Introduction to Marine Hydrodynamics (NA235)

(2014-2015, 2nd Semester)

Assignment No.5

(Seven problems, given on Apr 16, submitted on Apr 27, 2015)

Problem 1: Given velocity field of a flow:

$$u = y + 2z, \quad v = z + 2x, \quad w = x + 2y$$

Determine: (1) Vorticity field of the flow and the equation of vortex lines;

(2) Vortex strength passing a cross section with area $dS = 0.0001m^2$ on the plane $x + y + z = 1$.

Solution: Let the three components of the vorticity be $\Omega_x, \Omega_y, \Omega_z$, then:

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2 - 1 = 1$$

$$\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2 - 1 = 1$$

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2 - 1 = 1$$

So the vorticity is: $\vec{\Omega} = \vec{i} + \vec{j} + \vec{k}$

The equation of vortex lines is: $\dfrac{dx}{\Omega_x} = \dfrac{dy}{\Omega_y} = \dfrac{dz}{\Omega_z}$, then: $\begin{cases} \dfrac{dx}{1} = \dfrac{dy}{1} \\ \dfrac{dy}{1} = \dfrac{dz}{1} \end{cases}$

Integrating, $\begin{cases} x - y = C_1 \\ y - z = C_2 \end{cases}$

Let the unit normal vector of $x+y+z=1$ be $\vec{n}(l,m,n)$,

Because $F(x,y,z)=x+y+z-1=0$, then: $\frac{\partial F}{\partial x}=1,\quad \frac{\partial F}{\partial y}=1,\quad \frac{\partial F}{\partial z}=1$

$$l=\frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}}=\frac{1}{3}\sqrt{3}$$

Similarly, $m=n=\frac{1}{3}\sqrt{3}$

$\Omega_n=\vec{\Omega}\cdot d\vec{S}=(\vec{i}+\vec{j}+\vec{k})\cdot(\vec{i}+\vec{j}+\vec{k})\cdot\frac{1}{3}\sqrt{3}=\sqrt{3}$

So the strength of the vortex tube is:

$$\Gamma=\Omega_n dS=0.0001\times\sqrt{3}=0.000173m^2/s$$

Problem 2: A planar fluid flow is given in a polar coordinate system:

$$v_r=U_0(1-\frac{a^2}{r^2})\cos\theta,\quad v_\theta=-U_0(1+\frac{a^2}{r^2})\sin\theta+\frac{k}{r}$$

where a, k, U_0 are constants. Determine the velocity circulation around an arbitrary closed curve, which encloses the circle centered at the origin of radius $r=a$.

Solution: At $r=a$, $v_r=0$, which satisfies the no- penetration condition, so the circle can be regarded as a solid boundary, then the vorticity is:

$$\begin{aligned}\Omega&=\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\\&=U_0\frac{2a^2}{r^3}\sin\theta-\frac{k}{r^2}-\frac{U_0}{r}(1+\frac{a^2}{r^2})\sin\theta+\frac{k}{r^2}+\frac{U_0}{r}(1-\frac{a^2}{r^2})\sin\theta\\&=0\end{aligned}$$

The flow outside of the circle *r*=*a* is irrotational.

The velocity circulation around any closed curve *C* enclosing the circle $r = a$ is:

$$\Gamma_C = \Gamma_{r=a} = \int_0^{2\pi} [-U_0(1+\frac{a^2}{r^2})\sin\theta + \frac{k}{r}] r d\theta = 2\pi k$$

Problem 3: Given velocity distribution of a flow: $u = -\omega y, \quad v = \omega x$**.** Determine (1) Velocity circulation around the circle with a radius *R* and the vortex flux passing through the area surrounded by that circle; (2) Velocity circulation around closed curve *abcd* (see Figure 5-3) and the vortex flux passing through the area bounded by that curve.

