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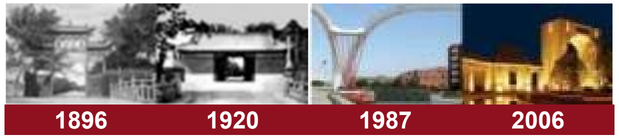
SHANGHAI JIAO TONG UNIVERSITY



Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Chapter 7
Water Waves

7.4 Airy Wave

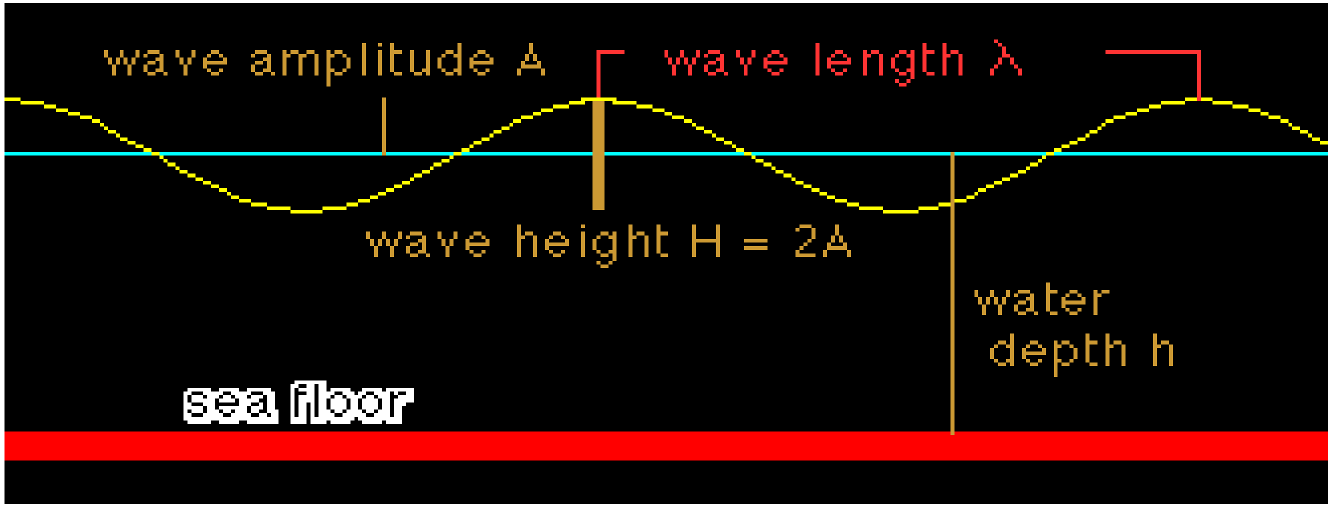
For Airy wave, we need to determine three parameters, $(A,\ k, \omega)$. In fact, only two of them, (A, k), need to be determined, since a dispersion relation exists, that will be introduced soon later in this section.

$$\eta = A\cos\left(kx - \omega t\right)$$

1. Airy wave is a 2-dimensional cosine function, known as cosine wave, sine wave or linear sinusoidal wave.

A -- **wave amplitude**; $H = 2A$ -- **wave height**.

λ -- **wave length**, the distance between two adjacent crests or troughs.



7.4 Airy Wave

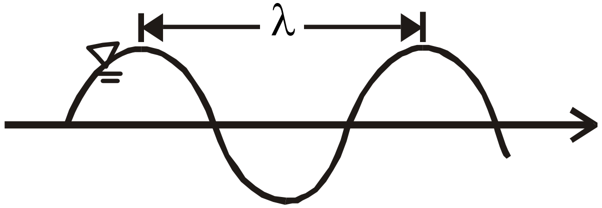
2. k : the wave number

Let $t = 0$, Airy wave becomes simply a cosine function of x.

$$\eta = A\cos(kx) \qquad kx = 2\pi n \quad (n \text{ is an integer})$$

For $n = 1$, it corresponds one wave form, that is, $x = \lambda$, therefore

$$k = \frac{2\pi}{\lambda}$$



k = wavenumber = $2\pi/\lambda$ $[L^{-1}]$

7.4 Airy Wave

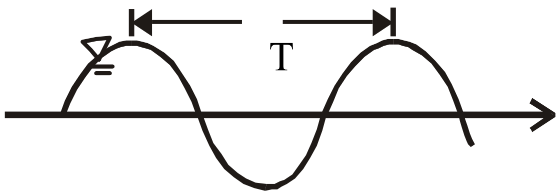
3. ω : circular frequency

Let $x = 0$, Airy wave becomes a cosine function of t.

$$\eta = A\cos(\omega t) \qquad \omega t = 2\pi n \quad (n \text{ is an integer})$$

For $n = 1$, it corresponds one period, $t = T = 1/f$, therefore

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad \text{where} \quad T = \frac{1}{f}$$



