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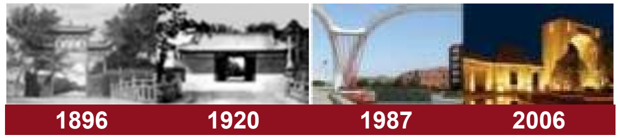



# Introduction to Marine Hydrodynamics

(NA235)

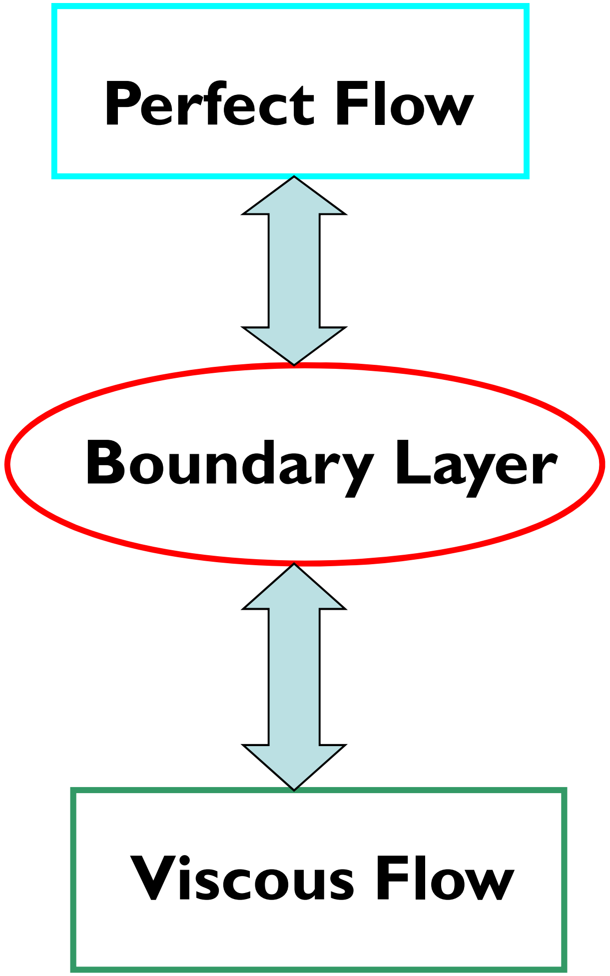
Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

# Chapter 9

# Boundary Layer Theory


Perfect Flow
Boundary Layer
Viscous Flow

# 9.1 Concept of Boundary Layer

**Viscous incompressible steady flow**

$$\boxed{\nabla \cdot \mathbf{V} = 0} \quad + \quad \boxed{\boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\frac{1}{\rho}\nabla p + \boldsymbol{g} + \nu \nabla^2 \boldsymbol{V}}$$

**Fundamental dimensions:**

**Length ($L$), Velocity ($U$), Pressure ($P$)**

$$\boldsymbol{V}^* \cdot \nabla^* \boldsymbol{V}^* = -(Eu)\nabla^* p^* + \left(\frac{1}{Fr^2}\right)\boldsymbol{g}^* + \left(\frac{1}{\mathrm{Re}}\right)\nabla^{*2}\boldsymbol{V}^*$$

$$Eu = \frac{P}{\rho U^2} \qquad Fr^2 = \frac{U^2}{gL} \qquad \mathrm{Re} = \frac{UL}{\nu}$$

# 9.1 Concept of Boundary Layer

**For most real flow, Re is large.**

$$\frac{1}{\mathrm{Re}} \to 0 \quad \textbf{It seems that} \quad \left(\frac{1}{\mathrm{Re}}\right) \nabla^{*2} \boldsymbol{V}^{*} \to 0$$

**For high Re (≈$10^6$～$10^9$), viscous force is far more less than inertial force. If we completely neglect viscous force, the *N-S equation* reduces to *Euler equation***

$$\boxed{\boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\frac{1}{\rho} \nabla p + \boldsymbol{g}}$$

**but it is not able to describe viscous flow any more.**

# 9.1 Concept of Boundary Layer

*Euler equation* only includes terms of 1st order derivatives, while *N-S equation* involves terms of 2nd order derivatives as well.

*Euler equation* is a 1st order equation for perfect flow, it is not able to fulfill *no-slip* condition on wall, which corresponds to 2 equations.

Therefore, simply neglecting viscous terms will be not able to obtain viscous flow near wall, as it can not fulfill *no-slip* condition.

How can the large Re flow be solved ?

