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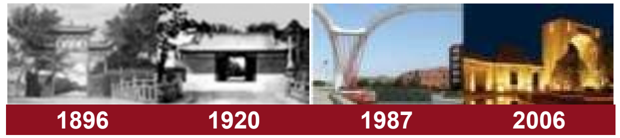

Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

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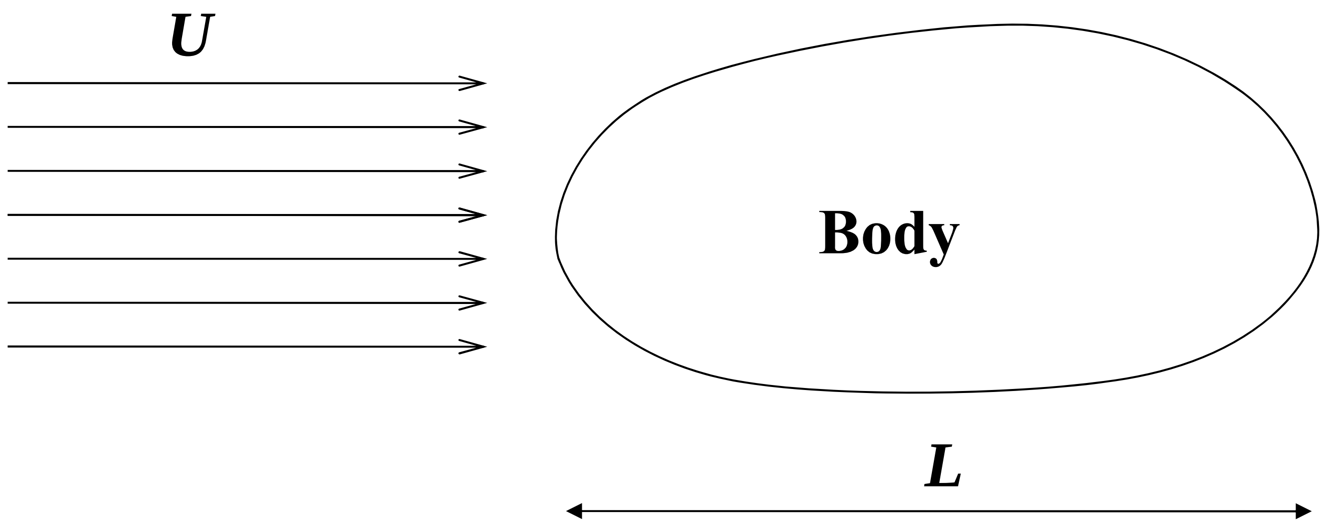
Chapter 8

Fundamental Theory of Viscous Incompressible Fluid Flow

8.4 Simplification of N-S Equation

In solution of *N-S equation*, for special cases, some terms may be of very small value relative to other terms, and less important, and become negligible. While N-S equation is written in a dimensionless form, as will be given later, we can simply determine whether a term is negligible or not. As an example, we look at an *unsteady flow past a body*.



We choose four ***Characteristic quantities*** below.

L – length of the body; U – uniform speed;

T – time; P – pressure at infinity.

8.4 Simplification of N-S Equation

In terms of these characteristic scales, physical quantities can be ***non-dimensionalized***.

$$\boldsymbol{V}^* = \boldsymbol{V}/U, \quad t^* = t/T, \quad \boldsymbol{x}^* = \boldsymbol{x}/L, \quad p^* = p/P, \quad \boldsymbol{g}^* = \boldsymbol{g}/g$$

And ***N-S equation*** is rewritten in these dimensionless quantities.

$$\left(\frac{L}{UT}\right)\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot \nabla^* \boldsymbol{V}^* = -\left(\frac{P}{\rho U^2}\right)\nabla^* p^* + \left(\frac{gL}{U^2}\right)\boldsymbol{g}^* + \left(\frac{\nu}{UL}\right)\nabla^{*2}\boldsymbol{V}^*$$

8.4 Simplification of N-S Equation

Apparently, importance of each term in N-S equation is determined by the relative value in parentheses in front of each term.

Strouhal number $$St = \frac{L}{UT} = \frac{\text{Local derivative}}{\text{Convective derivative}}$$

Euler number $$Eu = \frac{P}{\rho U^2} = \frac{\text{Pressure}}{\text{Inertial Force}}$$

Reynolds number $$Re = \frac{UL}{\nu} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

Froude number $$Fr = \frac{U^2}{gL} = \frac{\text{Inertial Force}}{\text{Gravity}}$$

8.4 Simplification of N-S Equation

According to the above 4 dimensionless numbers, we can classify different categories and simplify ***N-S equation*** respectively.

