Introduction to Marine Hydrodynamics (NA235)

(2014-2015, 2nd Semester)

Solutions to Assignment No.3

(Eight problems, submitted on March 30th, 2015)

Problem 1: Consider two flows:

$$(a)\begin{cases} v_x = 1 \\ v_y = 2 \end{cases}; \quad (b)\begin{cases} v_x = 4x \\ v_y = -4y \end{cases}$$

(1) Determine if the flow (a) has the stream function ψ. If does, solve the stream function and plot graph of the stream function;

(2) Determine if the flow (b) has the velocity potential φ. If does, solve φ, and plot the equipotential lines.

Solution: (1) For flow (a), $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$, which satisfies the continuity equation of a 2D incompressible flow, so there exists the stream function.

Based on: $\begin{cases} \frac{\partial \psi}{\partial y} = v_x = 1 \\ \frac{\partial \psi}{\partial x} = -v_y = -2 \end{cases}$,

the stream function is solved as: $\psi = y - 2x$ (the integral constant has been neglected).

From the definition of the streamline: $\psi = c$, so the streamline equation is: y-2x=c, the streamlines are a group of parallel straight lines with a slope of 2.

(2) Because the vorticity is $\Omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$, flow (b) is irrotational flow, with potential function.

From $\begin{cases} \frac{\partial \varphi}{\partial x} = v_x = 4x \\ \frac{\partial \varphi}{\partial y} = v_y = -4y \end{cases}$, the potential function is solved as: $\varphi = 2(x^2 - y^2)$ (the integral constant has been neglected).

Similarly, flow (b) satisfies the continuity equation of a 2D incompressible flow, so there exist the stream function.

From $\begin{cases} \frac{\partial \psi}{\partial y} = v_x = 4x \\ \frac{\partial \psi}{\partial x} = -v_y = 4y \end{cases}$, the stream function is $\psi = 4xy$. Thus, the equipotential lines are expressed as: $x^2 - y^2 = c$, the streamline equation is: $xy = c$. The equipotential lines and streamlines are two groups of hyperbolic lines.

Problem 2: Consider a plane flow field $v_x = 1 + 2t$, $v_y = 3 + 4t$. Determine: (1) Streamline equation; (2) at t=0, the shapes of three streamlines passing points (0, 0), (0, 1), (0, -1); (3) at t=0, the pathline equation of a fluid particle locating at point (0, 0).

Solution: (1) Because $\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$, it satisfies the continuity equation of a 2D incompressible flow, there exist the stream function.

From $\begin{cases}\frac{\partial \psi}{\partial y}=v_x=1+2t \\ \frac{\partial \psi}{\partial x}=-v_y=-3-4t\end{cases}$, the stream function is determined as: $\psi=(1+2t)y-(3+4t)x$ (the integral constant has been neglected). The streamline equation is: $(1+2t)y-(3+4t)x=c$.

(2) at t=0 and point (0, 0), the constant c is 0, so the streamline equation is: $y-3x=0$;

at t=0 and point (0, 1), the constant c is 1, so the streamline equation is: $y-3x-1=0$;

at t=0 and point (0, -1), the constant c is -1, so the streamline equation is: $y-3x+1=0$.

(3) From the velocity distribution: $\frac{dx}{dt}=v_x=1+2t$, $\frac{dy}{dt}=v_y=3+4t$

Thus, $x=t+t^2+c_1$, $y=3t+2t^2+c_2$

And because at t=0 and point (0, 0), the constants c_1=c_2=0.

The pathline equation is: $x=t+t^2$, $y=3t+2t^2$

Problem 3: The velocity components of an incompressible plane flow are given as: $v_x=1-y$, $v_y=t$. Determine: (1) at t=0, the pathline equation of a particle passing point (0, 0); (2) at t=1, the streamline equation of a particle passing point (0, 0).

