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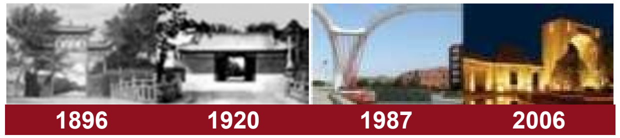

Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

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Chapter 10

Dimensional Analysis

10.1 Similitude

In ***Fluid Mechanics***, generally governing equations for a specific problem are given. But due to complexity, in most cases it is almost hopeless to solve them analytically, instead physical experiment is necessary.

Since any fluid flow is usually affected by many factors, it is impossible to perform experiment exhaustingly for all possible combinations of the factors. So we need a theory, based on which only least number of experiments need to be performed. In this chapter, ***dimensional analysis*** will be introduced. It provides a fundamental theory for physical experiment.

10.1 Similitude

Full Scale Experiment versus Model Experiment

Full Scale Experiment generally costs much, and in most cases is difficult to change parameters as will. Therefore, due to rich variety of flow phenomena there does not exist an easy way to determine dependence among the parameters involved in the phenomenon. Furthermore, in some cases full scale experiment is not able to perform.

Model Experiment is carried out on a scaled model on facility with specific function. **Wind tunnel**, **water channel**, **towing tank** and **ocean basin** are some facilities used in our major.

10.1 Similitude

Key Factors of Model Experiment

1. Scale Factor = Prototype Length / Model Length

   How can we choose a suitable scale factor?

2. How can we predict the prototype performance from the measured model experiment performance?

Essentially, it is a similarity problem. Model experiment should be ***similitude*** to the prototype problem.

10.1 Similitude

The origin of similarity principle came from geometry, where two similar triangles require ratios of the corresponding sides between them with identical value.

In *fluid mechanics*, the concept of similitude is similar to the geometry similarity. *Two flow fields are called similar, if and only at any corresponding instant at any corresponding location, ratios of the same kind physical quantity are of identical value.*

Three kind flow quantities are involved in *fluid mechanics*: ① ***geometric***, ② ***kinematic***, and ③ ***dynamic*** quantities.

Therefore, flow similarity involves three kinds of similitude, *i.e.* ① ***geometric similitude***, ② ***kinematic similitude*** and ③ ***dynamic similitude***.

10.1 Similitude

Geometric Similitude (Spatial Similitude)

Boundary of the scaled flow field is similar to the prototype boundary, *i.e.* model and prototype have the same shape, but with different size. Any linear quantity could be chosen as a representative, such as the length, width, height, radius, diameter or roughness, we just name some of them.

Length Scale Factor (Length Ratio)

$$C_l = \frac{l_p}{l_m}$$

Area Ratio

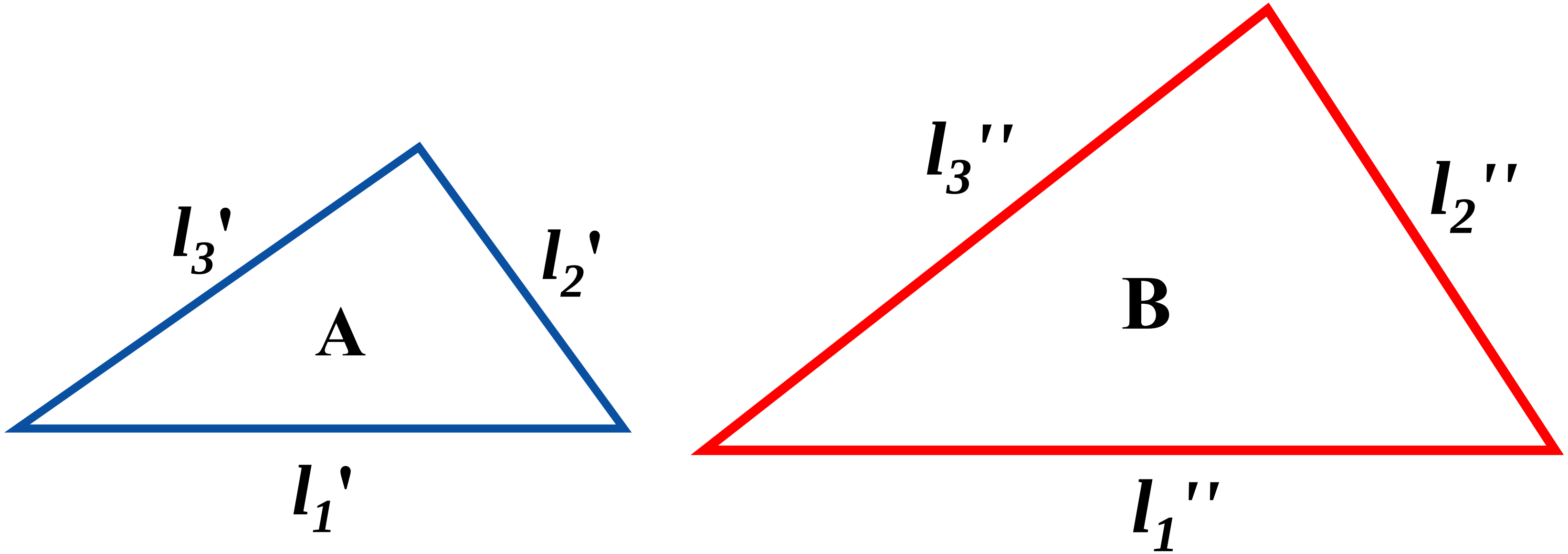
$$C_A = \frac{A_p}{A_m} = C_l^2$$

Volume Ratio

$$C_\Omega = \frac{\Omega_p}{\Omega_m} = C_l^3$$

Subscript *m* stands for the model, *p* for the prototype

10.1 Similitude



$$\frac{l_1''}{l_1'} = \frac{l_2''}{l_2'} = \frac{l_3''}{l_3'} = \frac{l_p}{l_m} = C_l$$

Representative of the Prototype Length

Scale factor

Representative of the Model Length

10.1 Similitude

Kinematic Similitude (Time Similitude)

Between two geometric similar flow fields, if both velocity fields and acceleration fields are similar, i.e. at corresponding location and instant velocities and accelerations are parallel and have an identical ratio, they are called kinematic similar. Thus, streamlines will also be geometric similar.