1) Large Reynolds number flow (Re>>1)

$$Re = \frac{UL}{\nu} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

Re>>1 ⇨ **viscous force is negligible**

N-S equation **is simplified to** ***Euler's equation***

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V} = \mathbf{g} - \frac{1}{\rho} \nabla p$$

2) Small Strouhal number Flow (St<<1)

$$St = \frac{L}{UT} = \frac{\text{Local derivative}}{\text{Convective derivative}}$$

St<<1 ⇒ **unsteady term is negligible**
⇒ **nearly steady flow**

$$\mathbf{V} \bullet \nabla \mathbf{V} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{V}$$

3) Small Reynolds number Flow (Re<<1)

$$Re = \frac{UL}{\nu} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

Re<<1 ⇨ **nonlinear convective term is negligible**

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{V}$$

It is a linear equation. *It may be solved analytically, provided initial and boundary conditions are simple enough.*

8.5 Some Simple Viscous Flows

$$\nabla \cdot \mathbf{V} = 0 \quad + \quad \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{V}$$

Generally, solution of ***N-S equation*** shows some difficulty. Most of them have to be solved by means of numerical methods. But for a few very simple flows, analytical solutions have been obtained from the ***simplified N-S equation*** derived by dimension analysis and/or physical investigations.

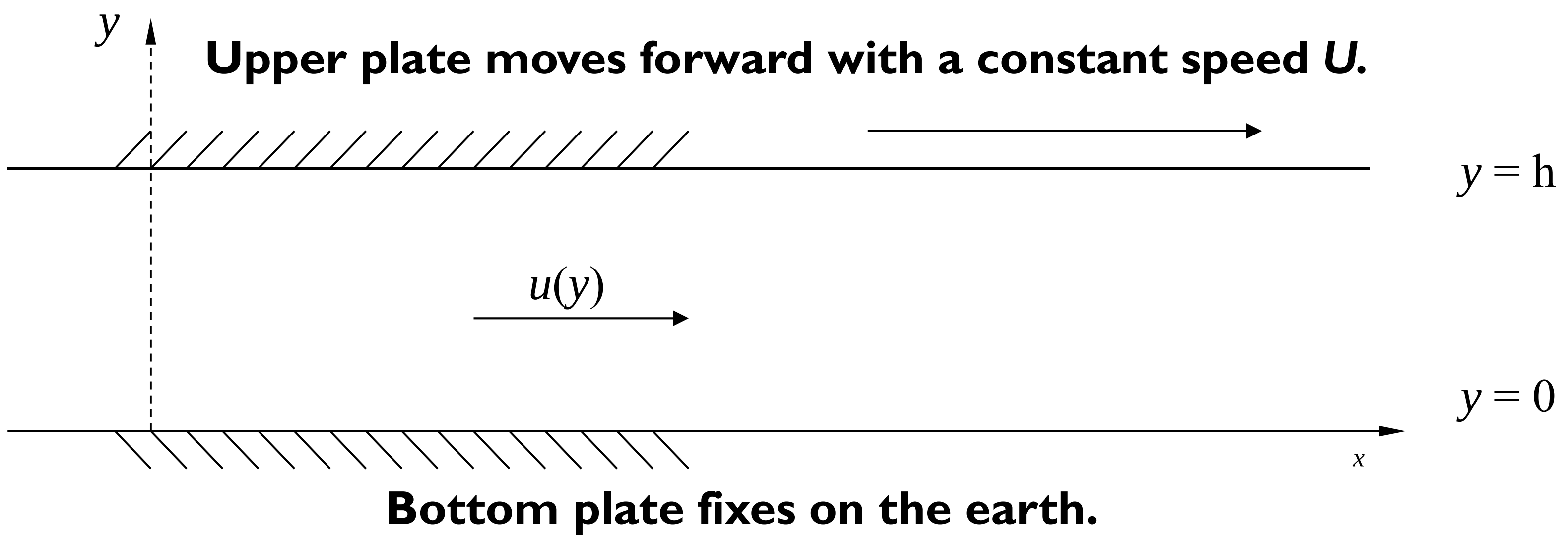
$$\left(\frac{L}{UT}\right)\frac{\partial \boldsymbol{V}^*}{\partial t^*} + \boldsymbol{V}^* \cdot \nabla^* \boldsymbol{V}^*$$

$$= -\left(\frac{P}{\rho U^2}\right)\nabla^* p^* + \left(\frac{gL}{U^2}\right)\boldsymbol{g}^* + \left(\frac{\nu}{UL}\right)\nabla^{*2}\boldsymbol{V}^*$$

Hereafter we shall introduce a few classical and well known simple viscous flows. For simplicity we consider ***steady flows***.

8.5 Some Simple Viscous Flows

Case (1) 2 Dimensional Plane *Poiseuille-Couette Flow*



From physical point of view, this flow is driven by 3 forces: (1) upper plate motion; (2) pressure gradient along *x*-axis, $\partial p / \partial x$; (3) the body force.