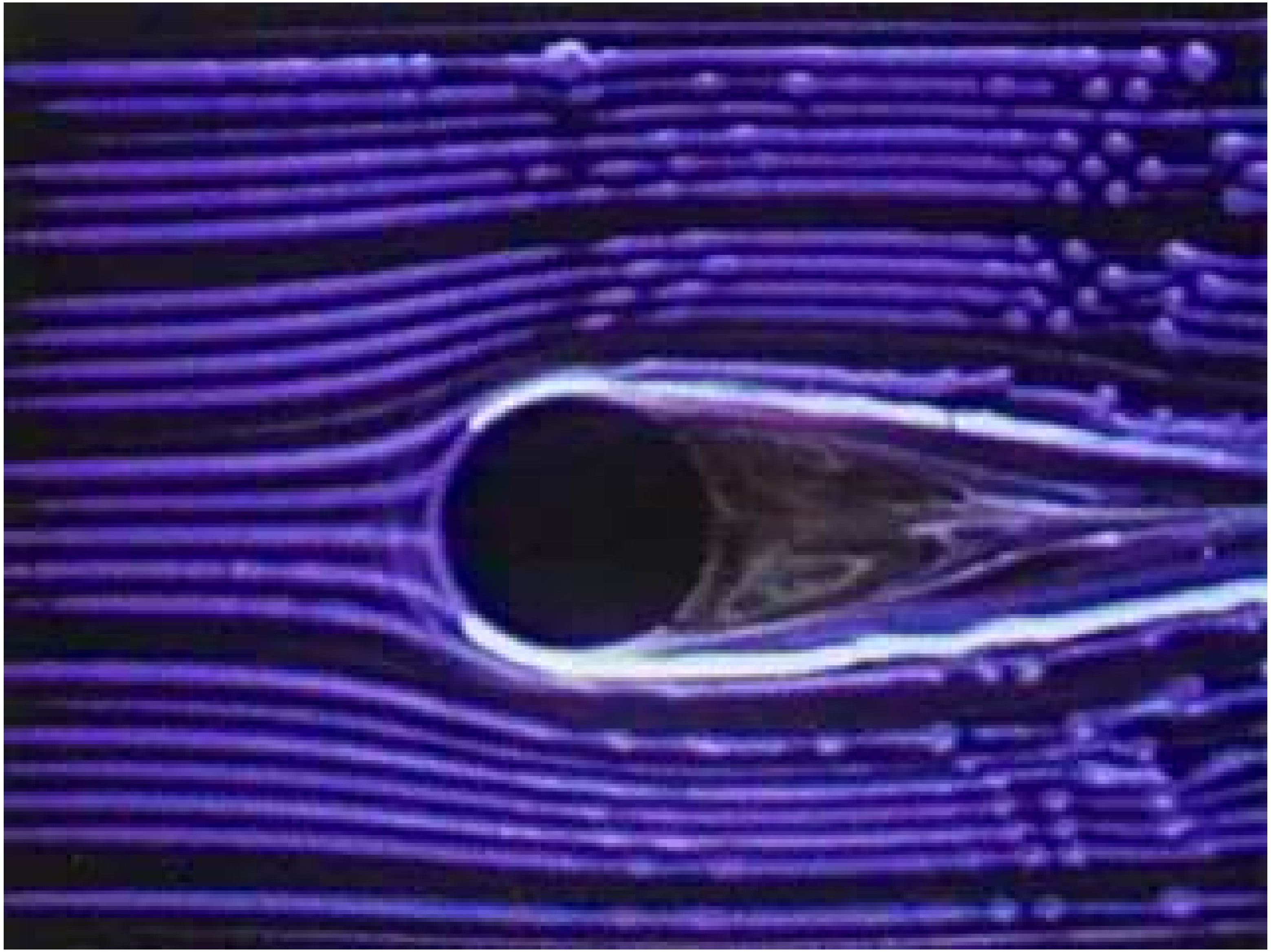
Until 1904, when German scientist Prandtl proposed Boundary theory, this difficulty had been reasonably solved .

# 9.1 Concept of Boundary Layer

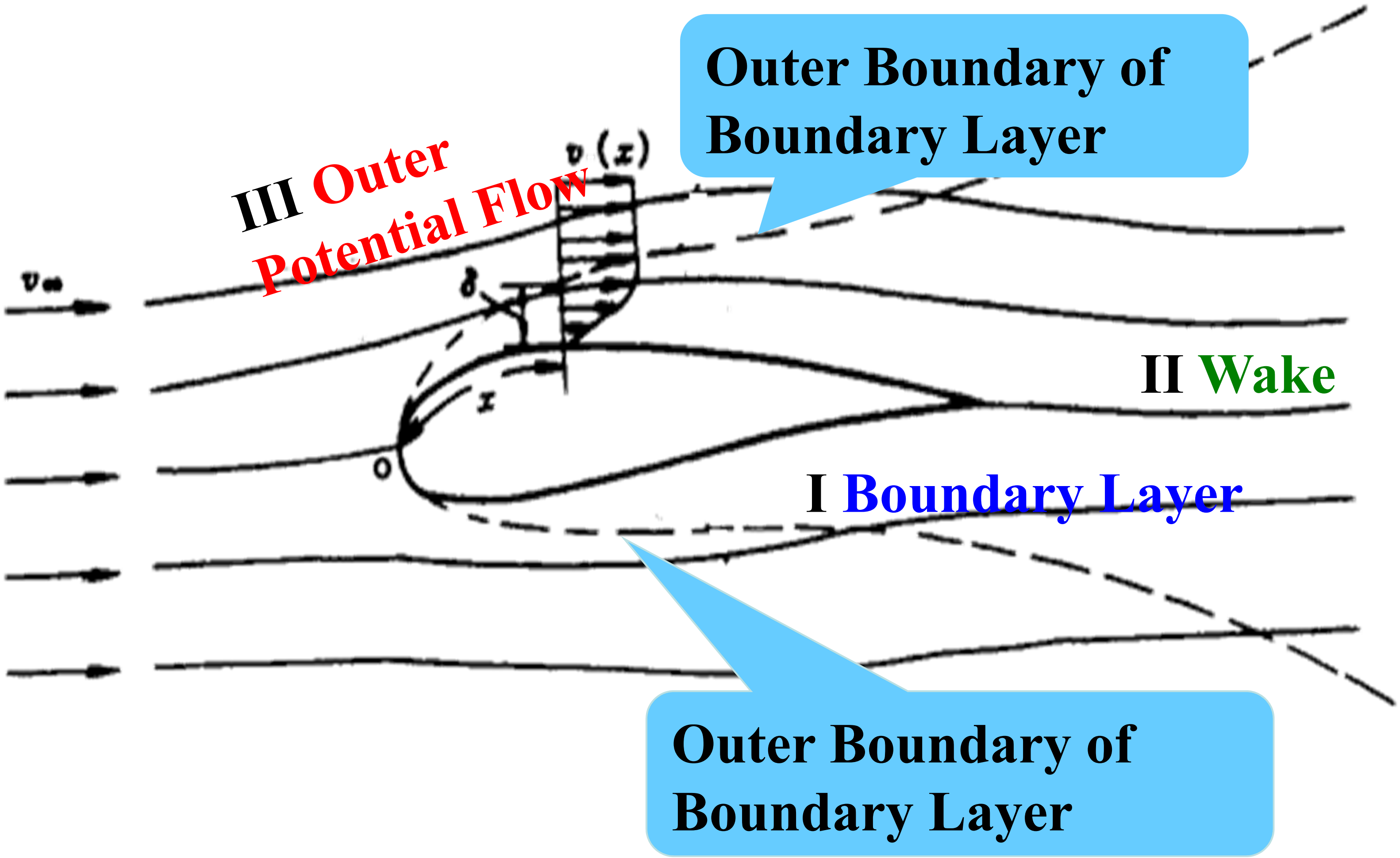
In 1904, Prandtl proposed ***Boundary Layer*** concept. For fluid with small viscosity, e.g. air and water, Reynolds number is generally very large. Effect of viscosity is limited in a thin layer near the wall, away from it viscosity can be fully neglected, where flow can be approximately treated as ideal potential flow. *The thin layer, where viscosity should be taken into account, is called boundary layer*.