Velocity Ratio

$$C_v = \frac{v_p}{v_m}$$

Time Ratio

$$C_t = \frac{t_p}{t_m} = \frac{L_p / v_p}{L_m / v_m} = \frac{C_l}{C_v}$$

Acceleration Ratio

$$C_a = \frac{a_p}{a_m} = \frac{v_p / t_p}{v_m / t_m} = \frac{C_v}{C_t} = \frac{C_v^{\ 2}}{C_l}$$

10.1 Similitude

Geometric similitude is a prerequisite for kinematic similitude. If two fields have identical time ratio, we say they are **time similitude**. It can be deduced from kinematic similitude, and vice versa. So **kinematic similitude is equivalent to time similitude**.

Scale factor of any kinematic quantity can be expressed as a combination of length ratio and time ratio.

Velocity Ratio $C_v = C_l C_t^{-1}$

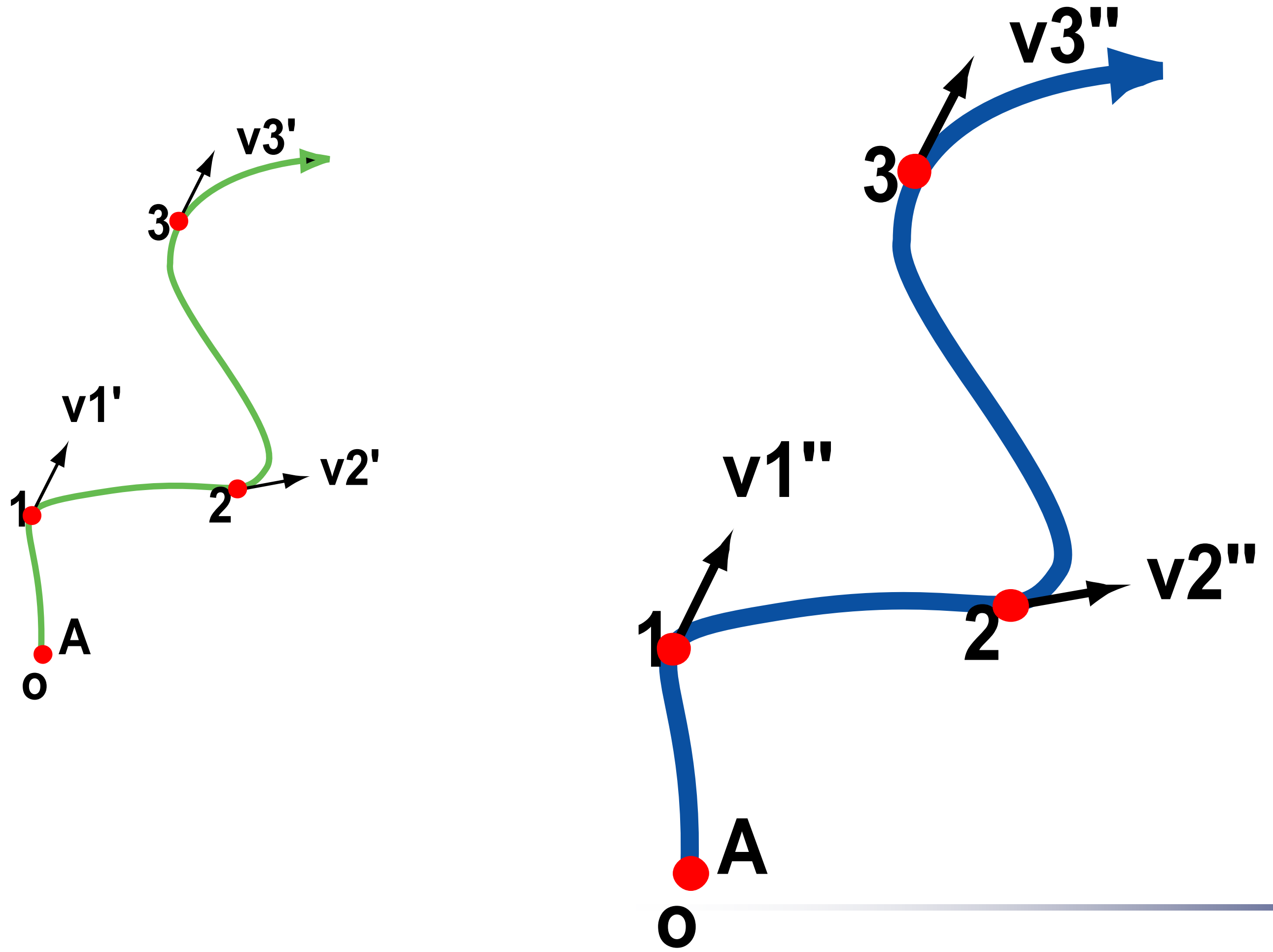
Acceleration Ratio $C_a = C_\upsilon C_t^{-1} = C_l C_t^{-2}$

Kinematic Viscosity Ratio $C_\nu = C_l^2 C_t^{-1}$

Flow Rate Ratio $C_Q = C_\Omega C_t^{-1} = C_l^3 C_t^{-1}$

10.1 Similitude

Orientations of the corresponding velocities and accelerations are either parallel or of an identical rotation, and magnitudes of are proportional with an identical ratio.



Dynamic Similitude (Force Similarity)

Corresponding forces are parallel, and their magnitudes are of identical ratio. The force could be any kind of force, such as gravity, pressure, viscous force and elastic force etc. Then the corresponding force polygons will be also geometric similar.

Generally, forces, applied to a fluid particle, involve gravity (F_G), pressure (F_p), viscous force (F_V), elastic force (F_E) and surface tension (F_T). If it moves with an acceleration, add an inertial force (F_I) to these forces will form a closed force polygon.