8.5 Some Simple Viscous Flows

Since upper plate is horizontally moving at a constant speed and the lower plate horizontally fixes on the earth, so fluid between the two plates will horizontally move with speed not varying along *x*-axis.

$$v = 0 \Longrightarrow \partial u / \partial x = 0 \Longrightarrow u = u(y)$$

This velocity distribution may be used to simplify ***N-S equation***.

$$x\text{-component:}\quad \frac{\partial \cancel{u}}{\cancel{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right) + g_x$$

$$\Rightarrow \nu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} - g_x \quad \cdots\cdots\cdots\cdots (1)$$

$$y\text{-component:}\quad \frac{\partial \cancel{v}}{\cancel{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right) + g_y$$

$$\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y = 0 \quad \cdots\cdots\cdots\cdots (2)$$

8.5 Some Simple Viscous Flows

Now all convective terms have disappeared. For any other ***unidirectional flow***, *where velocities are all parallel*, treatment will be similar.

From equation (2), the *y*-component equation, pressure is expressed as

$$p = p(x) + \rho g_y y$$

By integrating equation (1) twice for *y*, variation of velocity with *y* is further derived.

$$u(y) = \left(\frac{\partial p}{\partial x} - \rho g_x \right) \frac{y^2}{2\mu} + C_1 y + C_2$$

8.5 Some Simple Viscous Flows

The ***no-slip*** conditions on the upper and bottom plates say

$$u(y=0)=0 \qquad \text{(on the bottom plate)}$$

$$u(y=h)=U \qquad \text{(on the upper plate)}$$

Applying them to the velocity expression, constants C_1 and C_2 are determined.

$$C_1=\frac{1}{h}\left[U-\left(\frac{\partial p}{\partial x}-\rho g_x\right)\frac{h^2}{2\mu}\right], \quad C_2=0$$

Therefore, we finally obtain the **velocity** distribution

$$u(y)=\underbrace{\left(-\frac{\partial p}{\partial x}+\rho g_x\right)\frac{h^2}{2\mu}\left[\frac{y}{h}-\left(\frac{y}{h}\right)^2\right]}_{\text{Poiseuille Flow}}+\underbrace{U\frac{y}{h}}_{\text{Couette Flow}}$$

Poiseuille Flow ***Couette Flow***

8.5 Some Simple Viscous Flows

Shearing stress

$$\tau_{xy} = \mu \frac{du(y)}{dy} = \underbrace{\left(-\frac{\partial p}{\partial x} + \rho g_x\right)\left[\frac{h}{2} - y\right]}_{\text{linear stress distribution due to Poiseuille flow, zero stress at the centerline}} + \underbrace{\mu \frac{U}{h}}_{\text{constant stress due to Couette flow}}$$

Volume rate of unit thickness

$$Q = \int_0^h u\,dy = \left(-\frac{\partial p}{\partial x} + \rho g_x\right)\frac{h^3}{12\mu} + \frac{hU}{2}$$

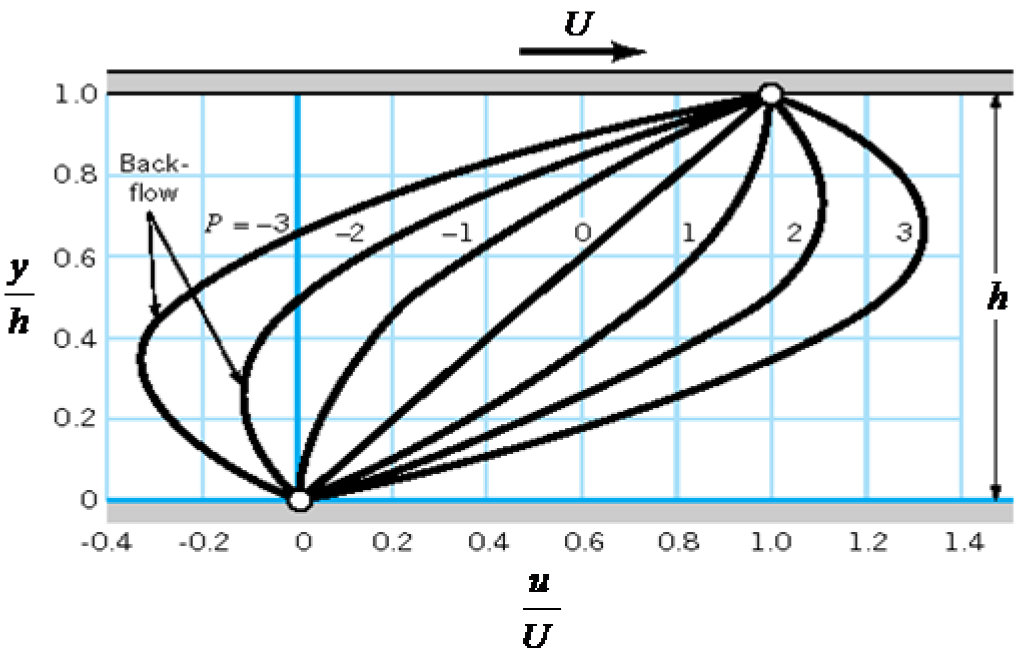
Average speed on vertical cross-section

$$\overline{u} = Q / h = \left(-\frac{\partial p}{\partial x} + \rho g_x\right)\frac{h^2}{12\mu} + \frac{U}{2}$$

