s$ along positive x-axis with a point vortex at the origin. If a stagnation point is at (0, -5), please ① determine the intensity of the vortex; ② calculate the velocity at (0, 5);

③ write down the equation of the streamline passing through the stagnation point.

Solution: ① Velocity potential of the flow is a superposition of the uniform flow and a point vortex at the origin, that is,

$$\phi = u_0 x - \frac{\Gamma}{2\pi}\theta$$

and the corresponding velocity is

$$u = \frac{\partial \phi}{\partial x} = u_0 + \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$$

$$v = -\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$$

Since point $(0,5)$ is a stagnation point, it results

$$\Gamma = -2\pi u_0 \frac{x^2+y^2}{y} = -2\pi \times 10 \times \frac{0^2+(-5)^2}{-5} = 100\pi\left(m^2/s\right).$$

② Velocity at point $(0,5)$ is directly evaluated as follows

$$u = 10 + \frac{100\pi}{2\pi}\frac{5}{0^2+5^2} = 20\left(m/s\right)$$

$$v = -\frac{100\pi}{2\pi}\frac{0}{0^2+5^2} = 0\left(m/s\right)$$

③ Streamline passing through the stagnation point $(0,-5)$ is a special case of the general streamlines.

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{u_0 + \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}} = \frac{dy}{-\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}}$$

that is,

$$\frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2} + u_0 y + C = 0 .$$

Substituting all the parameters and $x=0$, $y=-5$ in above expression, constant C is determined, then dividing out a common factor 50 on the left side, finally the streamline passing through the stagnation point $(0,-5)$ is written as

$$5\ln\sqrt{x^2+y^2}+y+\left(5-5\ln 5\right)=0.$$

Problem 5: A three dimensional axle symmetric flow of velocity potential

$$\varphi=U_0 r(1+\frac{a^3}{2r^3})\cos\theta$$

where U_0 and a are constants, θ is the polar angle from the symmetric axle, and r is the radial distance from the origin. ① Prove that for $r\geq a$ it is equivalent to the flow of a uniform flow past a fixed sphere of radius a. ② Determine positions on the sphere at which the velocities take maximum value and the value of U_0.

Solution: ① As radial distance is getting large enough, the velocity potential is dominated by the first term, that is,

$$\varphi=U_0 r(1+\frac{a^3}{2r^3})\cos\theta \underset{r\to\infty}{\doteq} U_0 r\cos\theta$$

that means that in the far field it is equivalent to a uniform flow with velocity U_0 along x-axis. Besides, since radial velocity at sphere $r=a$ is

$$u_r\big|_{r=a}=\frac{\partial\varphi}{\partial r}\bigg|_{r=a}=U_0(1-\frac{a^3}{r^3})\cos\theta\bigg|_{r=a}=0$$

equal to 0, it means that the sphere is an impermeable surface. As a summary, the given velocity potential represents a uniform flow with velocity U_0 along x-axis from far field passing an impermeable sphere of radius a.

② On the sphere, only the polar component of velocity is non-zero, that is,

$$u_\theta\big|_{r=a}=\frac{\partial\varphi}{r\partial\theta}\bigg|_{r=a}=-U_0(1+\frac{a^3}{2r^3})\sin\theta\bigg|_{r=a}=-\frac{3}{2}U_0\sin\theta.$$

At the uppermost and the lowest position of the sphere, i.e. $\theta=-\pi/2$, velocity takes the maximum value $(3/2)U_0$. At $\theta=\pm\sin^{-1}(2/3)$ and $\theta=\pi\pm\sin^{-1}(2/3)$, velocity takes value U_0.

Problem 6: An axle symmetric flow is generated by a point source of intensity m_1

= 60 m^3/s at the origin and another point source of intensity m_2 = 30 m^3/s at (0, 0, 2). Calculate velocities at (-1, -2, 0) and (1, 1, 1).

Solution: From the problem, symmetric axle of the flow coincides with *oz* axis. Velocity potential of the flow is as follows

$$\varphi = \varphi_1 + \varphi_2 = -\frac{m_1}{4\pi\sqrt{z^2 + r^2}} - \frac{m_2}{4\pi\sqrt{(z-2)^2 + r^2}}$$

From it velocity is immediately derived. Two components are non-zero. They are

$$v_z = \frac{\partial \varphi}{\partial z} = \frac{m_1 z}{4\pi(z^2 + r^2)^{3/2}} + \frac{m_2(z-2)}{4\pi[(z-2)^2 + r^2]^{3/2}}$$

$$v_r = \frac{\partial \varphi}{\partial r} = \frac{m_1 r}{4\pi(z^2 + r^2)^{3/2}} + \frac{m_2 r}{4\pi[(z-2)^2 + r^2]^{3/2}}.$$