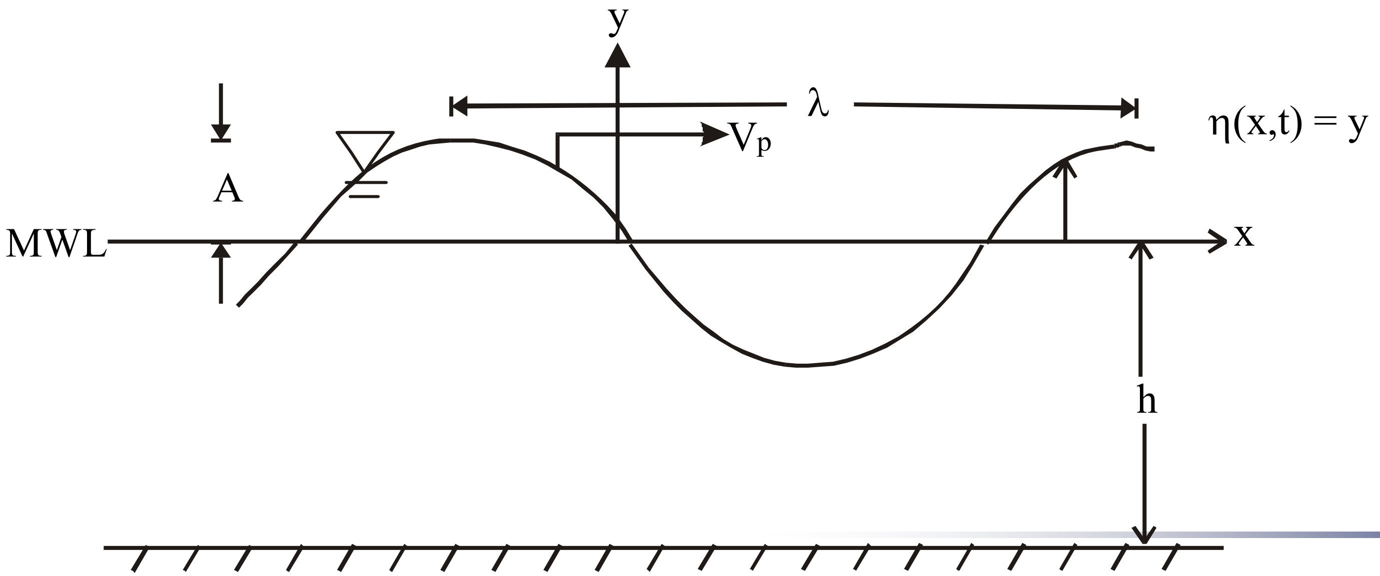
ω = frequency = $2\pi/T$ $[T^{-1}]$, e.g. rad/sec

7.4 Airy Wave

4. Phase velocity / Celerity (c or V_p): wave form moving velocity

Let's look at a fixed position in space where at an instant a crest located at. Then the crest moves forward. The duration until another crest arrives at that point is just one wave period. During this period, we can see the wave form moves forward just one wave length. So, velocity the wave form advances is

$$c = V_p = \frac{\lambda}{T}$$



7.4 Airy Wave

Airy wave is also known as sinusoidal wave. It is expressed as

$$\eta = A\cos(kx-\omega t) = A\cos\left[k\left(x-\frac{\omega}{k}t\right)\right] = A\cos\left[k(x-ct)\right]$$

where $$\frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T} = c$$

Phase velocity, wave length, wave period, wave frequency, circular frequency and wave number related with each other as follows:

$$c = \frac{\lambda}{T} = \frac{\omega}{k}$$

7.4 Airy Wave

5. Dispersion relation: relation between circular frequency and wave number.

Velocity potential
$$\phi = \frac{gA}{\omega}\sin(kx-\omega t)\frac{\cosh k(y+h)}{\cosh kh}$$

Free surface condition
$$g\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial t^2}=0 \qquad (\text{on } y=0)$$

Now, substituting the velocity potential in the linearized free surface condition, it yields

$$-\omega^2\cosh kh + gk\sinh kh = 0$$

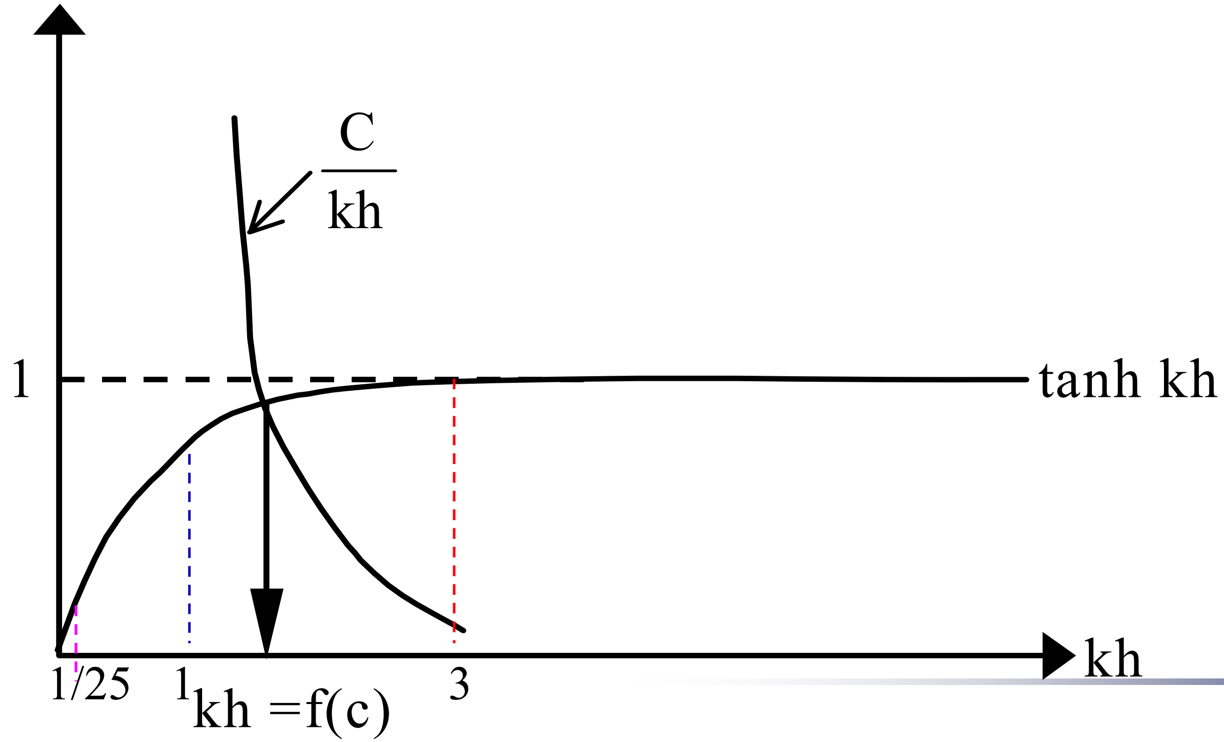
$$\omega^2 = gk\frac{\sinh kh}{\cosh kh} = gk\tanh kh$$