8.5 Some Simple Viscous Flows

If body force vanishes, and denote $P = -\dfrac{dp}{dx}\dfrac{h^2}{2\mu U}$, velocity distribution becomes

$$\frac{u(y)}{U} = P\frac{y}{h}\left[1-\left(\frac{y}{h}\right)\right]+\frac{y}{h}$$



Volume rate of unit thickness

$$Q = \frac{Uh}{2} + \frac{Uh}{6}P$$

8.5 Some Simple Viscous Flows

$$(1)\ P = 0 \quad \left(i.e.\ \ \frac{\mathrm{d}p}{\mathrm{d}x} = 0 \right)$$

$$\frac{u}{U} = \frac{y}{h} \quad \Longrightarrow \quad u = \frac{y}{h}U$$

This is the ***Couette flow***, where pressure gradient vanishes, only the upper plate applies a force to drive the flow. It results a linear velocity distribution.

$$(2)\ P > 0, \left(i.e.\ \frac{\mathrm{d}p}{\mathrm{d}x} < 0 \right)$$

This flow is driven by a ***favorable pressure gradient*** (opposed to an ***adverse pressure gradient***), where pressure gradually decreases along the flow. Velocities in the flow field direct to the opposite direction of the pressure gradient.

8.5 Some Simple Viscous Flows

(3) $P=-1$

$$\frac{u}{U}=-\frac{y}{h}(1-\frac{y}{h})+\frac{y}{h}=\left(\frac{y}{h}\right)^2$$

Velocity distribution along *y* is a parabola. At $y=0$, the parabola is tangent to the normal of the bottom plate. It corresponds to the largest ***adverse pressure gradient*** without backward flow area.

$$\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{2\mu U}{h^2}$$

⟹ ***limit pressure gradient*** without back flow

$\frac{\mathrm{d}p}{\mathrm{d}x}>0,$ called ***adverse pressure gradient***

8.5 Some Simple Viscous Flows

(4) $P < -1, \left(i.e.\ \frac{\mathrm{d}p}{\mathrm{d}x} > \frac{2\mu U}{h^2} \right)$

While the ***adverse pressure gradient*** is large enough, greater than the ***limit pressure gradient***, towing force on the upper plate transmitted to the neighborhood of the bottom plate becomes weaker than the adverse pressure gradient, and under the action of the adverse pressure gradient fluid there becomes moving backward. It forms a ***back flow***.

(5) $P = -3$

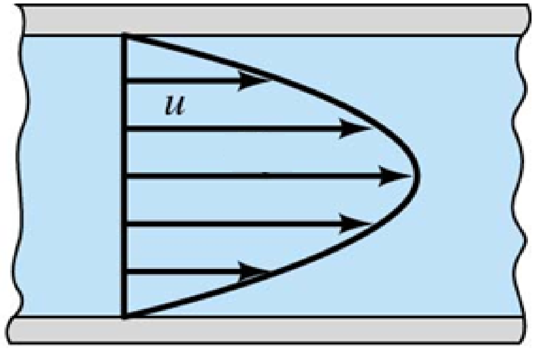
$$Q = \frac{Uh}{2} + \frac{Uh}{6} P = 0$$

In this case, contributions to Q due to ***adverse pressure gradient*** are balanced with those due to the upper plate motion. Then total volume rate through a vertical cross-section becomes zero. If P further decreases, P<-3, effect from adverse pressure gradient will be greater than the one from the upper plate motion, and volume rate will become negative, Q<0.

8.5 Some Simple Viscous Flows

As mentioned in the expression of u(y), this flow consists of two kinds of flows, ***Poiseuille flow*** and ***Couette flow***.

1) Poiseuille flow: Both plates are fixed, i.e., $U = 0$. The flow is solely driven by the pressure gradient.