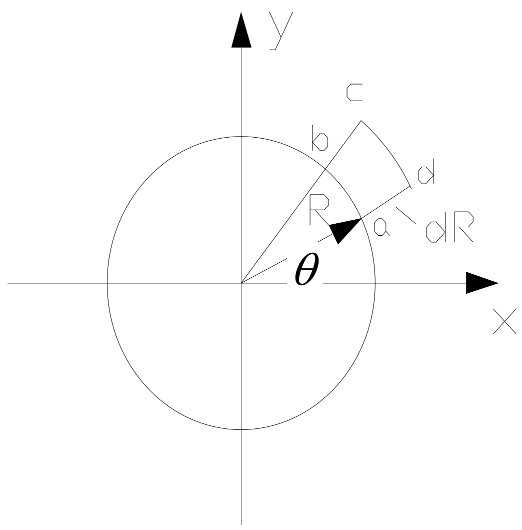


Figure 5-3

Solution: This is a plane flow, so the vorticity is a scalar quantity:

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega = const$$

which is independent of the coordinates. In polar coordinates, the velocity distribution is:

$$\begin{aligned} v_r &= u\cos\theta + v\sin\theta = -\omega r\sin\theta\cos\theta + \omega r\cos\theta\sin\theta \\ &= 0 \\ v_\theta &= -u\sin\theta + v\cos\theta = \omega r\sin^2\theta + \omega r\cos^2\theta \\ &= \omega r \end{aligned}$$

(1) Velocity circulation around a circle with a radius R is:

$$\Gamma_{r=R} = \int_0^{2\pi} v_\theta r d\theta = \omega R^2 \int_0^{2\pi} d\theta = 2\pi\omega R^2$$

Based on Stokes' theorem, the vortex flux is:

$$\phi = \iint_\sigma \Omega_n d\sigma = \Gamma_{r=R} = 2\pi\omega R^2$$

(2) For the closed curve $abcd$, because $\Omega = 2\omega = const$, the vortex flux is:

$\phi = \Omega\sigma = 2\omega R \cdot dR \cdot d\theta = \Gamma_{abcd}$

Problem 4: Suppose an ideal fluid is barotropic and under the action of body forces with potential Θ. Now if at an instant velocity field $\vec{V}$ of such a flow is irrotational, then verify that the corresponding local acceleration field $\frac{\partial \vec{V}}{\partial t}$ will be irrotational as well at any instant. Furthermore, derive the theorem that in that case vortex can be neither created nor destroyed.

Solution: From Euler equation: $\frac{\partial \vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}=\vec{F}-\frac{1}{\rho}\nabla P$,

where $(\vec{V}\cdot\nabla)\vec{V}=\nabla(\frac{V^2}{2})-\vec{V}\times\vec{\Omega}$

If the body force is potential, i.e., $\vec{F}=-\nabla\Theta$; the fluid is barotropic, i.e., $\nabla\Pi=\frac{1}{\rho}\nabla P$, then Euler equation can be transformed as:

$$\frac{\partial \vec{V}}{\partial t}=-\nabla(\frac{V^2}{2}+\Pi+\Theta)+\vec{V}\times\vec{\Omega}$$

Suppose the velocity field $\vec{V}$is irrotational at a time t=t_1, then:

$$(\frac{\partial \vec{V}}{\partial t})_{t_1}=-\nabla(\frac{V^2}{2}+\Pi+\Theta)+\vec{V}\times\vec{\Omega}$$

So at this time, $(\frac{\partial \vec{V}}{\partial t})_{t_1}$ is also irrotational (because it is potential).

For another time t2 after t1, the velocity $\vec{V}$can be expanded as:

$$\vec{V}_{t_2}=\vec{V}_{t_1}+(\frac{\partial \vec{V}}{\partial t})_{t_1}(t_2-t_1)+(\frac{\partial^2 \vec{V}}{\partial t^2})_{t_1}\frac{(t_2-t_1)^2}{2}+\cdots$$

If $\Delta t=t_2-t_1$is small enough, the high order terms can be neglected, i.e.,:

$$\vec{V}_{t_2}=\vec{V}_{t_1}+(\frac{\partial \vec{V}}{\partial t})_{t_1}(t_2-t_1)$$

Because $\vec{V}_{t_1}$, $(\frac{\partial \vec{V}}{\partial t})_{t_1}$ are all irrotational, $\vec{V}_{t_2}$is also irrotational. This in turn, can verify that $\vec{V}_{t_3}$, $\vec{V}_{t_4}$ …… are irrotational.