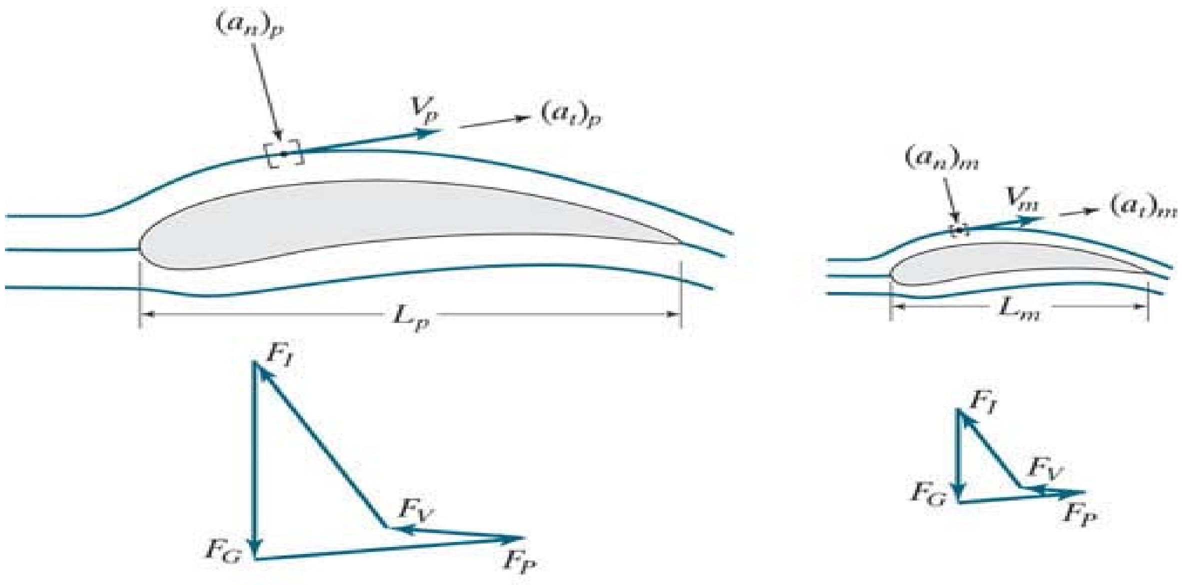
$$\mathbf{F}_G + \mathbf{F}_p + \mathbf{F}_V + \mathbf{F}_E + \mathbf{F}_T + \mathbf{F}_I = 0$$

$$\mathbf{F}_G + \mathbf{F}_p + \mathbf{F}_V + \mathbf{F}_E + \mathbf{F}_T = \sum \mathbf{F} = -\mathbf{F}_I$$

10.1 Similitude

For two flow fields of geometric and kinematic similitude, ***dynamic similarity*** is equivalent to that ratios of a force, applied to one fluid particle, to a corresponding force of the same kind, applied to the corresponding particle in another fluid field, will retain a constant, that is

$$C_F = \frac{F_{G_p}}{F_{G_m}} = \frac{F_{P_p}}{F_{P_m}} = \frac{F_{V_p}}{F_{V_m}} = \frac{F_{I_p}}{F_{I_m}} = \cdots$$



(a) Prototype (b) Model

10.1 Similitude

Force Ratio

$$C_F = \frac{F_p}{F_m} = \frac{F_{G_p}}{F_{G_m}} = \frac{F_{P_p}}{F_{P_m}} = \frac{F_{\nu_p}}{F_{\nu_m}} = \frac{F_{I_p}}{F_{I_m}} = \cdots$$

or written as

$$C_F = C_m C_a = (C_\rho C_l^{\,3})(C_l C_t^{-2}) = C_\rho C_l^2 C_v^2$$

10.1 Similitude

Boundary Condition Similitude

Boundary conditions should also be similar. Rigid body corresponds to rigid body. Free surface corresponds to free surface. In some sense, boundary condition similitude is similar to geometric similitude.

Initial Condition Similitude

Initial instant essentially is no difference with any other instant. Similar flow fields also require initial condition to be similar.

Similarity of boundary condition and initial condition are necessary conditions for similar flow fields. For steady flow, initial condition similitude is not explicitly required.

10.1 Similitude

Relations between Different Similitude

- **Geometric similitude is a prerequisite to kinematic and dynamic similitude.**
- **Dynamic similitude is an essential similitude to similar flow.**
- **Kinematic similitude is a result of geometric similitude and dynamic similitude.**
- **Similitude of boundary condition and initial condition is a necessary condition for similar flow fields.**

Therefore between similar flow fields, geometric, kinematic and dynamic similitude should all fulfill. Then fluid density ratio could be chosen as a dependent ratio and if so, it can be derived from other ratios.

Density Ratio:

$$C_\rho = \frac{\rho_p}{\rho_m} = \frac{F_p/(a_p\Omega_p)}{F_m/(a_m\Omega_m)} = \frac{C_F}{C_a C_\Omega} = \frac{C_F}{C_l^2 C_v^2}$$

10.1 Similitude

Usually fluid density is given and can not be changed as will. For example, water in towing tank is definite. Generally, ρ, l and v are selected as reference quantities. Ratios C_ρ, C_l and C_v are specified first, and other ratios are derived from them.

Torque M

$$C_M = \frac{(Fl)_p}{(Fl)_m} = C_\rho C_l^3 C_v^2$$

Pressure p

$$C_p = \frac{p_p}{p_m} = \frac{C_F}{C_A} = C_\rho C_v^2$$

Power N

$$C_N = \frac{(Fv)_p}{(Fv)_m} = C_F C_v = C_\rho C_l^2 C_v^3$$

Dynamic viscosity μ $C_\mu = C_\rho C_l C_v$

10.2 Similitude Law

Theoretically, a flow is uniquely determined by the governing equations with specified boundary and initial conditions. For two similar flow fields, one of them can be scaled to the other. So, governing equation and boundary and initial conditions can also be scaled from one to the other.

1) **Initial conditions**: specify velocity and pressure fields at some definite instant. For steady flow, it does not need to specify.
2) **Boundary conditions**: specify velocity and pressure on boundary of flow field. For unbounded flow fields, inlet and outlet are special boundaries and conditions there should also be specified.
3) **Geometric conditions**: roughness should be given as well.
4) **Physical parameters**: shape and size of the flow field, fluid density, viscosity, temperature and etc.

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10.2 Similitude Law

Similitude Principle: **Two flows are similar if and only if**

1) **governing equations after scaling are identical,**

2) **boundary and initial conditions after scaling are identical, involving**

- **initial conditions**
- **boundary conditions**
- **geometric conditions**
- **physical parameters**

3) **scale factors are determined by *dimensionless numbers*, which are combinations of physical parameters.**

So, for two flows of any kind, if any pair of dimensionless numbers is identical, then the two flows are similar.