Parabolic velocity pattern

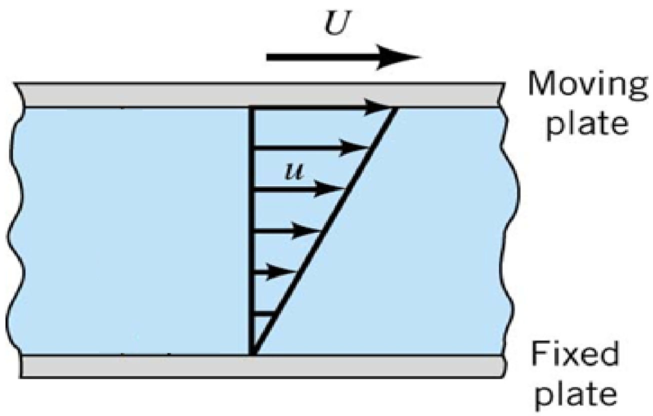
$$u(y) = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\left(y^2 - hy\right)$$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = 0 \Rightarrow y = \frac{h}{2} \quad \Longrightarrow \quad u_{\max} = -\frac{h^2}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}x}$$

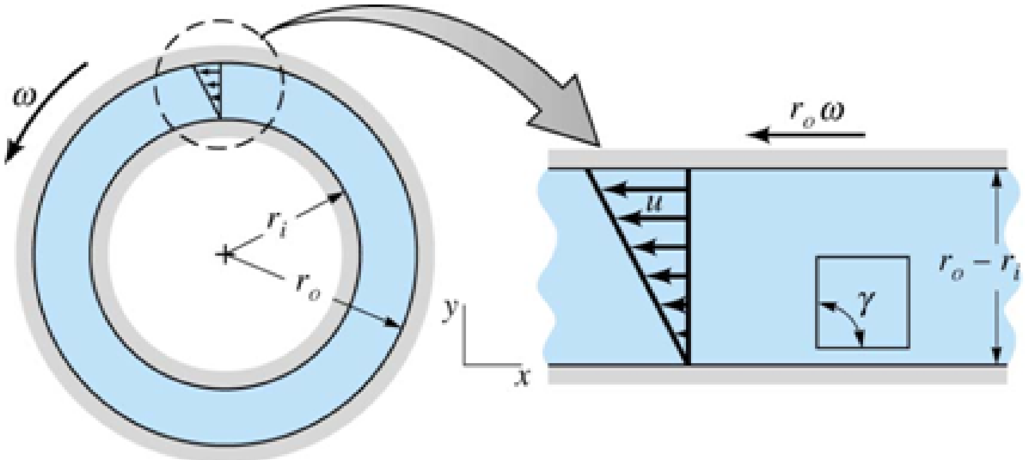
Average velocity over the cross-section:

$$\overline{u} = \frac{1}{h}\int_0^h u\,\mathrm{d}y = -\frac{h^2}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}$$

2) Couette flow: Pressure gradient along the flow is zero, the upper plate motion drive the flow.



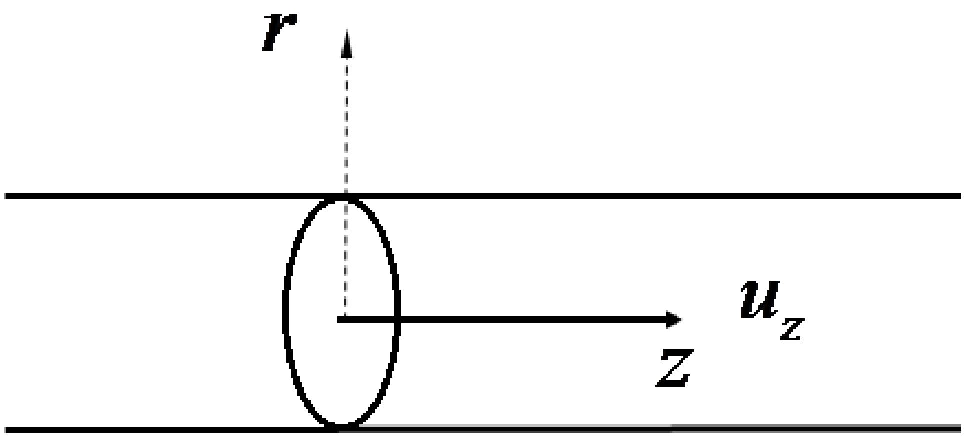
$$u(y) = \frac{y}{h}U$$



Case (2) Poiseuille flow in circular pipe

Fluid flows in a straight circular pipe. In a cylindrical coordinate system, since it is an ***axially-symmetric flow***, velocity is independent on the polar angle θ, i.e. $\partial/\partial\theta = 0$. In addition it is also a ***unidirectional flow*** as well. If the flow is driven by a pressure gradient, dp/dz, and denote velocity vector (u_r, u_θ, u_z), we have

$$u_r = u_\theta = 0, \quad u_z \neq 0$$



8.5 Some Simple Viscous Flows

Write down ***continuity equation***

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \qquad \Rightarrow \qquad u_z = u_z(r)$$

The z-component ***momentum equation*** becomes

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + g_z$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right) = \frac{r}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}$$

8.5 Some Simple Viscous Flows

By integrating the equation twice on r, expression of $u_z(r)$ is derived

$$u_z(r) = \frac{r^2}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z} + C_1 \ln r + C_2$$

Applying ***no-slip condition*** on the pipe wall to it, and considering the physical requirement of finite flow speed, constants C_1 and C_2 are thus determined.