Problem 5: Four vortices with an equal strength Γ initially located at (1, 0), (0, 1), (-1, 0), (0, -1) respectively. Determine the path for each of them.

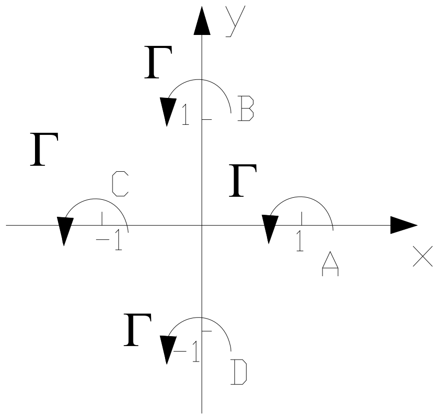


Figure 5-5

Solution: Because of the symmetry, only consider the motion of one of the vortex. Take the vortex A as an example, its velocity is induced by the vortices at points B, C, D.

$$V_{BA} = \frac{\Gamma}{2\pi \cdot \sqrt{2}} = \frac{\Gamma}{2\sqrt{2}\pi}$$

$$V_{DA} = \frac{\Gamma}{2\pi \cdot \sqrt{2}} = \frac{\Gamma}{2\sqrt{2}\pi}$$

$$V_{CA} = \frac{\Gamma}{2\pi \cdot 2} = \frac{\Gamma}{4\pi}$$

So the two components of velocity at point A are:

$$u_A = 0, \quad v_A = V_{CA} + \sqrt{V_{BA}^2 + V_{DA}^2} = \frac{\Gamma}{4\pi} + \frac{\Gamma}{2\pi} = \frac{3\Gamma}{4\pi}$$

Because of the symmetry, the origin is the center of gravity of the four vortices and it is a fixed point, Vortex A will rotate around the origin, the angular velocity is:

$$\omega = \frac{v_A}{r} = \frac{3\Gamma}{4\pi \cdot 1} = \frac{3\Gamma}{4\pi}$$

Thus, in polar coordinates, the motion equation of point A is:

$$r_A = 1, \quad \theta_A = \frac{3\Gamma}{4\pi} t$$

Similarly,

$$r_B = 1, \quad \theta_B = \frac{3\Gamma}{4\pi} t + \frac{\pi}{2}$$
$$r_C = 1, \quad \theta_C = \frac{3\Gamma}{4\pi} t + \pi$$
$$r_D = 1, \quad \theta_D = \frac{3\Gamma}{4\pi} t + \frac{3\pi}{2}$$

Problem 6: Suppose a circular vortex line, whose radius is *a*, and strength is Γ. Determine the induced velocity on the symmetry axis.

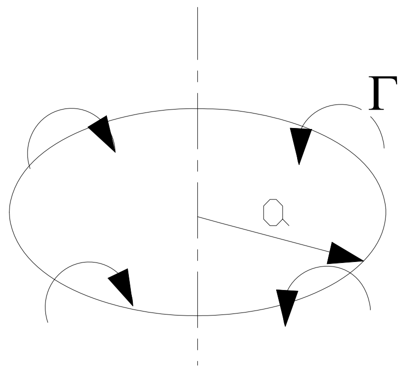


Figure 5-6

Solution: Take the symmetry axis is *z* axis, pointing upwards, and take the center of the vortex circle as the origin of the coordinates, as shown below:

Take an element *ds* on the circular vortex line, it induces a velocity to point M at z axis:

$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{d\vec{s}\times d\vec{r}}{r^3}$$

Its norm is: $\left|d\vec{V}\right| = \frac{\Gamma}{4\pi}\frac{ad\theta}{r^2} = \frac{\Gamma a d\theta}{4\pi(a^2+z^2)}$

