Introduction to Marine Hydrodynamics (NA235)

(2014-2015, 2nd Semester)

Solutions to Assignment No.2

(Eight problems, submitted on March 19th, 2015)

Problem 1: Consider a fluid perform an axial rotation at a constant angular acceleration (ε_0) like a rigid body. Express its position, velocity and acceleration from Lagrangian and Eulerian descriptions.

Solution:

(1) Lagrangian description

Since this is a two-dimensional axial rotation, it is more convenient to solved the problem in polar coordinates. Consider the initial position of a fluid particle is (r_0, θ_0), its position can be expressed as:

$$r = r_0$$
$$\theta = \theta_0 + \frac{1}{2}\varepsilon_0 t^2$$

The velocity is written as:

$$v_r = \frac{\partial r}{\partial t} = \frac{\partial r_0}{\partial t} = 0$$
$$v_\theta = r\frac{\partial \theta}{\partial t} = r_0 \varepsilon_0 t$$

And the acceleration is:

$$a_r = -\frac{v_\theta^2}{r} = -\frac{1}{r_0}(r_0\varepsilon_0 t)^2 = -r_0\varepsilon_0^2 t^2$$
$$a_\theta = \frac{\partial v_\theta}{\partial t} = r_0\varepsilon_0$$

In Cartesian coordinates, for a fluid particle locating at (x_0, y_0) at an initial time, its position is expressed as:

$$x = r\cos\theta = r_0\cos(\theta_0 + \frac{1}{2}\varepsilon_0 t^2)$$
$$y = r\sin\theta = r_0\sin(\theta_0 + \frac{1}{2}\varepsilon_0 t^2)$$

Where $r_0 = \sqrt{x_0^2 + y_0^2}$, $\tan\theta_0 = \frac{y_0}{x_0}$

The velocity is:

$$u = \frac{\partial x}{\partial t} = -r_0\varepsilon_0 t\sin(\theta_0 + \frac{1}{2}\varepsilon_0 t^2)$$
$$v = \frac{\partial y}{\partial t} = r_0\varepsilon_0 t\cos(\theta_0 + \frac{1}{2}\varepsilon_0 t^2)$$

The acceleration is:

$$a_x = \frac{\partial u}{\partial t} = -r_0\varepsilon_0\sin(\theta_0 + \frac{1}{2}\varepsilon_0 t^2) - r_0\varepsilon_0^2 t^2\cos(\theta_0 + \frac{1}{2}\varepsilon_0 t^2)$$
$$a_y = \frac{\partial v}{\partial t} = r_0\varepsilon_0\cos(\theta_0 + \frac{1}{2}\varepsilon_0 t^2) - r_0\varepsilon_0^2 t^2\sin(\theta_0 + \frac{1}{2}\varepsilon_0 t^2)$$

(2) Eulerian description

In polar coordinates, the velocity is expressed as:

$$v_r = 0$$
$$v_\theta = r\varepsilon_0 t$$

The acceleration is:

$$a_r = \frac{Dv_r}{Dt} + \frac{v_\theta^2}{r} = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{r\partial \theta} + \frac{v_\theta^2}{r} = r\varepsilon_0^2 t^2$$

$$a_\theta = \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{\partial v_\theta}{r\partial \theta} + \frac{v_r v_\theta}{r} = r\varepsilon_0$$

In Cartesian coordinates, the velocity is:

$$u = r\varepsilon_0 t \cdot (-\sin\theta) = -r\varepsilon_0 t \cdot \frac{y}{r} = -\varepsilon_0 yt$$

$$v = r\varepsilon_0 t \cdot \cos\theta = r\varepsilon_0 t \cdot \frac{x}{r} = \varepsilon_0 xt$$

The acceleration is:

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\varepsilon_0 y - \varepsilon_0^2 t^2 x$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \varepsilon_0 x - \varepsilon_0^2 t^2 y$$

Problem 2: Assume three velocity components in a three-dimensional velocity field are:

$$u = ax$$
$$v = ay$$
$$w = -2az$$

Where a is a constant. Verify that the streamline of this flow is an intersection of two curved surfaces $y^2 z = \text{const}$ and $\frac{x}{y} = \text{const}$.

Solution: The streamline equation is $\frac{dx}{ax} = \frac{dy}{ay} = \frac{dz}{-2az}$

i.e., $\frac{dx}{ax} = \frac{dy}{ay}, \quad \frac{dy}{ay} = \frac{dz}{-2az}$

Integrating these two equations to get:

$$\ln x = \ln y + \ln C_1\text{, i.e., } \frac{x}{y} = C_1$$

$$2\ln y + \ln z = \ln C_2\text{, i.e., } y^2 z = C_2$$

The streamline is thus an intersection of the two curved surfaces when C_1 and C_2 are constant: $y^2 z = \text{const}$ and $\frac{x}{y} = \text{const}$.