In this way, flow with large Reynolds number is divided into three different regions – inner *boundary layer*, outer *potential flow* and the rear *wake*.

# 9.1 Concept of Boundary Layer

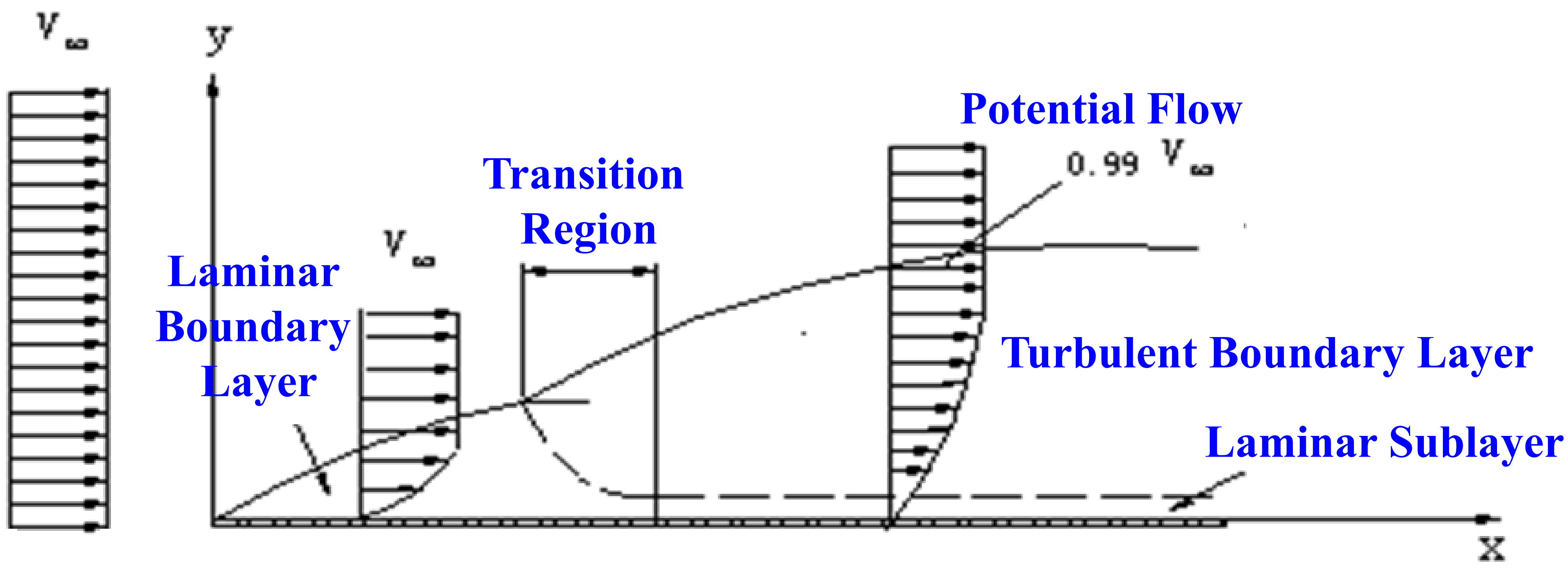


# 9.1 Concept of Boundary Layer



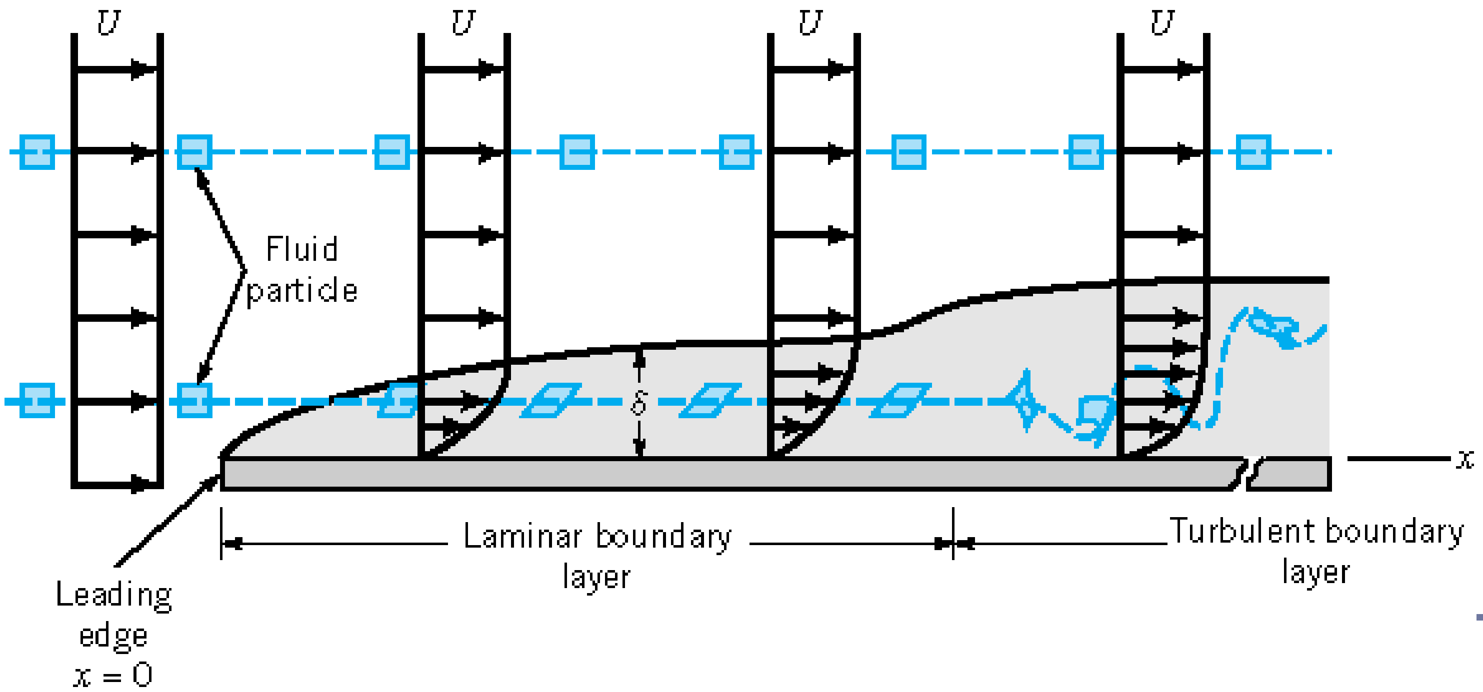
# 9.1 Concept of Boundary Layer

When viscous fluid flows past a body, due to the effect of fluid viscosity, there exists a thin layer (boundary layer) near the wall, where velocity dramatically varies. E.g., for flow past a horizontal plate, beside the plate a boundary layer forms.