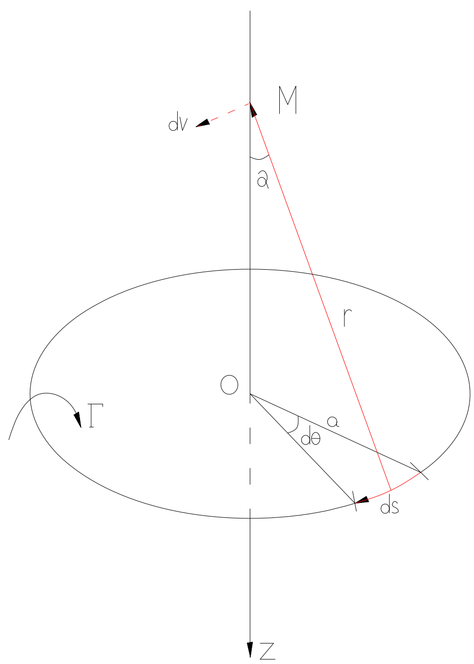
The projection of $d\vec{V}$ on z axis is:

$$\begin{aligned} dw &= \frac{-\Gamma a d\theta}{4\pi(a^2+z^2)}\sin\alpha \\ &= \frac{-\Gamma a d\theta}{4\pi(a^2+z^2)}\cdot\frac{a}{\sqrt{a^2+z^2}} \\ &= \frac{-\Gamma a^2 d\theta}{4\pi(a^2+z^2)^{3/2}} \end{aligned}$$

The induced velocity at point M by the whole circular vortex is:

$$w = \oint dw = -\int_0^{2\pi}\frac{-\Gamma a^2}{4\pi(a^2+z^2)^{3/2}}d\theta = \frac{-\Gamma a^2}{2(a^2+z^2)^{3/2}}$$

If the circular vortex is slipping downwards along z axis with an uniform velocity, then the induced velocity at point M decreases.



Problem 7: Two vortices at a distance r with strengths Γ_1 and Γ_2 respectively, of same magnitude $|\Gamma_1| \neq |\Gamma_2|$. Determine motions of these vortices for Γ_1 and Γ_2 with same or opposite signs.

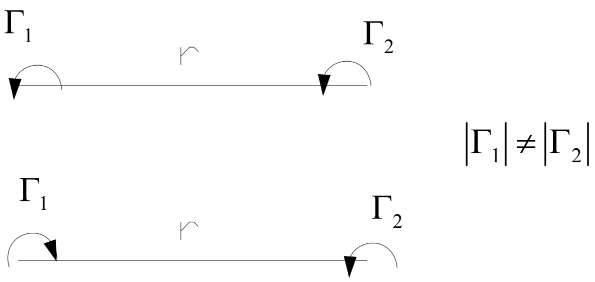


Figure 5-7

Solution: (1) Γ_1 and Γ_2 have the same sign

Take the location of Γ_1 as the origin of the coordinates, x axis is the direction along r to the right side, then the coordinates of Γ_1 and Γ_2 are Γ_1 (0, 0) and Γ_2 (r, 0), the coordinates of the center of gravity C is:

$$\xi_C = \frac{\Gamma_1 \xi_1 + \Gamma_2 \xi_2}{\Gamma_1 + \Gamma_2} = \frac{\Gamma_2 r}{\Gamma_1 + \Gamma_2} < r$$

$$\eta_C = \frac{\Gamma_1 \eta_1 + \Gamma_2 \eta_2}{\Gamma_1 + \Gamma_2} = 0$$

The center of gravity locates in between the two vortices. If $\Gamma_2 > \Gamma_1$, the center of gravity is close to Γ_2, the two vortices will rotate around the center of gravity C.

The induced velocity at Γ_1 is: $v_1 = \dfrac{\Gamma_2}{2\pi r}$

The angular velocity of the two vortices is: $\omega = \frac{v_1}{\xi_C} = \frac{\Gamma_1 + \Gamma_2}{2\pi r^2}$

(1) Γ_1 and Γ_2 have the opposite sign

Suppose $|\Gamma_1| < |\Gamma_2|$, the coordinates of Γ_1 and Γ_2 are Γ_1 (0, 0) and Γ_2 (r, 0), the coordinates of the center of gravity C is:

$$\xi_C = \frac{\Gamma_2 r}{\Gamma_1 + \Gamma_2} > r$$
$$\eta_C = 0$$

The two vortices rotate around the center of gravity.

The induced velocity at Γ_1 is: $v_1 = \frac{\Gamma_2}{2\pi r}$

The angular velocity of the two vortices is: $\omega = \frac{v_1}{\xi_C} = \frac{\Gamma_1 + \Gamma_2}{2\pi r^2}$