Solution: (1) Differential form of pathline equation is:

$$\begin{cases} \dfrac{\mathrm{d}x}{v_x} = \mathrm{d}t & \text{(a)} \\ \dfrac{\mathrm{dy}}{v_y} = \mathrm{d}t & \text{(b)} \end{cases}$$

From equation (b), $\mathrm{d}y = v_y \mathrm{d}t = t\mathrm{d}t$, integrating, we get: $y = \frac{1}{2}t^2 + c_1$

Because t=0, y=0, the constant c_1=0,

$$y = \frac{1}{2}t^2 \qquad \text{(c)}$$

From equation (a), $\mathrm{d}x = v_x \mathrm{d}t = (1-y)\mathrm{d}t$, substituting (c) into this equation: $\mathrm{d}x = (1-\frac{1}{2}t^2)\mathrm{d}t$

Integrating, we get: $x = t - \frac{1}{6}t^3 + c_2$

Because t=0, y=0, the constant c_2=0,

$$x = t - \frac{1}{6}t^3 \qquad \text{(d)}$$

Thus, the parameter equation of pathline at t=0 passing (0, 0) is (c)+(d):

$$\begin{cases} y = \dfrac{1}{2}t^2 \\ x = t - \dfrac{1}{6}t^3 \end{cases}$$

Eliminating t, pathline equation of a particle passing point (0, 0)is:

$$\frac{2}{9}y^3 - \frac{4}{3}y^2 + 2y - x^2 = 0$$

(2) The streamline equation is:

$$\frac{dx}{v_x}=\frac{dy}{v_y}\Rightarrow\frac{dx}{1-y}=\frac{dy}{t} \Rightarrow t dx=(1-y)dy$$, time t is a parameter.

Integrating, the streamline equation is: $tx=(y-\frac{y^2}{2})+c$

From t=1, x=y=0, c=0. So at t=1, the streamline equation of a particle passing point (0, 0) is: $x=\frac{1}{t}(y-\frac{y^2}{2})$

Problem 4: The velocity distribution of an incompressible plane flow is given as: $v_x=x^2+2x-4y,\quad v_y=-2xy-2y$. Determine if the flow: (1) satisfies the continuity equation; (2) is rotational; (3) has the velocity potential and stream function. If does, solve them.

Solution: (1) The condition of continuity equation is: $\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=(2x+2)+(-2x-2)=0$$

The flow satisfies the continuity equation.

(2) For the plane flow, the components of the angular velocity are:

$$\omega_x=\omega_y=0\ ,\quad \omega_x=\frac{1}{2}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})=\frac{1}{2}[-2y-(-4)]=2-y\neq 0$$

So the flow is rotational.

(3) Because the flow is rotational, there is no velocity potential.

The velocity components satisfy the continuity equation of a 2D incompressible flow, there is the stream function ψ:

$$\partial\psi/\partial x = -v_y, \partial\varphi/\partial y = v_x$$

1) Solve stream function ψ by method of undetermined coefficients

$$\partial\varphi/\partial y = v_x = x^2 + 2x - 4y$$

Integrating y, $\psi = \int (x^2 + 2x - 4y)dy = (x^2 + 2x)y - 2y^2 + f(x)$ (1)

This is the partial derivative of y, so the integral constant should be the function of x: $f(x)$, i.e., undetermined coefficient.

Besides, ψ satisfies $\partial\psi/\partial y = -v_y$, so from equation (1):

$$\partial\psi/\partial x = (2x+2)y + f^{'}(x) = -(-2x-2)y$$

$$f^{'}(x) = 0, f(x) = const$$

Substituting $f(x) = const$ into equation (1) and omitting the constant:

$$\psi = x^2 y + 2xy - 2y^2$$

2) Solve stream function ψ by method of integral path independence

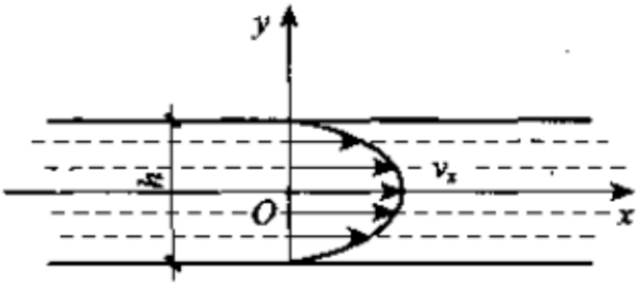
$$d\psi = -v_y dx + v_x dy$$

$$\psi = \int_L -v_y dx + v_x dy$$

Because the integral is independent of path, choose the path as: L: (0,0)-(x,0)-(x,y), then:

$$\begin{aligned}
\psi &= \int_{(0,0)}^{(x,y)} -v_y dx + v_x dy \\
&= \int_{(0,0)}^{(x,y)} -(-2xy - 2y)dx + (x^2 + 2x - 4y)dy \\
&= \int_{(0,0)}^{(x,0)} -(-2x\cdot 0 - 2\cdot 0)dx + \int_{(x,0)}^{(x,y)} (x^2 + 2x - 4y)dy \\
&= x^2 y + 2xy - 2y^2
\end{aligned}$$

Problem 5: Consider the flow between two parallel plates separated by distance $h = 2$ m, the velocity distribution is: $v_x = 10\times\left(\frac{1}{4}h^2 - y^2\right)$ (m/s), $v_y = 0$, axis x coincides with the center line of the two plates. Determine the stream function of the flow field and plot the streamlines in between the two plates.



Solution: using the method of integral path independence

$$\mathrm{d}\psi = -v_y\mathrm{d}x + v_x\mathrm{dy} = v_x\mathrm{dy}$$

$$\psi = \int_L 10\times\left(\frac{1}{4}h^2 - y^2\right)\mathrm{dy} \qquad L\text{: } (0,0)-(0,y)-(x,y)$$

$$= \int_{(0,0)}^{(0,y)} 10\times\left(\frac{1}{4}h^2 - y^2\right)\mathrm{dy} = 10y\left(\frac{1}{4}h^2 - \frac{1}{3}y^2\right)$$

The streamline equation is: $\psi = 10y\left(\frac{1}{4}h^2 - \frac{1}{3}y^2\right) = const$

Thus, $y = const$

The streamlines are a group of straight lines parallel to the center line.

Problem 6: The velocity potential of an incompressible plane potential flow is: $\varphi = 0.04x^3 + axy^2 + by^3$, units of x, y are m, unit of the potential function is m^2/s. (1) Determine the constants a, b; (2) Compute the pressure difference between points (0, 0) and (3, 4), assume the density of the fluid is 1300 kg/m^3.

Solution: (1) $\begin{cases} v_x = \dfrac{\partial \varphi}{\partial x} = 0.12x^2 + ay^2 \\ v_y = \dfrac{\partial \varphi}{\partial y} = 2axy + 3by^2 \end{cases}$ (a)

Because there is velocity potential, the flow must be irrotational, i.e.,

$$\Omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$$

In this problem, $\Omega = 2ay - 2ay = 0$, so the flow is irrotational and Ω is independent of a, b.

The flow has to satisfy the continuity equation, then:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.24x + (2ax + 6by) = (0.24 + 2a)x + 6by = 0$$

x, y are independent variables, so:

$$\begin{cases} 0.24 + 2a = 0 \\ b = 0 \end{cases} \quad \text{i.e.,} \quad \begin{cases} a = -0.12 \\ b = 0 \end{cases}$$

Substituting a and b into equation (a),

$$\begin{cases} v_x = 0.12x^2 - 0.12y^2 \\ v_y = -0.24xy \end{cases} \qquad \text{(b)}$$