That is the **dispersion relation.**

7.4 Airy Wave

If we want to calculate the wave number from a given water depth and a given circular frequency, it is difficult to derive an explicit expression. Instead, we can write the ***dispersion relation*** in a form below and try to find the intersection of two functions (curves), the left hand side function and the right side function of the equation.

$$C=\frac{\omega^2 h}{g}=(kh)\tanh(kh) \quad \Rightarrow \quad \frac{C}{kh}=\tanh(kh)$$



7.4 Airy Wave

Some characteristics of the related hyperbolic functions

f	deep water $kh > 3$	shallow water $kh \ll 1$
$f_0 = \tanh(kh)$	1	kh
$f_1(y) = \dfrac{\cosh k(y+h)}{\cosh kh}$	e^{ky}	1
$f_2(y) = \dfrac{\cosh k(y+h)}{\sinh kh}$	e^{ky}	$\dfrac{1}{kh}$
$f_3(y) = \dfrac{\sinh k(y+h)}{\sinh kh}$	e^{ky}	$1 + \dfrac{y}{h}$

7.4 Airy Wave

Equating function $f_1(kh)$ and function $f_2(kh)$, we can get solution, kh, of the ***dispersion relation***. Geometrically, the solution corresponds the intersection of the two curves of $f_1(kh)$ and $f_2(kh)$.

$$f_1(kh) = \frac{C}{kh}$$

$$f_2(kh) = \tanh(kh)$$

where $C = \dfrac{\omega^2 h}{g}$

$$\tanh kh = \frac{\sinh kh}{\cosh kh} = \frac{1-e^{-2kh}}{1+e^{-2kh}} \cong \begin{cases} kh & \text{if } kh \ll 1, \text{ or, } h \ll \lambda \\ & \text{(shallow water)} \\ 1 & \text{if } kh > 3, \text{ or, } h > \dfrac{\lambda}{2} \\ & \text{(deep water)} \\ & (\tanh 3 = 0.995) \end{cases}$$

7.4 Airy Wave

For a given water depth h, wave ***dispersion relation*** gives a correspondence between the wave circular frequency ω and the wave number $\boldsymbol{k}$.

circular frequency ($\boldsymbol{\omega}$) ⟷ wave number (k)

Since $c = \dfrac{\omega}{k}$ and $k = \dfrac{2\pi}{\lambda}$, the ***dispersion relation*** can be written as

$$c^2 = \frac{\omega^2}{k^2} = \frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)$$

Therefore, the dispersion relation also give the correspondence between phase velocity (c) and wave length (λ), provided water depth h is given.

phase velocity ($\boldsymbol{c}$) ⟷ wave length ($\boldsymbol{\lambda}$)

7.4 Airy Wave

Deep water waves:

If $kh > 3$, or, $\frac{2\pi}{\lambda} h > 3$, approximately $h > \frac{\lambda}{2}$, then

$$\tanh(kh) \to 1$$

it results the **deep water dispersion relation**

$$\omega^2 = gk \Rightarrow T^2 = \frac{2\pi\lambda}{g} \Rightarrow c_d^2 = \frac{g}{k} = \frac{g\lambda}{2\pi}$$

Therefore, $k\uparrow \Rightarrow \omega\uparrow;\quad \lambda\uparrow \Rightarrow T\uparrow;\quad \lambda\uparrow \Rightarrow c\uparrow.$

Since $\frac{h}{\lambda} > \frac{1}{2}$, deep water waves are also called short waves.