## Boundary Layer Thickness ---- δ

***Boundary layer thickness, δ, is defined as the distance from the wall (with zero velocity) to the place with velocity of 99% times the one of potential flow.*** **δ** increases downstream, because viscous friction reduces flow velocity, only farther away from the wall velocity could reach the 99% potential velocity there.



## Flow State of Boundary Layer

According to tests, similar to pipe flow, in boundary layer there also exist different flow states, laminar flow and turbulent flow. A ***laminar boundary layer*** means flow in the layer is pure laminar flow. A ***turbulent boundary layer*** specifies pure turbulent flow. There is a transitional region, where both laminar flow and turbulent flow co-exists.

Reynolds number, *Re*, is employed as a criterion to judge flow state of boundary layer. In evaluation of *Re*, distance, *x*, measured from the front edge is employed

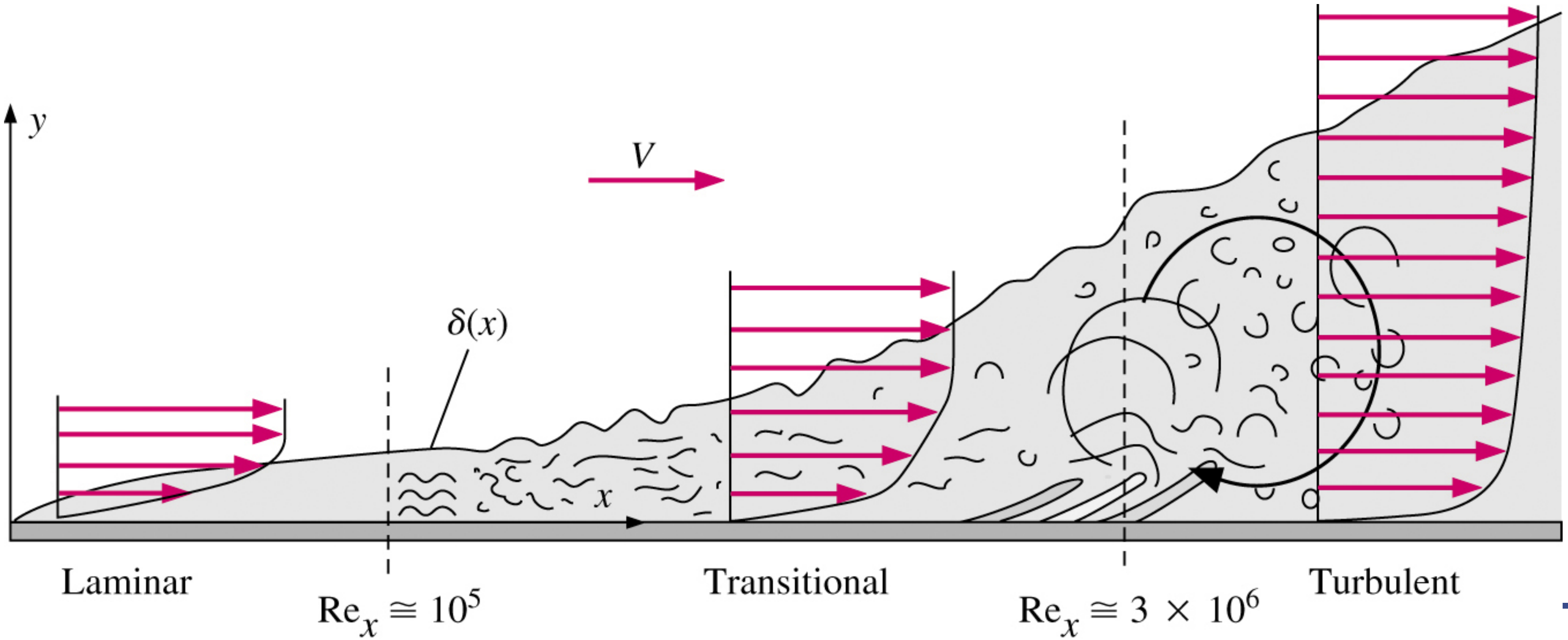
$$\boxed{\mathrm{Re} = \frac{V_\infty x}{\nu}}$$