Problem 3: the velocity field of a flow is given as:

$$\vec{V} = (x+1)t^2\vec{i} + (y+2)t^2\vec{j}$$

Determine the pathline and streamline equations at time t=1 and at point (2, 1).

Solution: (1) the differential equation of pathline is:

$$\frac{dx}{dt} = u = (x+1)t^2$$

$$\frac{dy}{dt} = v = (y+2)t^2$$

Integrating, yields:

$$\ln(x+1) = \frac{1}{3}t^3 + C_1$$

$$\ln(y+2) = \frac{1}{3}t^3 + C_2$$

Subtracting the second equation from the first one:

$$\ln(x+1)-\ln(y+2)=\ln C$$

$$\text{i.e.,}\quad x+1=C(y+2)$$

Substituting $x=2$, $y=1$ into the equation, we get: C=1

So the pathline equation at (2, 1) is: $x-y=1$

(2) The streamline equation is:

$$\frac{dx}{(x+1)t^2}=\frac{dy}{(y+2)t^2}$$

Eliminating t_2 and integrating, yields:

$$\ln(x+1)=\ln(y+2)+\ln C$$

$$\text{i.e.,}\quad x+1=C(y+2)$$

Substituting $x=2$, $y=1$ into the equation, we get: C=1

So the streamline equation at (2, 1) is: $x-y=1$

Problem 4: The velocity profile in a flow field is given as:

$$\begin{cases} u=yz+t \\ v=xz-t \\ w=xy \end{cases}$$

(1) Is the flow steady? (2) Determine the acceleration of the fluid particle through a field position (1, 1, 1).

Solution:

(1) u, v are dependent on time, therefore the flow is unsteady.

(2) The acceleration in the flow field is:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 1 + (xz - t)z + xy^2$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -1 + (yz + t)z + x^2 y$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = (yz + t)y + (xz - t)x$$

Substituting $x = 1$, $y = 1, z = 1$ into the equation above, yields:

$$a_x = 3 - t$$
$$a_y = 1 + t$$
$$a_z = 2$$

Problem 5: Verify that the acceleration field of an irrotational flow is a potential field.

Solution1:

Assume the velocity field is $\vec{V}(x, y, z, t)$ and the acceleration field is:

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}$$
$$= \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V}^2}{2}) - \vec{V} \times \nabla \times \vec{V}$$

For an irrotational flow, $\vec{\Omega} = \nabla \times \vec{V} = 0$, thus:

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V}^2}{2})$$

The curl of the equation above is:

$$\begin{aligned}\nabla\times\frac{D\vec{V}}{Dt} &= \nabla\times\frac{\partial\vec{V}}{\partial t}+\nabla\times\nabla(\frac{\vec{V}^2}{2}) \\ &= \frac{\partial}{\partial t}(\nabla\times\vec{V})+\nabla\times\nabla(\frac{\vec{V}^2}{2}) \\ &= 0\end{aligned}$$

An irrotational flow is a potential flow, so the acceleration field of an irrotational flow is a potential field.

Solution2:

The acceleration field of an irrotational flow is:

$$\begin{aligned}\frac{D\vec{V}}{Dt} &= \frac{\partial\vec{V}}{\partial t}+\nabla(\frac{\vec{V}^2}{2})-\vec{V}\times\nabla\times\vec{V} \\ &= \frac{\partial}{\partial t}(\nabla\varphi)+\nabla(\frac{\vec{V}^2}{2}) \\ &= \nabla(\frac{\partial\varphi}{\partial t})+\nabla(\frac{\vec{V}^2}{2}) \\ &= \nabla(\frac{\partial\varphi}{\partial t}+\frac{\vec{V}^2}{2})\end{aligned}$$

This means the acceleration of an irrotational flow is the gradient of a scalar function, thus this scalar function is the acceleration potential function.

Problem 6: Assume the velocity fields of two flows are:

$$(a)\ \vec{V}=(-\Omega y,\Omega x,0)$$
$$(b)\ \vec{V}=(-\Omega y/r^2,\Omega x/r^2,0)$$

Where Ω is a constant, and $r^2=x^2+y^2$. (1) Generate the streamline equations of these two flows; (2) Is the flow rotational or irrotational?

Determine the velocity potential of the irrotational flow.

Solution:

(a) $\vec{V} = -\Omega y\vec{i} + \Omega x\vec{j} + 0\vec{k}$, its streamline equation is:

$$\frac{dx}{-\Omega y} = \frac{dy}{\Omega x} = \frac{dz}{0}$$

Thus, $dz = 0$, i.e., $z = C_1$, $\Omega x dx = -\Omega y dy$

Integrating, yields: $x^2 + y^2 = C$

The streamlines are a series of concentric circles.

$\frac{\partial w}{\partial y} = 0 = \frac{\partial v}{\partial z}$, $\frac{\partial u}{\partial z} = 0 = \frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial x} = \Omega$, $\frac{\partial u}{\partial y} = -\Omega$, thus:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Omega - (-\Omega) = 2\Omega$$

The flow is rotational. The angular velocity in *z*-direction is Ω.