7.4 Airy Wave

Shallow water waves:

If $kh \ll 1$, or, $\frac{2\pi}{\lambda}h \ll 1$, that is, $h \ll \lambda$, generally, if $h < \lambda/25$, water depth can be considered small enough. Since

$$\tanh(kh) \approx kh$$

it results ***shallow water dispersion relation***.

$$\omega^2 = ghk^2 \Rightarrow T^2 = \frac{\lambda^2}{gh} \Rightarrow c_s^2 = gh$$

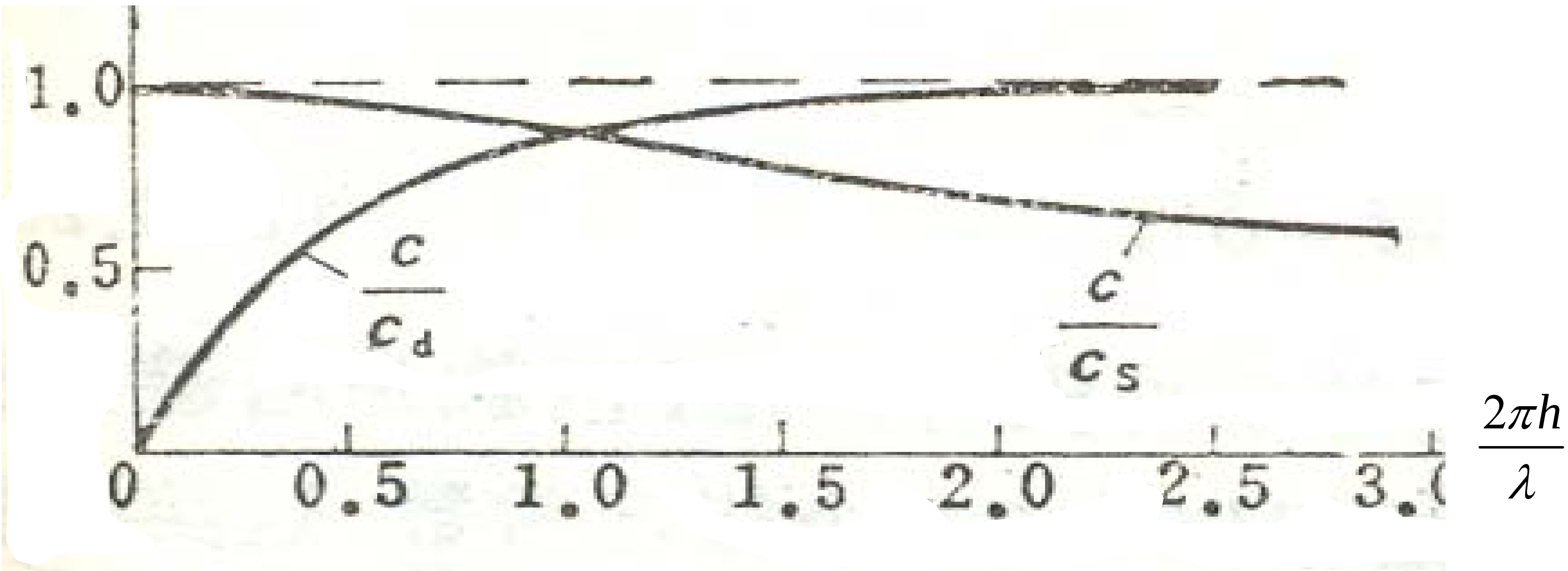
Therefore, $k\uparrow \Rightarrow \omega\uparrow, \lambda\uparrow \Rightarrow T\uparrow$, and c does not relate to wave length any more.

For $\frac{h}{\lambda} < \frac{1}{25}$, shallow water waves are also known as long waves.

7.4 Airy Wave

Generally for water depth in between ($\frac{1}{2} > \frac{h}{\lambda} > \frac{1}{25}$ or $\pi > \frac{2\pi h}{\lambda} > 0.08\pi$):

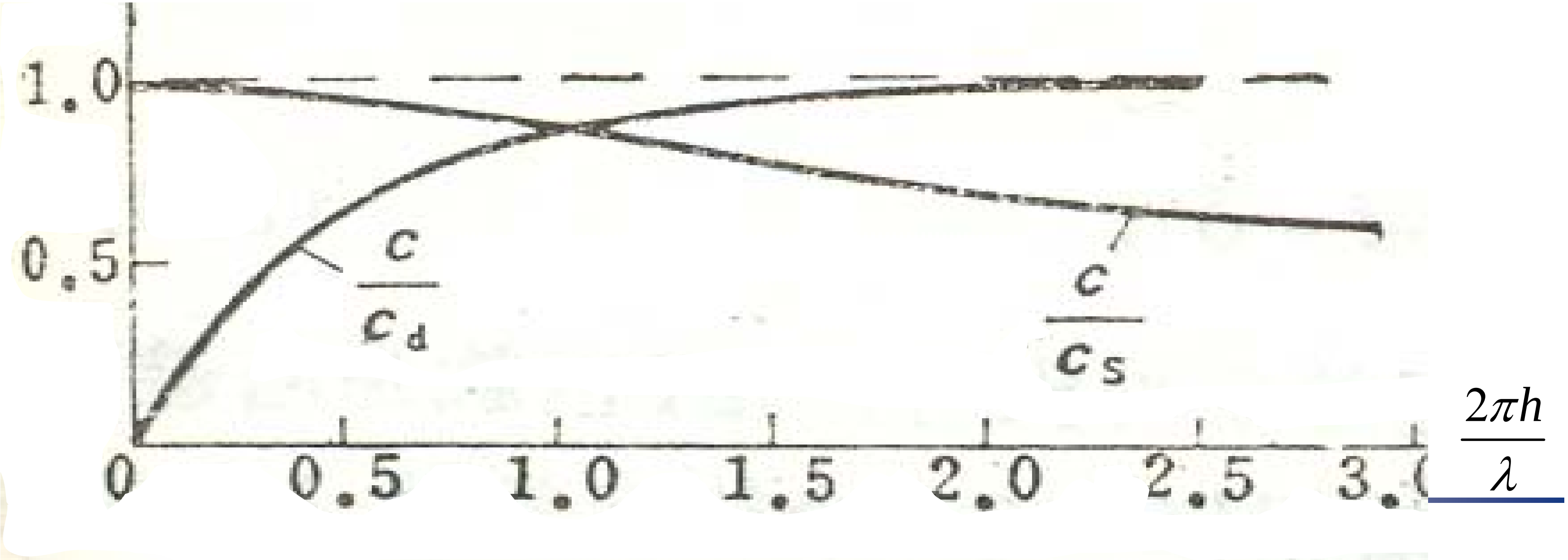
$$\frac{c}{c_d} = \sqrt{\tanh kh} = \sqrt{\tanh\left(\frac{2\pi h}{\lambda}\right)}, \qquad \frac{c}{c_s} = \sqrt{\frac{\tanh kh}{kh}} = \sqrt{\frac{\lambda}{2\pi h}\tanh\left(\frac{2\pi h}{\lambda}\right)}$$