**Critical Reynolds number**

$$Re_{xK} = \frac{V_\infty x_K}{\nu} = (3.5 \sim 5.0) \times 10^5$$

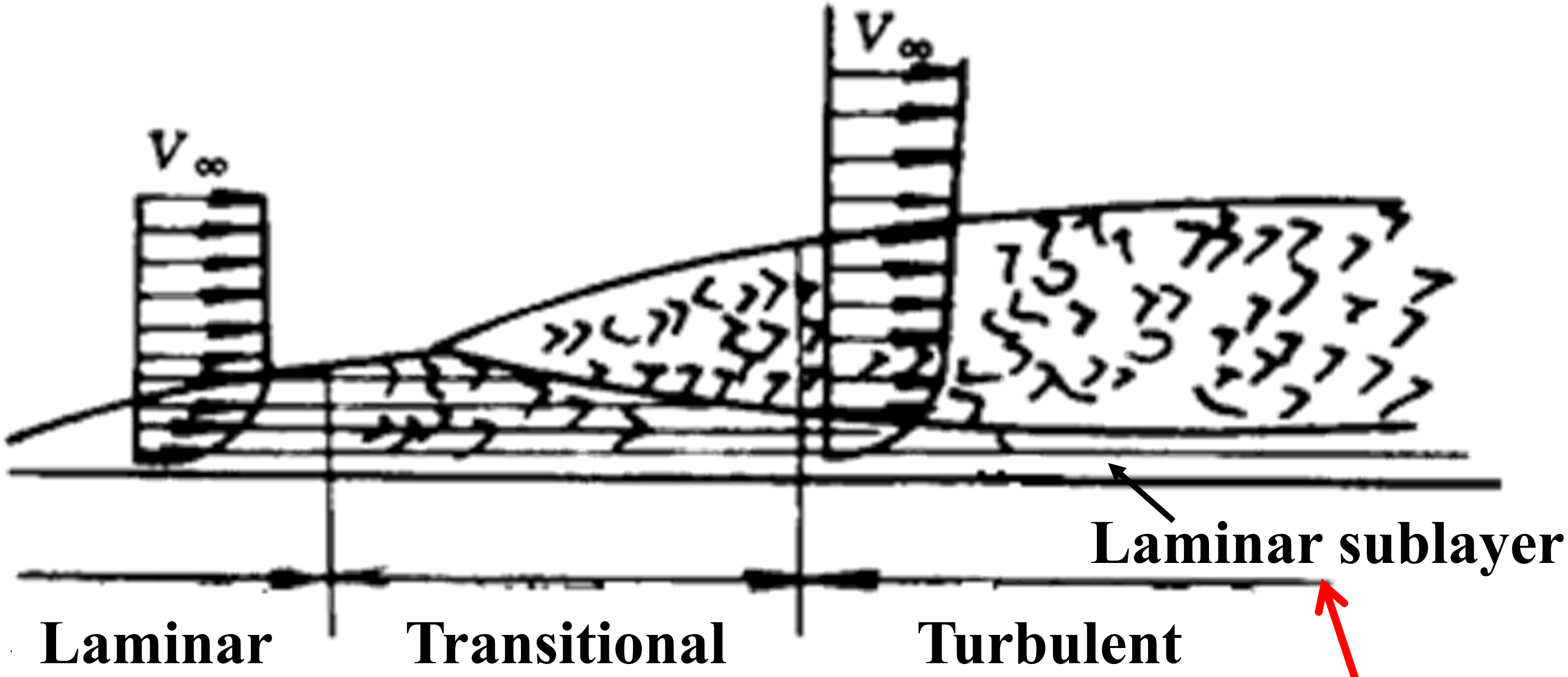
# 9.2 Description of Boundary Layer

In the starting area, very near to the leading edge, there exists a ***laminar boundary layer*** region. With the increase of distance, it finally develops to ***turbulent boundary layer***. In between, there is a ***transitional region***. The beginning point of the transitional region is called ***transitional point***.



# 9.2 Description of Boundary Layer

Even in transitional and turbulent region, due to the effect of the plate there still exists a very thin laminar flow near the wall. This laminar layer at the bottom of boundary layer is called ***laminar sublayer***.




Thickness related to *Re*

# 9.2 Description of Boundary Layer

## Characteristics of Boundary Layer

- **Very thin comparing with the length of body, *i*.e.,** $\delta \ll L$.
- **Large velocity gradient along boundary thickness**
- **Boundary thickness increases downstream.**
- **Pressure penetrates boundary layer due to small thickness.**
- **Viscous force is comparable with inertial force.**
- **In boundary layer, both laminar flow and turbulent flow are possible and may co-exist.**

# 9.3 Governing Equations of 2d Laminar Boundary Layer

**Assumptions**: ***2d flow***, ***incompressible***, ***steady***, ***no body force***.

***N-S equation*** is written as

$$\left.\begin{aligned}
v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \\
v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0
\end{aligned}\right\}$$

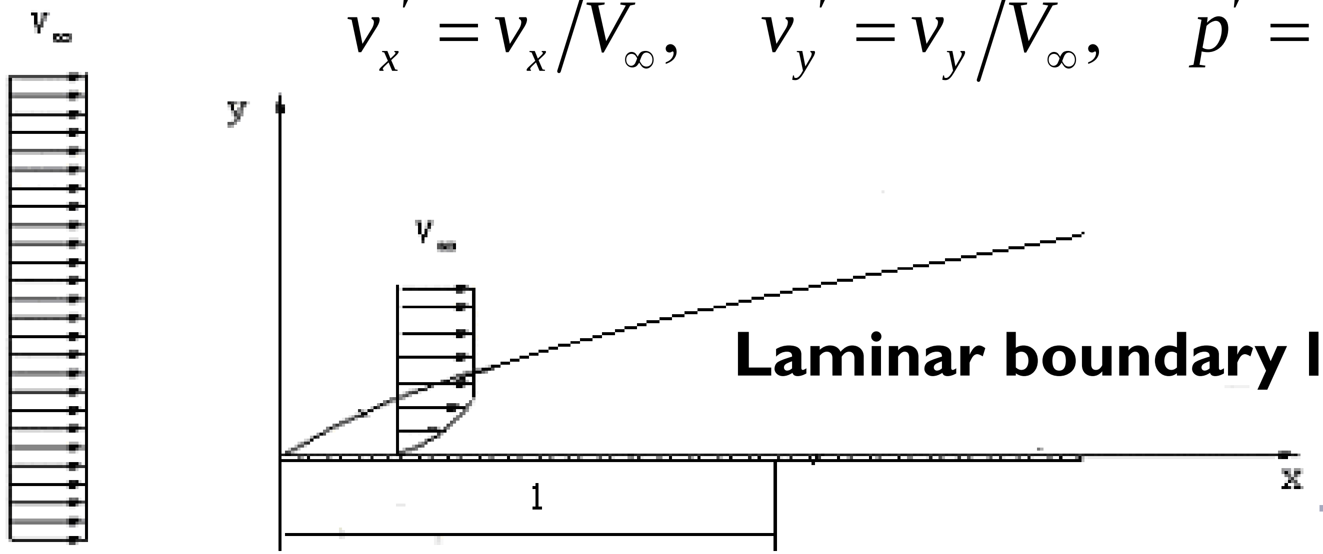
Based on characteristics of boundary layer, we shall see ***N-S equation*** can be greatly simplified. A boundary layer equation for laminar boundary layer will be derived by careful dimension analysis.