10.2 Similitude Law

Similitude Law is a relationship between similar flow fields. It is derived from geometric scale, kinematic scale and dynamic scale.

Methods to Derive Similitude Law

1. **Equation Analysis Method** -- Derived from the known governing equation and boundary conditions.

2. **Dimensional Analysis Method** -- Derived from parameters on which flow fields depend. This method is more general and more powerful. It can be applied to problems where governing equation is unclear.

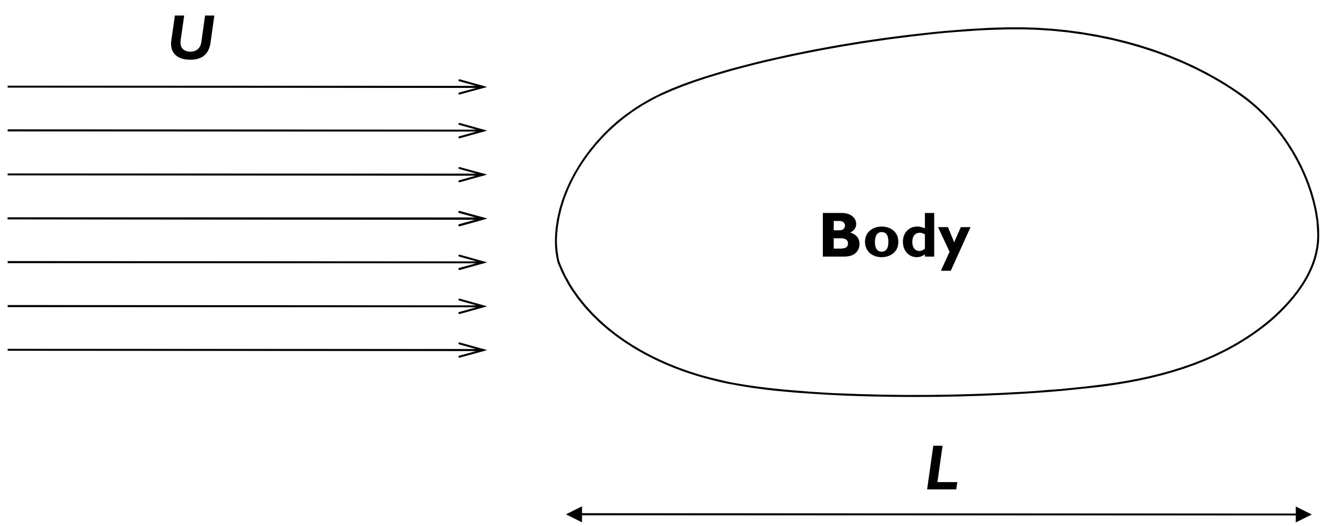
10.2 Similitude Law

Equation Analysis Method is efficient, reliable and powerful. If governing equations and boundary conditions are clear and given, number of quantities will be least without any redundancy.

For flow problems where governing equation or boundary conditions have not been established or are unclear. Equation Analysis Method would be useless and powerless.

10.2 Similitude Law

N-S equation is a general governing equation. For a specific flow, if some force is small enough in comparison with other forces, the corresponding term can be reasonably removed. ***Dimensionless expression*** of *N-S* equation will help us to do so. **As an example, we consider unsteady flow past a body below.**



Characteristic scales of this flow

Length (*L*), Time (*T*), Velocity (*U*) and Pressure (*P*).

10.2 Similitude Law

Nondimensionalization of physical quantities

$$\mathbf{V}^* = \mathbf{V}/U, \quad t^* = t/T, \quad \mathbf{x}^* = \mathbf{x}/L, \quad p^* = p/P, \quad \mathbf{g}^* = \mathbf{g}/g$$

Nondimensionalization of *N-S* equation

$$\left(\frac{L}{UT}\right)\frac{\partial \mathbf{V}^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = -\left(\frac{P}{\rho U^2}\right)\nabla^* p^* + \left(\frac{gL}{U^2}\right)\mathbf{g}^* + \left(\frac{\nu}{UL}\right)\nabla^{*2}\mathbf{V}^*$$

10.2 Similitude Law

Relative value of the dimensionless factors in front of each term determines the relative importance of that terms. If one factor is small enough, that term could be omitted without any essential harm to the solution.

$$\frac{L}{UT} = \text{Strouhal number (St)} = \frac{\text{Local Derivative}}{\text{Convective Derivative}}$$

$$\frac{P}{\rho U^2} = \text{Euler number (Eu)} = \frac{\text{Pressure}}{\text{Inertial Force}}$$

$$\frac{UL}{\nu} = \text{Reynolds number (Re)} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

$$\frac{U^2}{gL} = \text{Froude number (Fr}^2\text{)} = \frac{\text{Inertial Force}}{\text{Gravity}}$$

10.3 Equation Analysis Method

Similitude Law ---- Derived from *N-S* Equation

N-S equation for *prototype* (full scale) flow

$$\frac{\partial v_{pz}}{\partial t_p}+v_{px}\frac{\partial v_{pz}}{\partial x_p}+v_{py}\frac{\partial v_{pz}}{\partial y_p}+v_{pz}\frac{\partial v_{pz}}{\partial z_p}=-g_p-\frac{1}{\rho_p}\frac{\partial p_p}{\partial z_p}+\frac{\mu_p}{\rho_p}\left(\frac{\partial^2 v_{pz}}{\partial x_p{}^2}+\frac{\partial^2 v_{pz}}{\partial y_p{}^2}+\frac{\partial^2 v_{pz}}{\partial z_p{}^2}\right)$$

N-S equation for *model* flow

$$\frac{\partial v_{mz}}{\partial t_m}+v_{mx}\frac{\partial v_{mz}}{\partial x_m}+v_{my}\frac{\partial v_{mz}}{\partial y_m}+v_{mz}\frac{\partial v_{mz}}{\partial z_m}=-g_m-\frac{1}{\rho_m}\frac{\partial p_m}{\partial z_m}+\frac{\mu_m}{\rho_m}\left(\frac{\partial^2 v_{mz}}{\partial x_m{}^2}+\frac{\partial^2 v_{mz}}{\partial y_m{}^2}+\frac{\partial^2 v_{mz}}{\partial z_m{}^2}\right)$$