(b) $\vec{V} = -\Omega y / r^2\vec{i} + \Omega x / r^2\vec{j} + 0\vec{k}$, its streamline equation is:

$$\frac{dx}{-\Omega y / r^2} = \frac{dy}{\Omega x / r^2} = \frac{dz}{0}$$

Thus, $dz = 0$, i.e., $z = C_1$, $\frac{\Omega x}{r^2} dx = \frac{-\Omega y}{r^2} dy$

Eliminating Ω / r^2 and integrating, yields: $x^2 + y^2 = C$

The streamlines are the same as (a).

$\frac{\partial w}{\partial y}=0=\frac{\partial v}{\partial z}$, $\frac{\partial u}{\partial z}=0=\frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial x}=\frac{\Omega(y^2-x^2)}{(x^2+y^2)^2}=\frac{\partial u}{\partial y}$, so the flow is irrotational.

Integrating over $(0,0)\to(0,y)\to(x,y)$, the velocity potential is :

$$
\begin{aligned}
\varphi &= \int \frac{-\Omega y}{r^2}dx+\int \frac{\Omega x}{r^2}dy \\
&= -\Omega\int_0^x \frac{y}{x^2+y^2}dx+\Omega\int_0^y \frac{0}{0+y^2}dy \\
&= -\Omega y(\frac{1}{y}\arctan\frac{x}{y})+C \\
&= -\Omega\arctan\frac{x}{y}+C
\end{aligned}
$$

Problem 7: The velocity profile of a flow is given as:

$$
\begin{cases} u = a\,y(y^2-x^2) \\ v = a\,x(y^2-x^2) \end{cases}
$$

Where a is a constant. (1) Generate the streamline equation and plot the streamlines; (2) Is the flow rotational? If it is irrotational, determine the velocity potential function and plot the equipotential lines.

Solution:

(1) The streamline equation is:

$$
\frac{dx}{ay(y^2-x^2)}=\frac{dy}{ax(y^2-x^2)}
$$

Eliminating $a(y^2-x^2)$, yields: $xdx=ydy$

Integrating: $x^2-y^2=C$

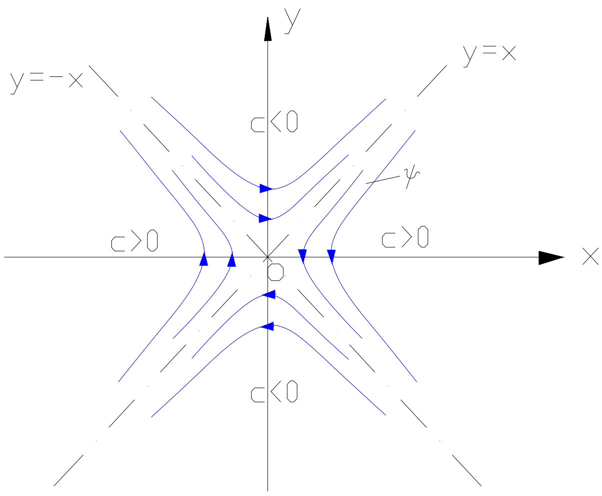
If C is set to various values, we get a family of streamlines (hyperbolas).

From the streamline equation, it can be noted that the straight line $y = \pm x$ is the asymptote of the streamlines.

The direction of the streamline can be determined from the velocity distribution $u = ay(y^2 - x^2),\ v = ax(y^2 - x^2)$

For $y > 0$, if $|y| > |x|$, $u > 0$; if $|y| < |x|$, $u < 0$;

For $y < 0$, if $|y| > |x|$, $u < 0$; if $|y| < |x|$, $u > 0$.



(2)
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} ax(y^2 - x^2) = a(y^2 - x^2) - 2ax^2 = a(y^2 - 3x^2)$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} ay(y^2 - x^2) = a(y^2 - x^2) + 2ay^2 = a(3y^2 - x^2)$$

$\frac{\partial v}{\partial x} \neq \frac{\partial u}{\partial y}$, so the flow rotational, there is no velocity potential.

Problem 8: Consider a viscous fluid flows through the surface of a flat plate. The velocity profile near the plate is given as: $u = u_0 \sin\frac{\pi y}{2a}$, where

u_0, a are constant, y is the distance to the plate. Determine the strain rates on the plate.

Solution:

From the velocity profile, the strain rates on the plate (y=0) are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}\Big|_{y=0} = 0, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}\Big|_{y=0} = 0, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}\Big|_{y=0} = 0,$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\Big|_{y=0} = 0, \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\Big|_{y=0} = 0,$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Big|_{y=0} = \frac{1}{2} u_0 \cos\left(\frac{\pi y}{2a}\right) \cdot \frac{\pi}{2a}\Big|_{y=0} = \frac{\pi u_0}{4a}$$