## Dimension Analysis

Consider a half infinitely long flat plate. The incoming flow is of speed $V_\infty$, a boundary layer forms near the plate with thickness, $\delta$, at a location away from the leading edge of distance $\ell$. Choose $V_\infty$ and $\ell$ as fundamental quantities, we can obtain following dimensionless quantities

$$x' = x/\ell, \quad y' = y/\ell, \quad \delta' = \delta/\ell,$$

$$v_x{}' = v_x/V_\infty, \quad v_y{}' = v_y/V_\infty, \quad p' = p/\left(\rho V_\infty^2\right)$$



# 9.3 Governing Equations of 2d Laminar Boundary Layer

Using these dimensionless quantities, ***N-S equation*** becomes

$$v'_x \frac{\partial v'_x}{\partial x'} + v'_y \frac{\partial v'_x}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_l}\left(\frac{\partial^2 v'_x}{\partial x'^2} + \frac{\partial^2 v'_x}{\partial y'^2}\right)$$

$$1\cdot 1 \qquad \delta'\cdot\frac{1}{\delta'} \qquad 1 \qquad (\delta')^2 \qquad 1 \qquad \frac{1}{\delta'^2}$$

$$v'_x \frac{\partial v'_y}{\partial x'} + v'_y \frac{\partial v'_y}{\partial y'} = -\frac{\partial p'}{\partial y'} + \frac{1}{\mathrm{Re}_l}\left(\frac{\partial^2 v'_y}{\partial x'^2} + \frac{\partial^2 v'_y}{\partial y'^2}\right)$$

$$1\cdot\delta' \qquad \delta'\cdot 1 \qquad \left(\frac{1}{\delta'}\right) \qquad (\delta')^2 \qquad \delta' \qquad \frac{1}{\delta'}$$

$$\frac{\partial v'_x}{\partial x'} + \frac{\partial v'_y}{\partial y'} = 0$$

$$1 \qquad 1$$

where

$$\mathrm{Re}_l = \frac{V_\infty \ell}{\nu}$$

is ***Reynolds number***.

# 9.3 Governing Equations of 2d Laminar Boundary Layer

As mentioned above, $\delta$ is small relative to $\ell$, i.e., $\delta \ll \ell$

or $\delta' = \delta/\ell \ll 1$. And $v_x \sim V_\infty$, $x \sim \ell$ and $y \sim \delta$.

Then, $v'_x \sim 1$, $x' \sim 1$ and $p' \sim 1$. Therefore, we have

$$\boxed{\frac{\partial v'_x}{\partial x'} \sim 1, \quad \frac{\partial^2 v'_x}{\partial x'^2} \sim 1, \quad \frac{\partial v'_x}{\partial y'} \sim \frac{1}{\delta'}, \quad \frac{\partial^2 v'_x}{\partial y'^2} \sim \frac{1}{\delta'^2}.}$$

From continuity, we have

$$\boxed{\frac{\partial v'_y}{\partial y'} = -\frac{\partial v'_x}{\partial x'} \sim 1}$$

So $v'_y \sim \delta'$ , therefore

$$\boxed{\frac{\partial v'_y}{\partial x'} \sim \delta', \quad \frac{\partial^2 v'_y}{\partial x'^2} \sim \delta', \quad \frac{\partial v'_y}{\partial y'} \sim 1, \quad \frac{\partial^2 v'_y}{\partial y'^2} \sim \frac{1}{\delta'}.}$$

Based on the above dimension analysis, inertial term in the 2nd equation is relatively small and can be neglected. The following terms are also small and can be neglected as well.

$$\frac{\partial^2 v'_x}{\partial x'^2}, \qquad \frac{\partial^2 v'_y}{\partial x'^2}, \qquad \frac{\partial^2 v'_y}{\partial y'^2}.$$

In this way, only one viscous term is left. It is the term

$$\frac{\partial^2 v'_x}{\partial y'^2}.$$

# 9.3 Governing Equations of 2d Laminar Boundary Layer

According to above dimension analysis, N-S equation is greatly simplified. Below is the result, namely ***boundary layer equations***

$$\begin{cases} v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu \dfrac{\partial^2 v_x}{\partial y^2} \\ \dfrac{\partial p}{\partial y} = 0 \\ \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0 \end{cases}$$

***Bernoulli's equation*** on the outer boundary

$$p_b + \rho V_\infty^{\ 2} / 2 = \text{const}$$

differentiate with $x$,

$$\frac{\mathrm{d}p_b}{\mathrm{d}x} = -\rho V_\infty \frac{\mathrm{d}V_\infty}{\mathrm{d}x}$$

So, boundary layer equation becomes

$$\boxed{\begin{cases} v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} = V_\infty \dfrac{\mathrm{d}V_\infty}{\mathrm{d}x} + \nu \dfrac{\partial^2 v_x}{\partial y^2} \\ \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0 \end{cases}}$$

# 9.3 Governing Equations of 2d Laminar Boundary Layer

For half infinitely long plate in uniform flow with ***favorable pressure gradient***, velocity on outer boundary is constant, that is

$$\frac{\mathrm{d}V_\infty}{\mathrm{d}x}=0$$

Then, ***boundary layer equation*** is further simplified

$$\boxed{\begin{cases} v_x\dfrac{\partial v_x}{\partial x}+v_y\dfrac{\partial v_x}{\partial y}=\nu\dfrac{\partial^2 v_x}{\partial y^2} \\ \dfrac{\partial v_x}{\partial x}+\dfrac{\partial v_y}{\partial y}=0 \end{cases}}$$

**Boundary conditions** of the boundary layer equation

**On** $y = 0$

$$v_x = v_y = 0$$

$$\frac{\partial^2 v_x}{\partial y^2} = -\frac{1}{\nu} V_\infty \frac{\mathrm{d}V_\infty}{\mathrm{d}x}$$

**On** $y = \delta$

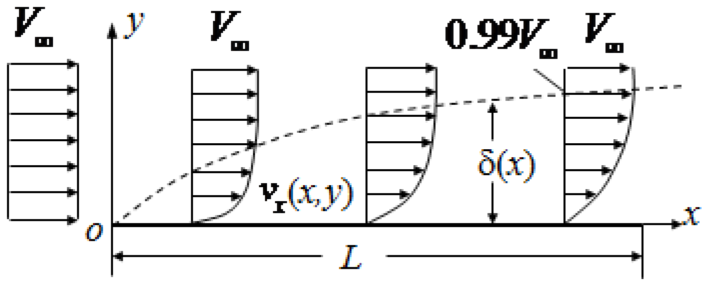
$$v_x = V_\infty$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2 v_x}{\partial y^2} = \frac{\partial^{(n)} v_x}{\partial y^{(n)}} = 0$$

# 9.4 Blasius Solution

## H. Blasius (1908)

**Assumptions:** ***Half infinite plate***, ***incompressible***, ***steady***, ***laminar flow***, ***no body force***, ***no pressure gradient***.