7.4 Airy Wave

For a fixed water depth, since $c_s = \sqrt{gh}$ is a constant, **phase velocity** c will decrease with **wave length** λ . That is, *longer waves travel faster, and shorter waves travel slower.*

For waves with fixed wave length, since $c_d = \sqrt{\dfrac{g\lambda}{2\pi}}$ is a constant, phase velocity c will increase with the water depth h. That is, *deep water waves travel faster, and shallow water waves travel slower.*



6. Velocity field

From velocity potential of the wave flows and ***dispersion relation***, velocity field of the wave flows can be readily derived.

$$\phi = \frac{gA}{\omega}\sin(kx-\omega t)\frac{\cosh k(y+h)}{\cosh kh}, \qquad \omega^2 = gk\tanh kh$$

$$u = \frac{\partial\phi}{\partial x} = \frac{Agk}{\omega}\cos(kx-\omega t)\frac{\cosh k(y+h)}{\cosh kh}$$

$$= A\omega\frac{\cosh k(y+h)}{\sinh kh}\cos(kx-\omega t)$$

$$v = \frac{\partial\phi}{\partial y} = \frac{Agk}{\omega}\sin(kx-\omega t)\frac{\sinh k(y+h)}{\cosh kh}$$

$$= A\omega\frac{\sinh k(y+h)}{\sinh kh}\sin(kx-\omega t)$$

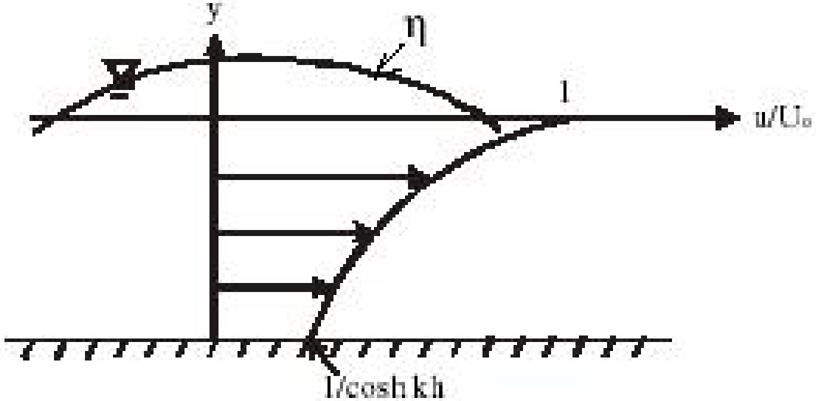
7.4 Airy Wave

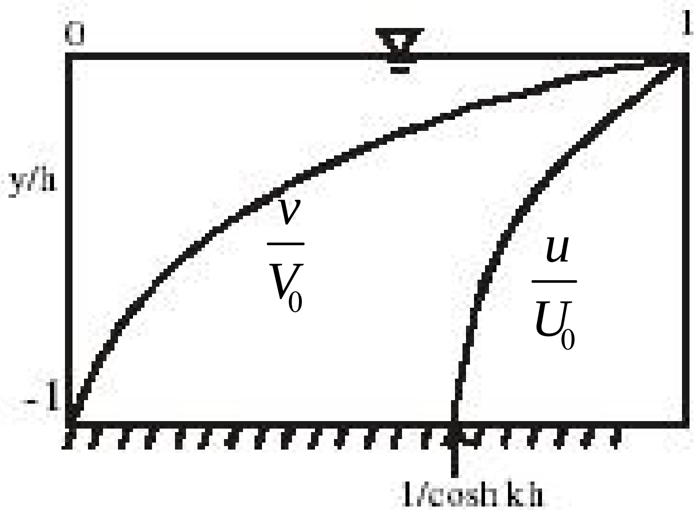
On $y = 0$:

$$U_0 = A\omega \frac{1}{\tanh kh} \cos(kx - \omega t), \qquad V_0 = A\omega \sin(kx - \omega t) = \frac{\partial \eta}{\partial t}$$

then, velocity can be written relative to that on y = 0, it gives

$$\frac{u}{U_0} = \frac{\cosh k(y+h)}{\cosh kh}, \qquad \frac{v}{V_0} = \frac{\sinh k(y+h)}{\sinh kh}$$