10.3 Equation Analysis Method

Geometric Scaling

$$x_p = C_l x_m, \quad y_p = C_l y_m, \quad z_p = C_l z_m$$

Kinematic Scaling

$$v_{px} = C_v v_{mx}, \quad v_{py} = C_v v_{my}, \quad v_{pz} = C_v v_{mz}$$

Dynamic Scaling

$$p_p = C_p p_m, \quad g_p = C_g g_m$$

Other Scaling

$$\rho_p = C_\rho \rho_m, \quad \mu_p = C_\mu \mu_m$$

The *N-S* equation for *prototype*

$$\frac{C_v}{C_t}\frac{\partial v_{mz}}{\partial t_m}+\frac{C_v^{\,2}}{C_l}\left(v_{mx}\frac{\partial v_{mz}}{\partial x_m}+v_{my}\frac{\partial v_{mz}}{\partial y_m}+v_{mz}\frac{\partial v_{mz}}{\partial z_m}\right)$$

$$=-C_g g_m-\frac{C_p}{C_l C_\rho}\frac{1}{\rho_m}\frac{\partial p_m}{\partial z_m}$$

$$+\frac{C_v C_\mu}{C_l^{\,2} C_\rho}\frac{\mu_m}{\rho_m}\left(\frac{\partial^2 v_{mz}}{\partial x_m^{\,2}}+\frac{\partial^2 v_{mz}}{\partial y_m^{\,2}}+\frac{\partial^2 v_{mz}}{\partial z_m^{\,2}}\right)$$

10.3 Equation Analysis Method

Then *model* flow is similar to *prototype* flow, if and only if

$$\frac{C_v}{C_t} = \frac{C^2{}_v}{C_l} = C_g = \frac{C_p}{C_l C_\rho} = \frac{C_v C_\mu}{C^2{}_l C_\rho}$$

(Similitude Law)

↓

Essentially identical *N-S* equations for *model* and for *prototype*

+

Identical Initial and boundary conditions

↓

Difference of solutions between *model* and *prototype* is only scale factor (scaling)

10.3 Equation Analysis Method

Modification of similitude laws

$$\frac{C_v}{C_t} = \frac{C_v^2}{C_l} = C_g = \frac{C_p}{C_l C_\rho} = \frac{C_v C_\mu}{C_l^2 C_\rho}$$

$$\div \frac{C_v^{\ 2}}{C_l}$$

$$\frac{C_l}{C_t C_v} = \frac{C_g C_l}{C_v^2} = \frac{C_p}{C_v^2 C_\rho} = \frac{C_l C_v C_\rho}{C_\mu} = 1$$

4 Dimensionless Numbers: St, Fr, Eu, Re

10.3 Equation Analysis Method

Reynolds Similitude Law (Re) -- Viscous Force Similitude

$$\frac{C_\rho C_v C_l}{C_\mu} = 1 \Rightarrow \frac{\rho_p U_p L_p}{\mu_p} = \frac{\rho_m U_m L_m}{\mu_m} = \boxed{\frac{\rho UL}{\mu} = \frac{UL}{\nu} = \mathrm{Re}}$$

$$\text{Reynolds Number (Re)} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

【Application】 For flows where viscous force is an important force, Re similitude should be fulfilled. Skin frictional resistance of ships is an example. Submarine is another, as surface water waves does not affect the flow field around it. Other examples such as the laminar pipe flow, flow in tunnels and flow around underwater vehicles and etc.

10.3 Equation Analysis Method

Reynolds Similitude Law — ***For flows where viscous forces are to be similar, if and only if Reynolds numbers are identical.***

Since

$$\frac{C_\rho C_v C_l}{C_\mu} = 1$$

scales C_ρ **,** C_v **,** C_l **and** C_μ **can not be arbitrarily chosen. Only part of them can be freely chosen, but the remaining ones are determined from *Reynolds similitude law*. For example, if model and prototype use same fluid (say water), then**

$$C_\rho = C_\mu = 1 \quad \Rightarrow \quad C_v = \frac{1}{C_l}$$

10.3 Equation Analysis Method

Froude Similitude Law (Fr) -- Gravitational Force Similitude

$$\frac{C_v^2}{C_g C_l} = 1 \quad \Rightarrow \quad \frac{U_p^2}{g_p L_p} = \frac{U_m^2}{g_m L_m} = \boxed{\frac{U^2}{gL} = Fr^2}$$

$$\text{Froude Number (Fr)} = \frac{\text{Inertial Force}}{\text{Gravitational Force}}$$

【Application】 For flows where gravitational force is an important force, Fr similitude should be fulfilled. For flows with free surface, gravitational force will be a dominant force. Flow over weir, nozzle flow, channel flow, turbulent pipe flow and tunnel flow are some examples of this kind of flows.

10.3 Equation Analysis Method

Froude Similitude Law — *Flows where gravitational forces are to be similar, if and only if Froude numbers are identical.*

Since

$$\frac{C_v^2}{C_g C_l} = 1$$

scale factors in the equation can not be arbitrarily chosen. Only part of them can be freely chosen, but the remaining ones are determined from *Froude similitude law*. Since gravitational forces are generally the same, it results

$$g_p = g_m \quad \Rightarrow \quad C_g = \frac{g_p}{g_m} = 1 \quad \Rightarrow \quad C_v = C_l^{1/2}$$

10.3 Equation Analysis Method

Euler Similitude Law (Eu) -- Pressure Similitude

$$\frac{C_p}{C_\rho C_v^2} = 1 \Rightarrow \frac{p_p}{\rho_p U_p^2} = \frac{p_m}{\rho_m U_m^2} = \boxed{\frac{p}{\rho U^2} = Eu}$$

$$\text{Euler Number (Eu)} = \frac{\text{Pressure}}{\text{Inertial Force}}$$

In *N-S* equation, only pressure gradient appears, that is, flows are affected by pressure difference △p, rather than a single pressure at a point. It is reasonable to write Eu as

$$Eu = \frac{\Delta p}{\rho U^2}$$

10.3 Equation Analysis Method

Euler Similitude Law — *Flows where pressures are to be similar, if and only if Euler numbers are identical.*

In most cases, pressure similitude is not a prerequisite similitude for similar flows, but is a result of similar flows. Eu similitude is a dependant similitude. If other main similitude laws are fulfilled, pressure similitude is spontaneously satisfied.