At point $(-1,-2,0)$, $r = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$, $z = 0$, the velocity components are evaluated

$$v_z = \frac{60 \times 0}{4\pi\left[0^2 + \left(\sqrt{5}\right)^2\right]^{3/2}} + \frac{30 \times (0-2)}{4\pi\left[(0-2)^2 + \left(\sqrt{5}\right)^2\right]^{3/2}} = -\frac{5}{9\pi} = -0.177(m/s)$$

$$v_r = \frac{60 \times \sqrt{5}}{4\pi\left[0^2 + \left(\sqrt{5}\right)^2\right]^{3/2}} + \frac{30 \times \sqrt{5}}{4\pi\left[(0-2)^2 + \left(\sqrt{5}\right)^2\right]^{3/2}} = \frac{54 + 5\sqrt{5}}{18\pi} = 1.153(m/s)$$

At point $(1,1,1)$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $z = 1$, the velocity components are evaluated

$$v_z = \frac{60 \times 1}{4\pi\left[1^2 + \left(\sqrt{2}\right)^2\right]^{3/2}} + \frac{30 \times (1-2)}{4\pi\left[(1-2)^2 + \left(\sqrt{2}\right)^2\right]^{3/2}} = \frac{5}{2\pi\sqrt{3}} = 0.459(m/s)$$

$$v_r = \frac{60 \times \sqrt{2}}{4\pi\left[1^2 + \left(\sqrt{2}\right)^2\right]^{3/2}} + \frac{30 \times \sqrt{2}}{4\pi\left[(1-2)^2 + \left(\sqrt{2}\right)^2\right]^{3/2}} = \frac{15\sqrt{2}}{2\pi\sqrt{3}} = 1.949(m/s).$$

Problem 7: A sphere is fixed in a uniform flow field. If *a* is the radius of the sphere, U_0 and P_0 are the speed and pressure of the uniform flow, find the maximum and minimum pressures on the sphere and their positions.

Solution: In a spherical coordinate system with origin at the center of the sphere, velocity potential of the flow is expressed as

$$\varphi = U_0 r(1+\frac{a^3}{2r^3})\cos\theta$$

where r is the radial distance from the origin and θ is the meridian angle from the axis parallel to the direction of the uniform flow. The corresponding velocity is derived from the velocity potential as follows,

$$v_r = \frac{\partial\varphi}{\partial r} = U_0(1-\frac{a^3}{r^3})\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -U_0(1+\frac{a^3}{2r^3})\sin\theta.$$

On the sphere, $r=a$, we have $v_r = 0$ and $v_\theta = -\frac{3}{2}U_0\sin\theta$. According to Bernoulli's equation, pressure on the sphere is written as

$$P = P_0 + \rho\frac{U_0^2}{2} - \rho\frac{v_\theta^2}{2} = P_0 + \rho\frac{U_0^2}{2}\left(1-\frac{9}{4}\sin^2\theta\right).$$

It takes maximum value $P_{\max} = P_0 + \frac{1}{2}\rho U_0^2$ at $\theta = 0$ and $\theta = \pi$. It takes minimum value $P_{\min} = P_0 - \frac{5}{8}\rho U_0^2$ at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

Problem 8: Given $\varphi = V_0(r+\frac{a^2}{r})\cos\theta$ the velocity potential of a circular flow. Please calculate the resultant hydrodynamic force on the semi-circle in Figure 6-2.

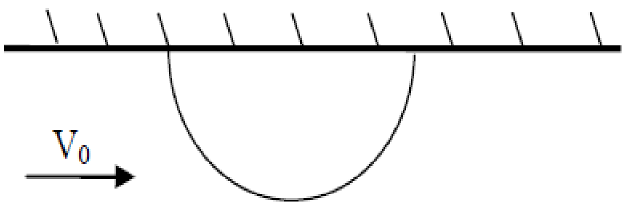

Figure 6-2

Solution: From the velocity potential, velocity on the semi-circle, $r=a$ and $\theta\in[-\pi,0]$, can be immediately evaluated as follows

$$u_\theta\big|_{r=a} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\bigg|_{r=a} = -V_0(1+\frac{a^2}{r^2})\sin\theta\bigg|_{r=a} = -2V_0\sin\theta,$$

while $u_r\big|_{r=a} = 0$. According to Bernoulli's equation, dynamic pressure on the semi-circle is expressed as

$$p(\theta) = -\frac{1}{2}\rho u_\theta^2 = -2\rho V_0^2\sin^2\theta,$$

which is normal to and inward to the semi-circle. Due to horizontal symmetry, the resultant dynamic force is vertical and evaluated below,

$$F_y = \int_{-\pi}^{0} p(\theta)\sin\theta\cdot a d\theta = \int_{-\pi}^{0}\left(-2\rho V_0^2\sin^2\theta\right)\left|\sin\theta\right|\cdot a d\theta = -\frac{8}{3}a\rho V_0^2.$$

That is, the resultant dynamic force is vertically downward of magnitude $\frac{8}{3}a\rho V_0^2$.