$$u_z(r=0) \text{ is finite} \quad \Rightarrow \quad C_1 = 0$$

$$u_z(r=R) = 0 \ (\textit{no-slip} \text{ condition}) \quad \Rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z}$$

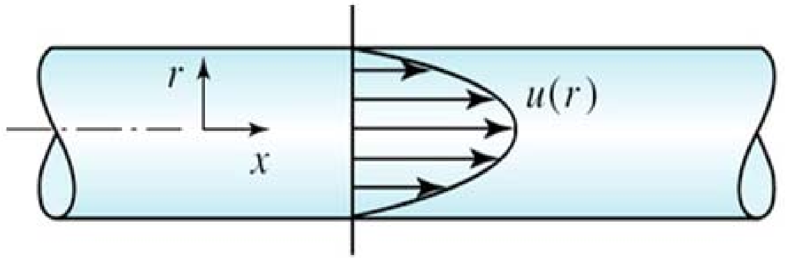
8.5 Some Simple Viscous Flows

Finally, velocity expression is obtained.

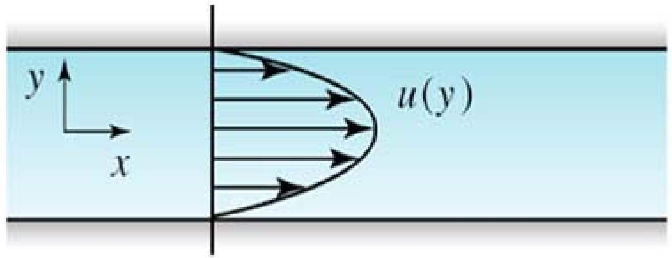
$$u_z(r) = -\frac{dp}{dz}\frac{R^2}{4\mu}\left[1-\frac{r^2}{R^2}\right]$$

The **maximum velocity** appears at the centerline of the pipe.

$$u_{max} = u_z(r=0) = -\frac{R^2}{4\mu}\frac{dp}{dz}$$



Flow inside a circular pipe



Flow between two parallel plates

Volume rate

$$Q = \int_A u_z \mathrm{d}A = 2\pi \int_0^R u_z r \mathrm{d}r = -\frac{\pi R^4}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}z}$$

Poiseuelle's law: *For flow inside a circular pipe driven by a constant pressure gradient,* ***flow rate*** *through a cross section is proportional to the pressure gradient and the 4th power of the pipe radius, but inversely proportional to the* ***dynamic viscosity*** *of the fluid.*

8.5 Some Simple Viscous Flows

Average velocity over the cross-section

$$\bar{u} = Q/A = \left. -\frac{\pi R^4}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \middle/ \pi R^2 \right. = -\frac{R^2}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{u_{\max}}{2}$$

Shear stress on the wall

$$\tau_w = -\mu \left.\frac{\mathrm{d}u_z}{\mathrm{d}r}\right|_{r=R} = -\frac{R}{2}\frac{\mathrm{d}p}{\mathrm{d}z} = 4\mu\frac{\bar{u}}{R}$$

8.5 Some Simple Viscous Flows

For a circular pipe of length *L*, ***pressure drop*** between two ends is equal to the product of the constant pressure gradient and the length it experiences

$$\Delta p = -\left(\mathrm{d}p / \mathrm{d}z\right) L$$

It is consumed by the fluid viscosity, and can be expressed as the ***head loss due to friction***, *i.e.* the equivalent pressure head of the fluid

$$h_f = \Delta p / \rho g$$

On the other hand, pressure gradient can be expressed by cross-sectional average velocity, and then **Darcy-Weisbach equation** is derived. It is an expression of the ***head loss due to friction***

$$h_f = \lambda \frac{L}{D} \frac{\overline{u}^2}{2g}$$

where *D* is the diameter of the circular pipe, *D* = 2*R*, or generally ***hydraulic diameter*** for non-circular cylinder.

$\lambda = \frac{64}{\mathrm{Re}}$ **: *Darcy friction factor*.** $\mathrm{Re} = \frac{\rho D \bar{u}}{\mu}$ **: *Reynolds number*.**