Re similitude and Fr similitude are generally considered. Only when there exists some negative pressure (lower than atmospheric pressure) flow region, especially in case cavitation occurs, Euler similitude becomes dominant, and should be taken into account firstly. A *cavitation number*, σ, (an alternative of Eu number) is introduced.

$$\sigma = \frac{p - p_v}{\frac{1}{2}\rho v^2}$$

Strouhal Similitude Law (St) -- Unsteady Force Similitude

$$\frac{C_v C_t}{C_l} = 1 \quad \Rightarrow \quad \frac{U_p T_p}{L_p} = \frac{U_m T_m}{L_m} = \boxed{\frac{UT}{L} = St}$$

$$\text{Strouhal Number (St)} = \frac{\text{Local Derivative}}{\text{Convective Derivative}}$$

Ratio of local acceleration to convective acceleration. The former relates to unsteadiness, the latter represents non-uniformity.

10.3 Equation Analysis Method

Strouhal Similitude Law — *Flows, where unsteady forces are to be similar, if and only if Strouhal numbers are identical.*

For model experiment with unsteady forces, it should obey Strouhal similitude law.

If an unsteady flow is a periodic oscillation or wavelike motion with frequency *f*, then St can be expressed in another form

$$St = \frac{U}{fL}$$

and *St similitude*

$$\frac{U_p}{f_p L_p} = \frac{U_m}{f_m L_m}$$

10.3 Equation Analysis Method

In summary, four similitudes are derived from *N-S* equation.

$$\frac{UL}{\nu} = \text{Reynolds Number (Re)} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

$$\frac{U^2}{gL} = \text{Froude Number (Fr}^2\text{)} = \frac{\text{Inertial Force}}{\text{Gravitational Force}}$$

$$\frac{p}{\rho U^2} = \text{Euler Number (Eu)} = \frac{\text{Pressure}}{\text{Inertial Force}}$$

$$\frac{U}{fL} = \text{Strouhal Number (St)} = \frac{\text{Local Derivative}}{\text{Convective Derivative}}$$

Dimensional Analysis Method

For complicate flows, whose governing equation is unclear, dimensional analysis method will be powerful. It collects main physical quantities the flow relates, analyses dimensions of them, assembles dimensionless numbers and finally establishes a few equations that relate those dimensionless numbers.

Dimensional analysis method only partially solve the flow problem. Relationship between dimensionless numbers has to be determined by other means, say experiments. Also, effectiveness of this method largely depends on the exhaustive collection of physical quantities. Any incompleteness may cause incorrect conclusion.

10.4 Dimensional Analysis

Dimensional analysis method is another method to explore the law governing flows experimentally. Especially effective to the complicate flows, difficult to be investigated theoretically.

Equation analysis

Governing equation is known

Scaling from model to prototype, or vice versa

Similitude Laws

Dimensional analysis

Based on dimensional analysis of physical quantities involved in the flow problem, according to dimension consistency deduce similitude laws logically.

Characteristics of Dimensional Analysis Method

Detail knowledge on the flow is not necessary. We only need to know

- **Fundamental laws the problem should obey**
- **Quantities on the boundary**
- **Quantities involved in the problem.**

Units

Units are used to specify scale of physical quantities. For example, unit of length could be *m*, *cm* or *mm*, and unit of time be *hour*, *minute* or *second*.

Units are specified by some official organizations. For example, the first definition of length unit *metre* was given by French Academy of Science in 1791. *One metre equals ten millionth of the distance from the North Pole to the Equator*. A more precise definition was given in 1960 by the 11th CGPM. *One metre equals 1650763.73 wavelengths of the radiation corresponding to the transition between the* $2P^{10}$ *and* $5P^{5}$ *quantum levels of the krypton-86 atom in vacuum*.

10.5 Concepts of Dimensional Analysis

Dimension

Dimension expresses kind of physical quantities. For example, quantities measured in *m*, *cm* and *mm* all belong to length, denoted in L. Those measured in *hour*, *minute* and *second* belong to time, denoted in T. Those measured in *kilogram* and *gram* belong to mass, denoted in *M*.

Apparently, dimension is essential to physical quantities. There is a correspondence of units to dimension. One dimension could be expressed in different units, but one unit only corresponds to one dimension.

10.5 Concepts of Dimensional Analysis

Dimension and Units

In Fluid Mechanics there are a number of quantities, such as length, time, mass, force, velocity, acceleration, viscosity, pressure etc. *A quantity is identified by its dimension and measured with units.*

For example, length is a geometric linear quantity (the dimension), defined as linear span between two points. It is measured with units – *metre*, *centimetre*, *foot* or *light year*. Dimension and Units are two facets of physical quantity.

10.5 Concepts of Dimensional Analysis

Basic Dimensions and Derived Dimensions

In international standard unit system (SI units), 7 basic units are specified. Among them, 3 units are related to *Fluid Mechanics* – *metre* (*m*) as length unit, *kilogram* (*kg*) as mass unit, and *second* (*s*) as time unit. The corresponding dimensions [L], [M], [T] are the basic dimensions (reference dimension) of *Fluid Mechanics*.

Dimensions of other quantities are derived from basic dimensions. For example, dimension of velocity is derived from dimensions of length and time, that is, $[u]=[L][T]^{-1}$. *Dimensions derived from basic dimensions are called* derived dimensions.

10.5 Concepts of Dimensional Analysis

As soon as basic dimensions are chosen, quantities can be expressed as a monomial, each factor is a power of a single dimension. It is called **expression of derived dimension**. Below is a list of derived dimensions used in *Fluid Mechanics*.