(2) Determine the pressure difference: using Bernoulli equation

The plane potential flow deals with the ideal, incompressible, irrotational and steady plane flow, so the integral of steady-flow differential equation of ideal fluids is valid in the flow field:

$$z+\frac{p}{\rho g}+\frac{v^2}{2g}=c \qquad \text{(g)}$$

According to equation (b), at point (0, 0), $\begin{cases} v_x = 0 \\ v_y = 0 \end{cases}$

$$v^2 = 0$$

at point (3, 4), $\begin{cases} v_x = 0.12\times 3^2 - 0.12\times 4^2 = -0.84\text{m/s} \\ v_y = -0.24\times 3\times 4 = 2.88\text{m/s} \end{cases}$

$$v^2 = 9\text{m}^2/\text{s}^2$$

From equation (g),

$$\left.(z+\frac{p}{\rho g}+\frac{v^2}{2g})\right|_{(0,0)} = \left.(z+\frac{p}{\rho g}+\frac{v^2}{2g})\right|_{(3,4)}$$

$$0+\frac{p_{(0,0)}}{\rho g}+0 = 0+\frac{p_{(3,4)}}{\rho g}+\frac{9}{2g}$$

$$\Delta p = p_{(0,0)} - p_{(3,4)} = \rho g(\frac{9}{2g})$$

$$\Rightarrow \qquad = 1300\times 9.807\times\frac{9}{2\times 9.807} = 5850\ \text{Pa}$$

$$= 5.85\ \text{kPa}$$

Problem 7: Consider an incompressible plane flow, the module (norm) of its velocity vector is: $q=\sqrt{x^2+y^2}$ **.** The streamline equation of the flow is: $y^2 - x^2 = c$, where c is a constant. Determine the velocity distribution of this flow.

Solution: Let the stream function: $\psi = A(y^2 - x^2)$ $\quad A:\text{const}$ (a)

The streamline equation is: $\psi = A(y^2 - x^2) = c$, i.e., $y^2 - x^2 = c$

$$\begin{cases} v_x = \dfrac{\partial \psi}{\partial y} = 2Ay \\ v_y = -\dfrac{\partial \psi}{\partial x} = 2Ax \end{cases} \tag{b}$$

The module (norm) $q = \sqrt{v_x^2 + v_y^2} = \sqrt{(2Ay)^2 + (2Ax)^2} = 2|A|\sqrt{x^2 + y^2}$

Consider $q = \sqrt{x^2 + y^2}$, then $2|A|\sqrt{x^2 + y^2} = \sqrt{x^2 + y^2}$, $\quad 2|A| = 1$

$$A = \pm\frac{1}{2}$$

Substituting into equation (b), thus: $\begin{cases} v_x = y \\ v_y = x \end{cases}$ or $\begin{cases} v_x = -y \\ v_y = -x \end{cases}$

Problem 8: The stream function is known as: $\psi = x^2 - y^2$. (1) Determine velocity potential φ; (2) Neglect the mass force, determine the pressure distribution in the flow field.

Solution: (1) From the stream function, the velocity distribution can be determined:

$$\begin{cases} v_x = \dfrac{\partial \psi}{\partial y} = -2y \\ v_y = -\dfrac{\partial \psi}{\partial x} = -2x \end{cases}$$

So the vorticity is $\Omega_z = \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} = -2 - (-2) = 0$, the flow is irrotational, it has the potential function, thus:

$$\begin{cases} \frac{\partial \varphi}{\partial x} = v_x = -2y \\ \frac{\partial \varphi}{\partial y} = v_y = -2x \end{cases}$$

The velocity potential function is solved as: $\varphi = -2xy$ (the integral constant has been neglected).

(2) According to Bernoulli equation:

$$\frac{V^2}{2} + \frac{p}{\rho} + gz = c$$

Substituting the velocity distribution into it and neglecting the mass force, the pressure distribution in the flow field is:

$p = -2\rho(x^2 + y^2) + c$, c is a constant.