Case (3) Flow past a sphere at small Reynolds number

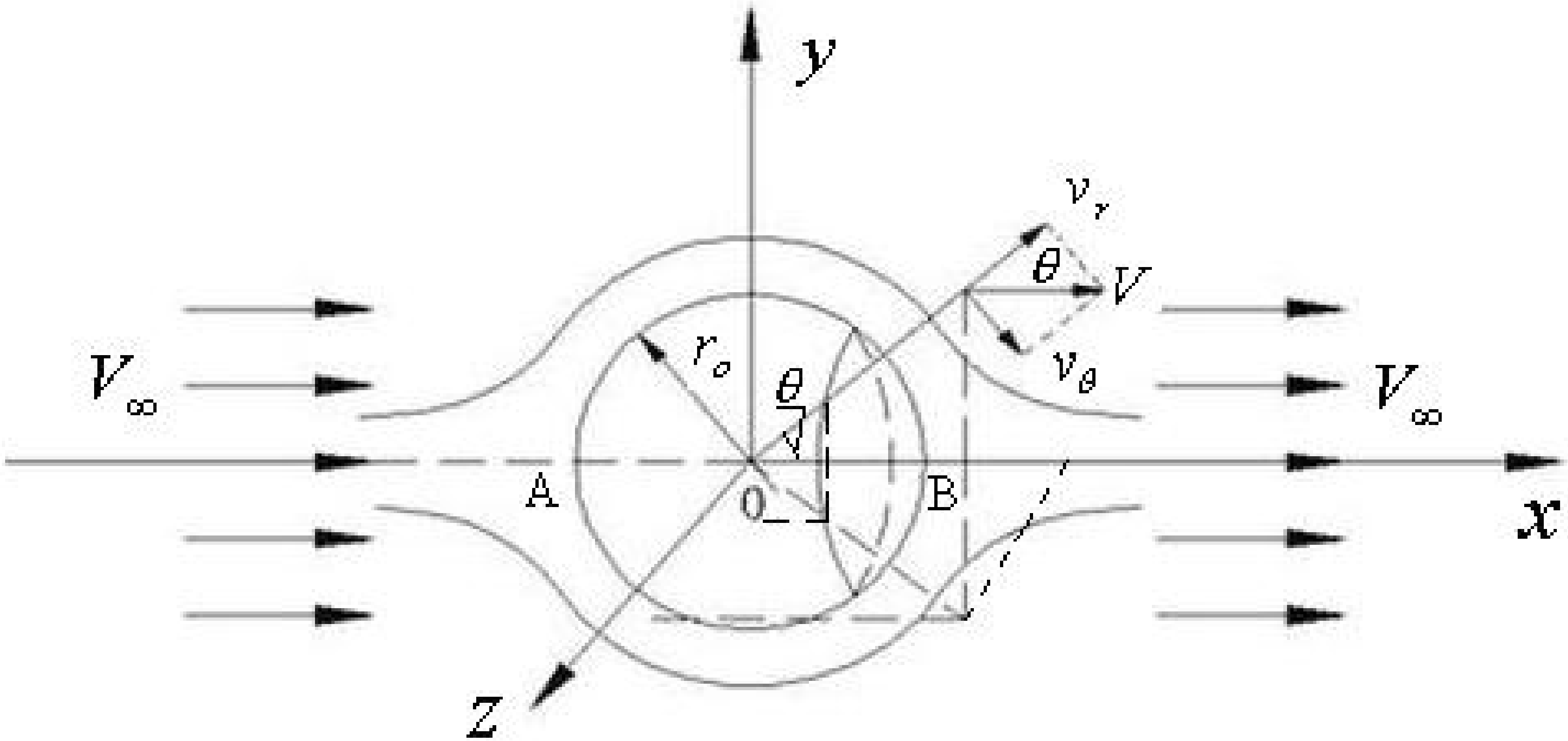
According to dimension analysis, for flows with small Reynolds number, inertial force is negligible. This kind of flows is called **Stokes flow**, or **creep flow**, where both the size of the flow field and value of velocity are small. The flow inside a bearing clearance is an example.

After removing inertia forces from ***N-S equation***, a simple ***Stokes equation*** results.

$$\left.\begin{aligned}
\frac{\partial p}{\partial x} &= \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \\
\frac{\partial p}{\partial y} &= \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) \\
\frac{\partial p}{\partial z} &= \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)
\end{aligned}\right\}$$

8.5 Some Simple Viscous Flows

As the figure shows, consider a uniform flow with velocity V_∞ at infinity along *x*-axis flows past a fixed small sphere of radius r_0



8.5 Some Simple Viscous Flows

In spherical coordinate system, since the flow is axially symmetry with x-axis, physical quantities will do not vary with the polar angle around x-axis.

$$V_r = V_r(r,\theta),\quad V_\theta = V_\theta(r,\theta),\quad V_\varphi = 0,\quad p = p(r,\theta)$$

And **Stokes equations** are written as

$$\frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{2V_r}{r} + \frac{V_\theta \mathrm{ctg}\,\theta}{r} = 0$$

$$\frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial V_r}{\partial r} + \frac{\mathrm{ctg}\,\theta}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{2V_r}{r^2} - \frac{2\mathrm{ctg}\,\theta}{r^2}V_\theta\right)$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial V_\theta}{\partial r} + \frac{\mathrm{ctg}\,\theta}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2 \sin^2\theta}\right)$$

Boundary conditions

at $r = r_0$, $$V_r(r_0, \theta) = V_\theta(r_0, \theta) = 0$$

at infinity $$V_r \to V_\infty \cos\theta, \quad V_\theta \to -V_\infty \sin\theta$$