Velocity $[V]=\left[LT^{-1}\right]$

Acceleration $[a]=\left[LT^{-2}\right]$

Density $[\rho]=\left[ML^{-3}\right]$

Force $[F]=\left[LMT^{-2}\right]$

Pressure and Stress

$$[p]=\left[ML^{-1}T^{-2}\right]$$

Power $[N]=\left[ML^{2}T^{-3}\right]$

Dynamic Viscosity

$$[\mu]=\left[ML^{-1}T^{-1}\right]$$

Kinematic Viscosity

$$[\nu]=\left[L^{2}T^{-1}\right]$$

Work and Energy

$$[W]=[E]=\left[ML^{2}T^{-2}\right]$$

Dimension Expression

If we denote dimension of q as [*q*], it is generally expressed as

$$[q] = [M]^{\alpha} [L]^{\beta} [T]^{\gamma}$$

For $\alpha = 0$, $\beta = 0$, $\gamma = 0$, ***q*** **is dimensionless.**

For $\alpha = 0$, $\beta \neq 0$, $\gamma = 0$, ***q*** **is geometric.**

For $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$, ***q*** **is kinematic.**

For $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, ***q*** **is dynamic.**

Dimensionless Quantity

Dimensionless quantity is a constant or a constant quantity without unit. π, the ratio of a circle's circum-ference to its diameter, is a constant.

In dimension expression of a quantity *q*, if all exponents of basic dimensions are zero (α=β=γ=0), q is a dimensionless quantity.

Dimensionless quantity might be ratio of two quantities with same dimension, such as linear strain $\varepsilon = \Delta l / l$, $[\varepsilon] = [L]/[L] = 1$. It may be a combination of product and/or quotient of several quantities with different dimensions. For example, in pipe flow, Reynolds number is a combination of mean cross velocity, v, diameter of pipe, d, and kinematic viscosity, ν.

$$[\mathrm{Re}] = [\frac{vd}{\nu}] = \frac{([L][T]^{-1})[L]}{[L]^2[T]^{-1}} = 1$$

Characteristics of Dimensionless Quantity

1. Objective

As mentioned above, for any quantity, if it has dimension, it will accompany with a unit. With different choice of units, the numeric value of the quantity will be different. Therefore, the numeric value and unit is individual. But for dimensionless quantity, the situation is completely different. The value is unique – change of units won't change its value. Since selection of units is decided by our decision (subjective), while dimensionless quantity is objective, which is independent with the choice of units, so it is superior to quantities with dimension, which are subjective.

2. Independent on Law of Physics

Since dimensionless quantity is a constant, the numerical value does not change with different units, and does not vary among similar flows. For similar flows, no matter how large or small the scaling factors are, dimensionless quantity will keep identical. In model experiments, dimensionless quantity (say, Re) is used as a criterion to judge flow similarity.

3. Capable to Be Operated in Transcendental Functions

Quantities with dimension can only be operated in rational functions, but meaningless in transcendental functions, such as logarithmic, exponential and triangular functions, while dimensionless quantities without such a limitation. Compression of isothermal compressible gas is an example.

$$W = p_1 V_1 \ln\left(\frac{V_2}{V_I}\right)$$

where the ratio, V_2 / V_1 **, of the original volume to the compressed volume is a dimensionless quantity. It is reasonable to operate in logarithm.**

10.5 Concepts of Dimensional Analysis

Dimension Homogeneity

In an equation of physics law, each term has the same dimension.

In other words, for an equation of physics law, the corresponding dimension equation is homogeneous.

Dividing an equation of physics law by any term of it will make the equation dimensionless.

10.5 Concepts of Dimensional Analysis

Example: In momentum equation of viscous incompressible flow for *x*-component, dimension of every term is LT^{-2}**.**

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x = \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}$$

Example: Dimension of terms in Bernoulli's equation is the length, [*L*].

$$z_1 + \frac{p_1}{\rho g} + \frac{\alpha_1 \upsilon_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{\alpha_2 \upsilon_2^2}{2g} + h_w$$

For any equation, if it is a correct expression of a physics law, it obeys dimension homogeneity.

Deduction of Dimension Homogeneity

(1) *An equation, correctly expresses a physics law, can be modified to another equation of dimensionless quantities.* This is because each term in the original equation is of same dimension, division of any term to the equation does not change its correctness, but the equation becomes a dimensionless equation.

(2) *Dimension homogeneity reflects essential relationship among quantities related to a physical problem.* Because relationship among quantities in a physics problem is definite. The equation is just the expression of the definite relation. Dimensional analysis method is based on this principle and is an important discover in mechanics at the beginning of the last century.

10.6 Rayleigh Method

Dimension analysis method is subdivided into two methods.

Rayleigh Method is suitable to simple problem.

Π Theorem (or, Buckingham Method) is a general method.

Joint with experiment, dimension analysis method can make clear the law of flow lacking governing equation beforehand. The derived law of flow is extensively general for the same kind of flow.

10.6 Rayleigh Method

Rayleigh Method writes down a conceptual relation for a specific physical problem, where all quantities are involved completely.

$$f(q_1, q_2, q_3, \cdots q_n) = 0$$

where some quantity, q_i, is expressed as a product of the others.

$$q_i = K q_1{}^a q_2{}^b \cdots\cdots q_{n-1}{}^p$$

or in a dimension form

$$[q_i] = [q_1]^a [q_2]^b \cdots\cdots [q_{n-1}]^p$$

10.6 Rayleigh Method

Expressing every quantity in a product of basic quantity powers, exponents *a*, *b*, …, *p* are then determined from the dimension homogeneity.

Here is an example of application of Reyleigh Method.

Example 1. A circular pipe steady flow of incompressible fluid. Find the expression of head loss, $\Delta p / l$, along one unit length.

Solution Now solve this problem step by step.

(1) Analyze quantities in this problem. This is the key step.

Apparently wall roughness ε affects drag.

10.6 Rayleigh Method

Pipe length, diameter, mean velocity V affects drag too. Besides, fluid properties, such as density ρ and viscosity ν also effect.

In this way, totally 7 quantities, $\Delta p, l, d, V, \rho, \nu$ and ε , effect. For drag per unit length, $\frac{\Delta p}{l}$, following equation is summarized

$$f\left(\frac{\Delta p}{l}, d, V, \rho, \nu, \varepsilon\right) = 0$$

(2) Assume dimension relation

$$\frac{\Delta p}{l} = K d^{\alpha} V^{\beta} \nu^{\gamma} \rho^{\delta} \varepsilon^{\kappa}$$

where α, β, γ,δ and κ are constants to be determined, while *K* is a constant coefficient.