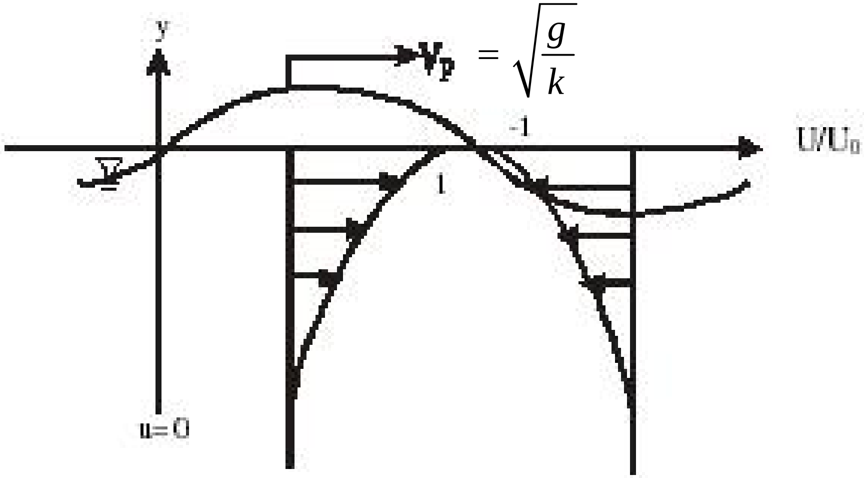


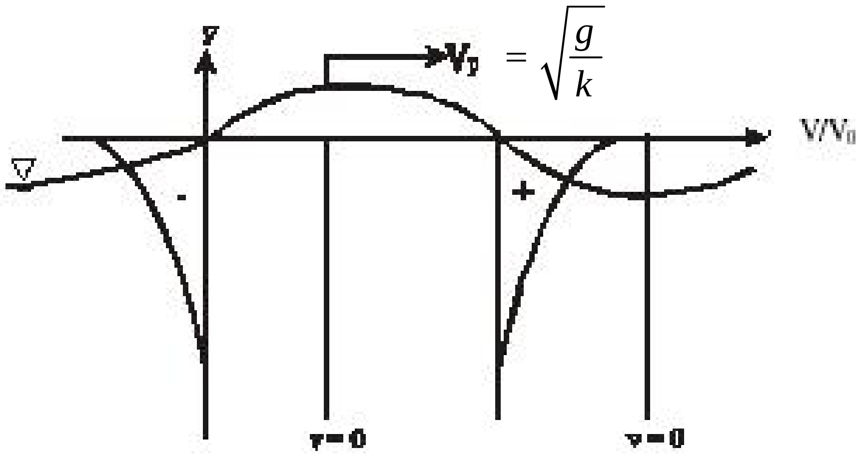


7.4 Airy Wave

For ***deep water waves***, $kh > 3$, we have

$$\frac{u}{U_0} = \frac{\cosh k(y+h)}{\cosh kh} \approx e^{ky}, \qquad \frac{v}{V_0} = \frac{\sinh k(y+h)}{\sinh kh} \approx e^{ky}$$