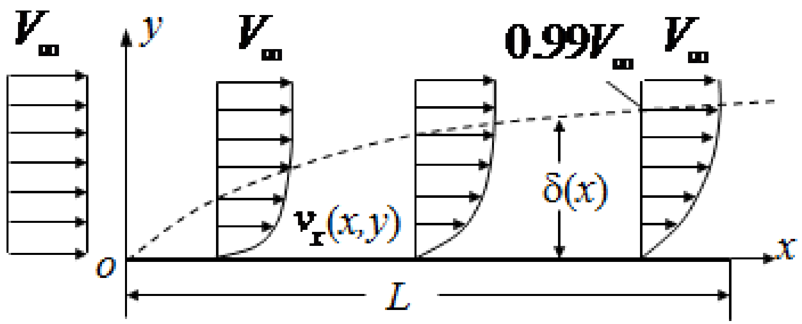

$$\left.\begin{aligned}
&\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0\\
&v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=\nu\frac{\partial^2 v_x}{\partial y^2}\\
&y=0,\quad v_x=v_y=0\\
&y\to\infty,\quad v_x=V_\infty
\end{aligned}\right\}
\Rightarrow
\boxed{\begin{aligned}&\frac{v_x}{V_\infty}=F\left(\frac{y}{\delta}\right)=F(\eta)\\ &f(\eta)\equiv\int_0^\eta F(\eta)d\eta\end{aligned}}
\Rightarrow
\begin{cases}2f'''+ff''=0\\ \eta=0,\quad f'=0,\ f=0\\ \eta\to\infty,\ f'=1\end{cases}$$

**Boundary Layer Equation** — **Similar Velocity Profile** — **3rd Order ODE**

# 9.4 Blasius Solution

## Similar Velocity Profile regardless of the Location

It is in the sense that though the scale factors of velocity and vertical distance could vary with location, but relation of scaled velocity to scaled distance is the same for any cross-section.




**Introduce stream function $\psi$**

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

**Boundary Layer Equation**

$$\begin{cases} y = 0: & \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial x} = 0 \\ y = \infty: & \dfrac{\partial \psi}{\partial y} = V_\infty \end{cases}$$

# 9.4 Blasius Solution

**Blasius choose $\delta$ and $V_\infty$ as scale factors for distance and velocity.**

$$\frac{\delta}{x} \sim \sqrt{\frac{\nu}{V_\infty x}}$$

$$\eta \sim y/\delta$$

$$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \eta}\dfrac{\partial \eta}{\partial x} = -\dfrac{y}{2x}\sqrt{\dfrac{V_\infty}{\nu x}}\dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial \eta}\dfrac{\partial \eta}{\partial y} = \sqrt{\dfrac{V_\infty}{\nu x}}\dfrac{\partial}{\partial \eta} \\ v_x = \dfrac{\partial \psi}{\partial y} = \sqrt{\dfrac{V_\infty}{\nu x}}\dfrac{\partial \psi}{\partial \eta} \\ v_y = -\dfrac{\partial \psi}{\partial x} = \dfrac{y}{2x}\sqrt{\dfrac{V_\infty}{\nu x}}\dfrac{\partial \psi}{\partial \eta} \end{cases}$$

$$\frac{v_x}{V_\infty} = \frac{1}{\sqrt{\nu x V_\infty}}\frac{\partial \psi}{\partial \eta}$$

$$f(\eta) = \frac{\psi}{\sqrt{\nu x V_\infty}}$$

# 9.4 Blasius Solution

$$\begin{cases} v_x = \dfrac{\partial \psi}{\partial y} = V_\infty f' \\[2ex] v_y = -\dfrac{\partial \psi}{\partial x} = \dfrac{V_\infty}{2}\sqrt{\dfrac{\nu}{V_\infty x}}(\eta f' - f) \\[2ex] \dfrac{\partial v_x}{\partial x} = \dfrac{\partial^2 \psi}{\partial x \partial y} = -\dfrac{V_\infty}{2x}\eta f'' \\[2ex] \dfrac{\partial v_x}{\partial y} = \dfrac{\partial^2 \psi}{\partial y^2} = V_\infty \sqrt{\dfrac{V_\infty}{\nu x}} f'' \\[2ex] \dfrac{\partial^2 v_x}{\partial y^2} = \dfrac{\partial^3 \psi}{\partial y^3} = V_\infty \left(\dfrac{V_\infty}{\nu x}\right) f''' \end{cases}$$

$$\begin{cases} 2f''' + ff'' = 0 \\ \eta = 0, \quad f' = 0, f = 0 \\ \eta \to \infty, f' = 1 \end{cases}$$

- Write $f(\eta)$ as a power series of $\eta$ at point $\eta = 0$ .
- Coefficients are determined from the boundary condition.

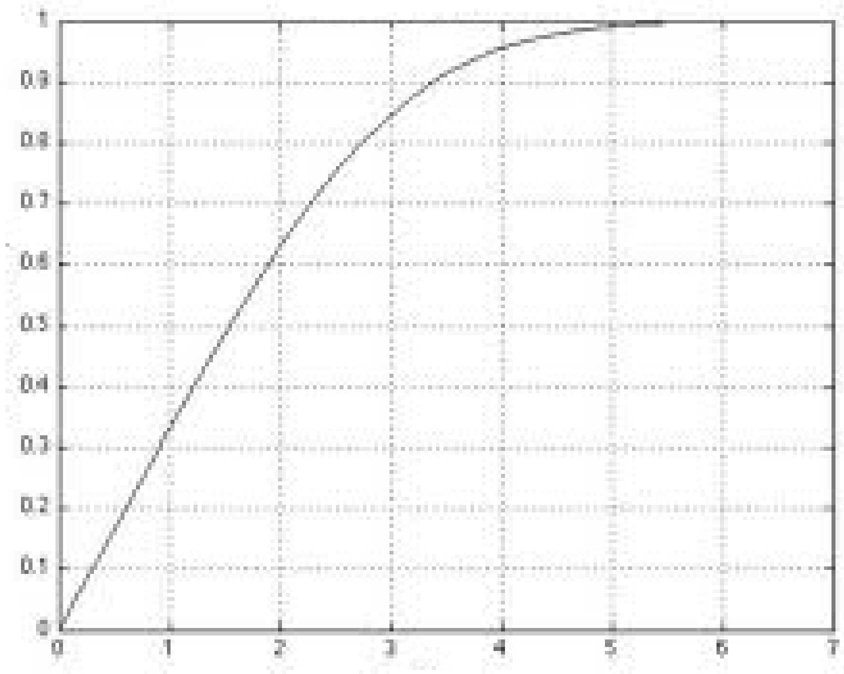
**L. Howarth (1938) gave out a numeric result with good accuracy.**