By use of ***variable separation method***, we assume that the unknown velocity components and pressure be formally written as

$$\begin{cases} V_r(r, \theta) = f(r)\cos\theta \\ V_\theta(r, \theta) = -g(r)\sin\theta \\ p(r, \theta) = \mu h(r)\cos\theta \end{cases}$$

8.5 Some Simple Viscous Flows

Substituting these expressions in ***Stokes equations***, it derives a set of ***ordinary differential equations*** (or, simply **ODE**) on the unknown functions, $f(r)$, $g(r)$ and $h(r)$.

$$\begin{cases} h' = f'' + \dfrac{2}{r} f' - \dfrac{4(f-g)}{r^2} \\ \dfrac{h}{r} = g'' + \dfrac{2}{r} g' + \dfrac{2(f-g)}{r^2} \\ f' + \dfrac{2(f-g)}{r} = 0 \end{cases}$$

8.5 Some Simple Viscous Flows

Accordingly, the **boundary conditions** become

$$f\left(r_0\right)=0,\quad g\left(r_0\right)=0,\quad f\left(\infty\right)=V_\infty,\quad g\left(\infty\right)=V_\infty$$

From the above **ODE**s, an **ODE** on function f is obtained.

$$r^3 f^{(4)}+8r^2 f^{(3)}+8rf''-8f'=0$$

It can be immediately verified that following are four **particular solutions** of the above **ODE**.

$$f_1=r^2,\quad f_2=1,\quad f_3=\frac{1}{r},\quad f_4=\frac{1}{r^3}$$

Then, **general solution** of function f can be expressed as

$$f\left(r\right)=\frac{C_1}{r^3}+\frac{C_2}{r}+C_3+C_4r^2$$

8.5 Some Simple Viscous Flows

Applying the **general solution** of f to **Stokes equations**, other two functions have general solutions

$$g(r) = -\frac{C_1}{2r^3} + \frac{C_2}{2r} + C_3 + 2C_4 r^2$$

$$h(r) = \frac{C_2}{r^2} + 10C_4 r$$

Satisfaction of the **boundary conditions** determines the four constants.

$$C_1 = \frac{1}{2}V_\infty r_0^2, \quad C_2 = -\frac{3}{2}V_\infty r_0, \quad C_3 = V_\infty, \quad C_4 = 0$$

8.5 Some Simple Viscous Flows

Finally, substituting back to the expressions of velocity and pressure, we obtain the solution of the problem.

$$\begin{cases} V_r(r,\theta) = V_\infty \cos\theta \left(1 - \dfrac{3r_0}{2r} + \dfrac{r_0^3}{2r^3}\right) \\ V_\theta(r,\theta) = -V_\infty \sin\theta \left(1 - \dfrac{3r_0}{4r} - \dfrac{r_0^3}{4r^3}\right) \\ p(r,\theta) = -\dfrac{3r_0 \mu V_\infty}{2r^2} \cos\theta \end{cases}$$

8.5 Some Simple Viscous Flows

According to the ***constitutive law***, stresses can then be obtained.

$$\begin{cases} \sigma_{rr} = -p + 2\mu \dfrac{\partial V_r}{\partial r} \\ \tau_{r\theta} = \mu \left(\dfrac{1}{r} \dfrac{\partial V_r}{\partial \theta} + \dfrac{\partial V_\theta}{\partial r} - \dfrac{V_\theta}{r} \right) \\ \tau_{r\varphi} = \mu \left(\dfrac{1}{r} \dfrac{\partial V_\varphi}{\partial r} + \dfrac{1}{r \sin\theta} \dfrac{\partial V_r}{\partial \varphi} - \dfrac{V_\varphi}{r} \right) \end{cases}$$

Since the flow is symmetry, we have

$$V_\varphi = 0, \quad \frac{\partial}{\partial \varphi} = 0$$

On the sphere surface, stresses have expressions

$$\begin{cases} \sigma_{rr}\big|_{r=r_0} = \left(-p + 2\mu \dfrac{\partial V_r}{\partial r}\right)_{r=r_0} = \dfrac{3\mu V_\infty}{2r_0}\cos\theta \\ \tau_{r\theta}\big|_{r=r_0} = \left(\mu \dfrac{\partial V_\theta}{\partial r}\right)_{r=r_0} = \dfrac{3\mu V_\infty}{2r_0}\sin\theta \end{cases}$$

The resultant force will be parallel to the uniform flow, and of magnitude

$$F_D = \oiint_S \left(\sigma_{rr}\cos\theta - \tau_{r\theta}\sin\theta\right) dS$$

$$= \int_0^\pi \frac{3\mu V_\infty}{2r_0}\left(\cos^2\theta + \sin^2\theta\right) 2\pi r_0^2 \sin\theta d\theta$$

$$= 2\pi\mu r_0 V_\infty + 4\pi\mu r_0 V_\infty = 6\pi\mu r_0 V_\infty$$