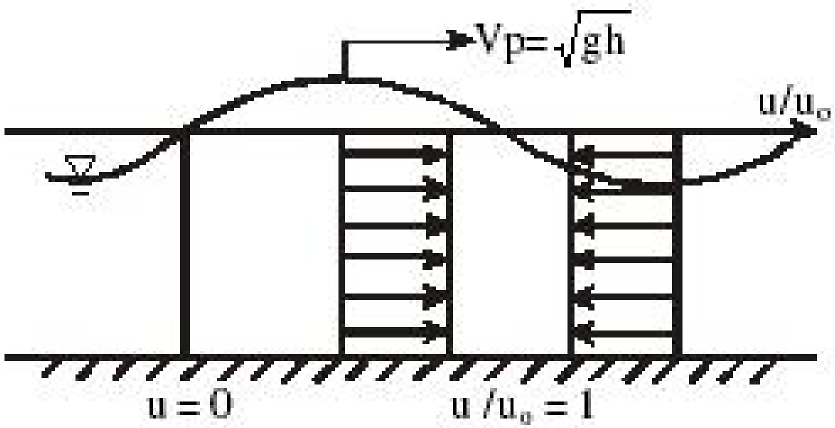


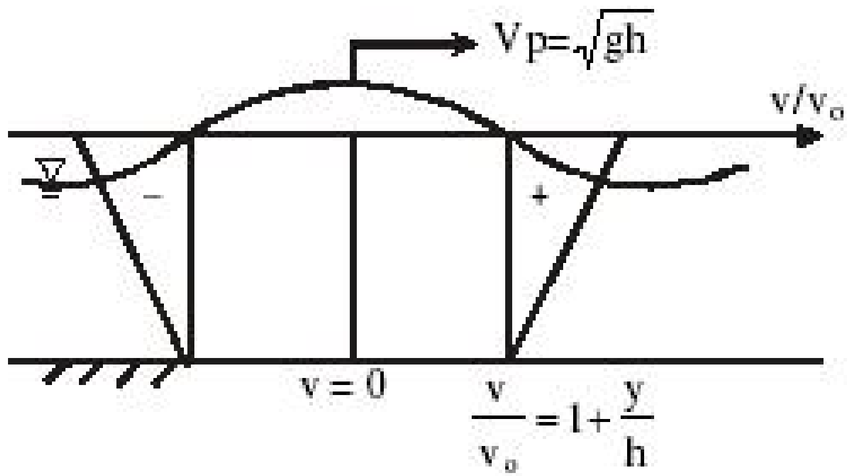
7.4 Airy Wave

For ***shallow water waves***, $kh \ll 1$, we have

$$\frac{u}{U_0} = \frac{\cosh k(y+h)}{\cosh kh} \approx 1, \quad \frac{v}{V_0} = \frac{\sinh k(y+h)}{\sinh kh} \approx 1 + \frac{y}{h}$$

$$u = \frac{A\omega}{kh}\cos(kx-\omega t) = \eta\sqrt{\frac{g}{h}}, \qquad v = A\omega\left(1+\frac{y}{h}\right)\sin(kx-\omega t)$$





7.4 Airy Wave

Characteristics of Airy waves

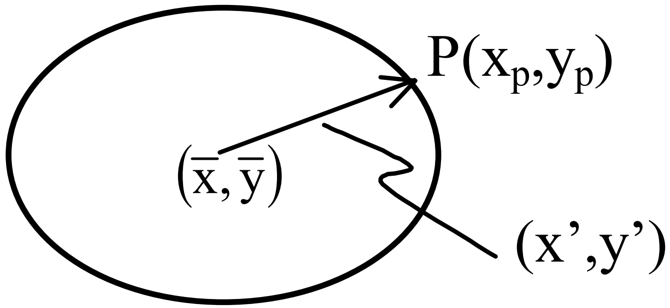
	deep water $kh > 3$	shallow water $kh \ll 1$
dispersion relation	$\omega^2 = gk$ $c_d^2 = \frac{g}{k} = \frac{g\lambda}{2\pi}$	$\omega^2 = ghk^2$ $c_s^2 = gh$
velocity	$\frac{u}{U_0} \approx e^{ky}$ $\frac{v}{V_0} \approx e^{ky}$	$\frac{u}{U_0} \approx 1$ $\frac{v}{V_0} \approx 1 + \frac{y}{h}$

7. Particle orbit

At time t, fluid particle P takes the position $(x_P(t), y_P(t))$, and if we denote its mean position is $\left(\overline{x}_P, \overline{y}_P\right)$ and

$$x_P(t) = \overline{x}_P + x'_P(t)$$

$$y_P(t) = \overline{y}_P + y'_P(t)$$



Since velocity is a time derivative of position, that is,

$$u_P = \frac{dx_P}{dt} = u\left(\overline{x}, \overline{y}, t\right) + \frac{\partial u\left(\overline{x}, \overline{y}, t\right)}{\partial x} x' + \frac{\partial u\left(\overline{x}, \overline{y}, t\right)}{\partial y} y' + \cdots$$

remaining the main first term and integrating the equation, it gives

$$x_P = \overline{x} + \int u\left(\overline{x}, \overline{y}, t\right) dt = \overline{x} + \int A\omega \frac{\cosh k\left(\overline{y} + h\right)}{\sinh kh} \cos\left(k\overline{x} - \omega t\right) dt$$

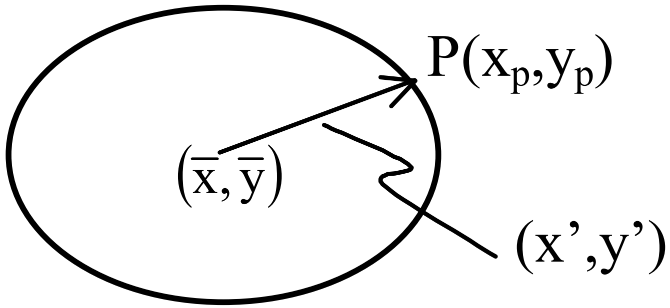
$$= \overline{x} - A \frac{\cosh k\left(\overline{y} + h\right)}{\sinh kh} \sin\left(k\overline{x} - \omega t\right)$$

7. Particle orbit

At time t, fluid particle P takes the position $(x_P(t), y_P(t))$, and if we denote its mean position $\left(\overline{x}_P, \overline{y}_P\right)$ and

$$x_P(t) = \overline{x}_P + x'_P(t)$$

$$y_P(t) = \overline{y}_P + y'_P(t)$$



Since velocity is a time derivative of position, that is,

$$u_P = \frac{dx_P}{dt} = u\left(\overline{x}, \overline{y}, t\right) + \frac{\partial u\left(\overline{x}, \overline{y}, t\right)}{\partial x} x' + \frac{\partial u\left(\overline{x}, \overline{y}, t\right)}{\partial y} y' + \cdots$$

remaining the main first term and integrating the equation, it gives

$$x_P = \overline{x} + \int u\left(\overline{x}, \overline{y}, t\right) dt = \overline{x} + \int A\omega \frac{\cosh k\left(\overline{y} + h\right)}{\sinh kh} \cos\left(k\overline{x} - \omega t\right) dt$$

$$= \overline{x} - A\frac{\cosh k\left(\overline{y} + h\right)}{\sinh kh} \sin\left(k\overline{x} - \omega t\right)$$