# 9.4 Blasius Solution

| $\eta = y\sqrt{\frac{V_\infty}{\nu x}}$ | $f(\eta)$ | $f'(\eta)=\frac{v_x}{V_\infty}$ | $f''(\eta)$ | $\eta = y\sqrt{\frac{V_\infty}{\nu x}}$ | $f(\eta)$ | $f'(\eta)=\frac{v_x}{V_\infty}$ | $f''(\eta)$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 0 | 0 | 0.33206 | 4.8 | 3.08534 | 0.98779 | 0.02187 |
| 0.4 | 0.02656 | 0.13277 | 0.33147 | 5.0 | 3.28329 | 0.99155 | 0.01591 |
| 0.8 | 0.10611 | 0.26471 | 0.32739 | 5.2 | 3.48189 | 0.99425 | 0.01134 |
| 1.2 | 0.23895 | 0.39378 | 0.31659 | 5.6 | 3.88031 | 0.99748 | 0.00543 |
| 1.6 | 0.42023 | 0.51676 | 0.29667 | 6.0 | 4.27964 | 0.99898 | 0.00240 |
| 2.0 | 0.65003 | 0.62977 | 0.26675 | 6.4 | 4.67938 | 0.99961 | 0.00098 |
| 2.4 | 0.92230 | 0.72899 | 0.22809 | 6.8 | 5.07928 | 0.99987 | 0.00037 |
| 2.8 | 1.23099 | 0.81152 | 0.18401 | 7.2 | 5.47925 | 0.99996 | 0.00013 |
| 3.2 | 1.56911 | 0.87609 | 0.13913 | 7.6 | 5.87924 | 0.99999 | 0.00004 |
| 3.6 | 1.92954 | 0.92333 | 0.09809 | 8.0 | 6.27923 | 1.00000 | 0.00001 |
| 4.0 | 2.30576 | 0.95552 | 0.06424 | 8.4 | 6.67923 | 1.00000 | 0.00000 |
| 4.4 | 2.69238 | 0.97587 | 0.03897 | 8.8 | 7.07923 | 1.00000 | 0.00000 |

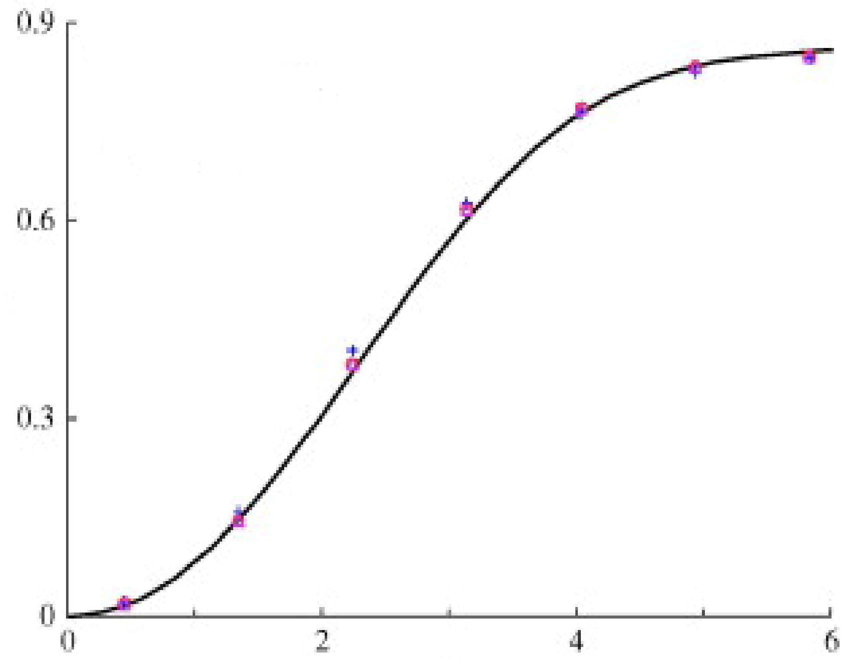
# 1.Velocity Profile of Blasius Solution

$$f'(\eta)=\frac{v_x}{V_\infty}$$




$$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$$

$$\frac{v_y}{V_\infty}\sqrt{\frac{V_\infty x}{\nu}}=\frac{1}{2}(\eta f'-f)$$




$$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$$

## 2. Thickness of Boundary Layer

$$\delta \approx 5.0\sqrt{\frac{\nu x}{V_\infty}}$$

**Proportional to $\sqrt{x}$ , agree well with the qualitative analysis.**

## 3. Shear Stress

$$\tau = \mu\left(\frac{\partial v_x}{\partial y}\right)_{y=0} = \mu\sqrt{\frac{V_\infty^3}{\nu x}}f''(0) = 0.332\mu V_\infty\sqrt{\frac{V_\infty}{\nu x}}$$

**Shear stress coefficient**

$$C_x = \frac{\tau}{\frac{1}{2}\rho V_\infty^2} = 0.664\sqrt{\frac{\nu}{V_\infty x}} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

## 4. Drag and Drag Coefficient

$$D_f = \int_0^L \tau \mathrm{d}x = \frac{0.664 \rho V_\infty^2 L}{\sqrt{\mathrm{Re}_L}}$$

### Drag Coefficient

$$C_f = \frac{D_f}{\frac{1}{2}\rho V_\infty^2 L} = \frac{1.328}{\sqrt{\mathrm{Re}_L}} \qquad (100 < \mathrm{Re}_L < 5\times 10^5)$$

### 2nd Order Approximation (Yung-Huai Kuo)

$$C_f = \frac{1.328}{\sqrt{\mathrm{Re}_L}} + \frac{4.12}{\mathrm{Re}_L} \qquad (1 < \mathrm{Re}_L < 5\times 10^5)$$

郭永怀（1909－1968）

## 5. Remarks on Blasius Solution

### Validation

**For finite length plate, it was validated by wind tunnel test carried by *Nikuradse* (1942).**

### Application

- Estimation of frictional drag,
- Calibration of velocity probe,
- Verification of analysis method and code of boundary layer solution,
- Expression of laminar boundary layer at the front edge of an object in solution of turbulent boundary layer.