10.6 Rayleigh Method

(3) Write down dimension expression for every quantity

$$\left[\frac{\Delta p}{l}\right] = ML^{-2}T^{-2}$$

$$[d] = L$$

$$[V] = LT^{-1}$$

$$[\nu] = L^2T^{-1}$$

$$[\rho] = ML^{-3}$$

$$[\varepsilon] = L$$

10.6 Rayleigh Method

(4) Substitute in dimension relation

$$ML^{-2}T^{-2} = L^{\alpha}(LT^{-1})^{\beta}(L^{2}T^{-1})^{\gamma}(ML^{-3})^{\delta}L^{\kappa}$$

$$= M^{\delta}L^{\alpha+\beta+2\gamma-3\delta+\kappa}T^{-\beta-\gamma}$$

(5) Equate dimension exponents (dimension homogeneity) and solve the unknown exponents

$$\delta = 1$$

$$\alpha + \beta + 2\gamma - 3\delta + \kappa = -2$$

$$-\beta - \gamma = -2$$

10.6 Rayleigh Method

Among 5 unknowns in 3 equations, 3 unknowns can be expressed by other 2 unknowns

$$\alpha = -1 - \gamma - \kappa, \qquad \beta = 2 - \gamma, \qquad \delta = 1$$

(6) Substitute in the assumed expression, assemble quantities with same exponent together to form similarity number

$$\frac{\Delta p}{l} = K d^{-1-\gamma-\kappa} V^{2-\gamma} \nu^{\gamma} \rho \varepsilon^{\kappa} = \mathrm{K} \frac{\rho V^2}{d} \left(\frac{\nu}{Vd}\right)^{\gamma} \left(\frac{\varepsilon}{d}\right)^{\kappa}$$

10.6 Rayleigh Method

or written as

$$\frac{\Delta p}{\rho} = \lambda \frac{l}{d} \frac{V^2}{2}$$

where

$$\lambda = f(\mathrm{Re}, \frac{\varepsilon}{d}) = 2K(\frac{\nu}{Vd})^{\gamma}(\frac{\varepsilon}{d})^{\kappa}$$

λ is the *friction factor*, and $\mathrm{Re} = \frac{Vd}{\nu}$.

That is, *friction factor*, λ, of circular pipe flow is determined by Re and *roughness* ε of the pipe wall. This result is consistent with the Nikuradse diagram.

10.6 Rayleigh Method

Value of constant *K*, exponents γ and κ can not be determined by dimension analysis itself. They are usually determined from experiments.

For a pipe with a given roughness ε, in order to investigate effects of d, V, ρ, ν on friction factor λ, if we roughly choose 10 different values for each quantity, totally 10^4 times experiments have to be performed. But according to the result of dimension analysis, since λ is a function of Re, only 10 times experiments is enough. Furthermore, we need not change all the 4 quantities, instead, change any one of them, say velocity *V*, is enough, provided Re changes. This demonstrates the power of dimension analysis method.

10.6 Rayleigh Method

Rayleigh Method is effective to relative simple flow with fewer quantities. For complex flow with many quantities, say *n*, Rayleigh Method will be less effective. Since in Fluid Mechanics, there are only 3 basic dimensions, *n*-3 unknowns will be undetermined. Selection of the n-3 from the total n quantities will become problem.

10.7 Buckingham Π Theorem

Π Theorem: For a physical problem, number of independent dimensionless quantities is equal to difference of the number of dimensional quantities the problem involves from the number of basic dimensions.

Denote n the number of dimensional quantities and m the basic dimensions, then n – m independent dimensionless quantities result, and are related with each other. Historically, dimensionless quantities are denoted in symbol Π with subscripts. Accordingly, Buckingham theorem is simply called Π theorem.

10.7 Buckingham Π Theorem

Suppose X_1, X_2, … , X_n are n quantities with dimension, such as velocity, density, pressure and so on. They are related with each other in a dimensional homogeneous equation.

$$F(X_1, X_2, X_i, \cdots, X_n) = 0$$

It can be expressed as a relation of dimensionless quantities.

$$f(\Pi_1, \Pi_2, \Pi_j, \cdots, \Pi_{n-m}) = 0$$

where every Π_i represents an individual dimensionless quantity, which is a product of several dimensional quantities with their own exponents.

10.7 Buckingham Π Theorem

In Π theorem, dimensionless quantities, Π_i , are **similarity numbers**. Operations on Π_i , e.g. reciprocal, power, and sum, difference, product and quotient with constant and/or other dimensionless quantities, will generate new similarity numbers.

- **Similarity number to the n-th power**

$$\left(\frac{gl}{v^2}\right)^{-1/2} = \frac{1}{Fr}$$

- **Product of two similarity numbers**

$$Fr^2\,\mathrm{Re}^2 = \frac{gl}{v^2}\left(\frac{\rho v l}{\mu}\right)^2 = \frac{g\rho^2 l^3}{\mu^2} = Ga$$

10.7 Buckingham Π Theorem

- **Product with dimensionless number**

$$Ga\left(\frac{\rho-\rho_0}{\rho}\right)=\frac{g\rho^2 l^3}{\mu^2}\left(\frac{\rho-\rho_0}{\rho}\right)=\frac{g\rho(\rho-\rho_0)l^3}{\mu^2}=Ar$$

- **Sum and difference of similarity numbers**

$$\pi_1=\left(\frac{\sigma}{g\rho_1 l^2}\right)^{-1},\ \pi_2=\left(\frac{\sigma}{g\rho_2 l^2}\right)^{-1},\quad (\pi_1-\pi_2)^{-1}=\frac{\sigma}{g(\rho_1-\rho_2)l^2}=We$$

- **Replacement of a quantity in similarity number with an increment of the quantity**

$$Eu=\frac{p}{\rho v^2},\qquad Eu=\frac{\Delta p}{\rho v^2}$$

Tenth Assignment

- **The tenth assignment can be downloaded from following website:**

Website： ftp://public.sjtu.edu.cn

Username： dcwan

Password： 2015mhydro

Directory： IntroMHydro2015-Assignments

- **Five problems**

- **Given on June 18th**