Problem 9: A circular cylinder of radius *R* and length *L* is suspended at a fixed point O with thin light ropes *OA* and *OB*. Denote A and B the two end points of the axle of the circular cylinder. Points O, A and B forms a isosceles triangle, i.e., $\overline{OA} = \overline{OB}$. Denote o the midpoint of the axle of the circular cylinder, it is given that $\overline{oO} = l$. Now axle *AB* of the circular cylinder is globally rotating around the fixed point O at an angular velocity Ω, and concurrently the circular cylinder is locally rotating around its axle *AB* at an angular velocity ω. Given weight of the cylinder is *G*, fluid density is ρ, and $l \gg R$. Calculate tensions applied to the ropes *OA* and *OB* respectively.

Solution: For the global rotation, a centrifugal force will be applied to the rope, and it is directly estimated, similar to a mass point moves around a point at a constant angular velocity, as

$$F_C = \frac{G}{g}\frac{V^2}{l} = \frac{G}{g}\frac{(l\Omega)^2}{l} = \frac{Gl\Omega^2}{g}$$

where g is the gravitational acceleration.

On the other hand, due to local rotation and the viscosity, fluid particles on the surface of the circular cylinder will be move with the cylinder at a speed of $u = \omega R$ and it will cause a circulation, $\Gamma = 2\pi R u = 2\pi R^2 \omega$, around the cylinder. Since $l \gg R$, globally, the cylinder is approximately moving at a constant speed, $U = \Omega l$. As a result, a lift force

$$F_\omega = \rho U \Gamma \cdot L = \rho \Omega l \cdot 2\pi R^2 \omega \cdot L = 2\pi \rho \omega \Omega l R^2 L$$

is exerted on the cylinder. Direction of this lift force will be changed from centrifugal to centric alternatively.

Furthermore, as the cylinder is globally rotating around the fixed point O, it will accompany a centric acceleration $a_C = U^2/l = \Omega^2 l$, and an added mass, $\lambda_{CC} = \rho \pi R^2 L$, has to be taking into account. As a result, it will cause a centrifugal force

$$F_\lambda = \lambda_{CC} a_C = \rho \pi R^2 L \cdot \Omega^2 l = \rho \pi l L R^2 \Omega^2.$$

Now put above 3 forces together, rope oO will be applied a tension, i.e. centrifugal force, of magnitude

$$\begin{aligned} F &= F_C \pm F_\omega + F_\lambda \\ &= \frac{G l \Omega^2}{g} \pm 2\pi \rho \omega \Omega l R^2 L + \rho \pi l L R^2 \Omega^2 \\ &= \left[\frac{G}{g} + \rho \pi R^2 L \left(1 \pm 2 \cdot \frac{\omega}{\Omega} \right) \right] \cdot \Omega^2 l \\ &= \left[\frac{G}{g} + \frac{\Delta}{g} \left(1 \pm 2 \cdot \frac{\omega}{\Omega} \right) \right] \cdot \Omega^2 l \\ &= \left[\frac{G + \Delta}{g} \pm 2 \cdot \frac{\Delta}{g} \cdot \frac{\omega}{\Omega} \right] \cdot \Omega^2 l \end{aligned}$$

where $\Delta = \rho g \pi R^2 L$ is the weight of the fluid displaced by the circular cylinder.

Here, rope oO does not exist, instead rope OA and rope OB suspend the cylinder. We can immediately find the tension force on the rope OA and rope OB from the resultant force F based on the geometric relations.

$$\begin{aligned} F_{OA} = F_{OB} &= \frac{\sqrt{l^2 + (L/2)^2}}{l} \frac{F}{2} = \frac{\sqrt{4l^2 + L^2}}{l} F \\ &= \left[\frac{G + \Delta}{g} \pm 2 \cdot \frac{\Delta}{g} \cdot \frac{\omega}{\Omega} \right] \cdot \Omega^2 \cdot \sqrt{4l^2 + L^2} \end{aligned}$$