7.4 Airy Wave

Similarly,

$$v_P = \frac{dy_P}{dt} = v(\overline{x}, \overline{y}, t) + \frac{\partial v(\overline{x}, \overline{y}, t)}{\partial x} x' + \frac{\partial v(\overline{x}, \overline{y}, t)}{\partial y} y' + \cdots$$

keeping the first term only and integrating the equation, it results

$$y_P = \overline{y} + \int v(\overline{x}, \overline{y}, t) dt = \overline{y} + \int A\omega \frac{\sinh k(\overline{y} + h)}{\sinh kh} \sin(k\overline{x} - \omega t) dt$$

$$= \overline{y} + A \frac{\sinh k(\overline{y} + h)}{\sinh kh} \cos(k\overline{x} - \omega t)$$

Specifically, on the mean free surface, $\overline{y} = 0$, the unknown wave elevation is obtained.

$$y_P = A\cos(k\overline{x} - \omega t) = \eta$$

7.4 Airy Wave

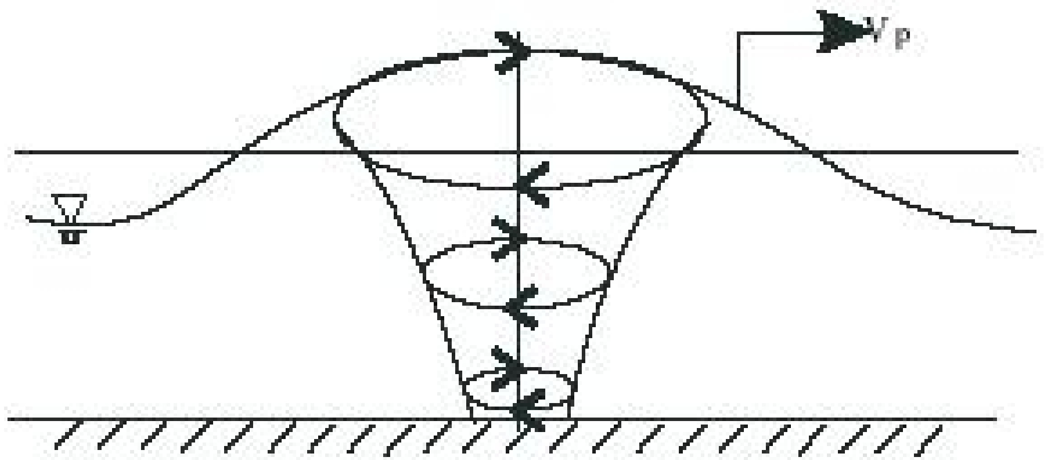
Then, the ***particle orbit*** of particle P is obtained.

$$\frac{\left(x_P-\overline{x}\right)^2}{a^2}+\frac{\left(y_P-\overline{y}\right)^2}{b^2}=1$$

where

$$a=A\frac{\cosh k\left(\overline{y}+h\right)}{\sinh kh},\quad b=A\frac{\sinh k\left(\overline{y}+h\right)}{\sinh kh}$$

Particle orbits are ***ellipses.*** Both major axis and minor axis decrease with the increase of the depth below the free surface.

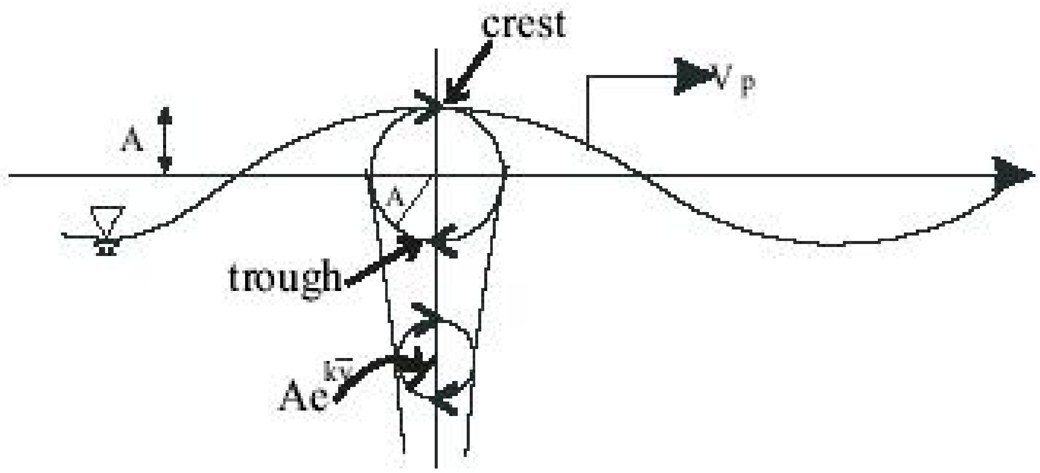


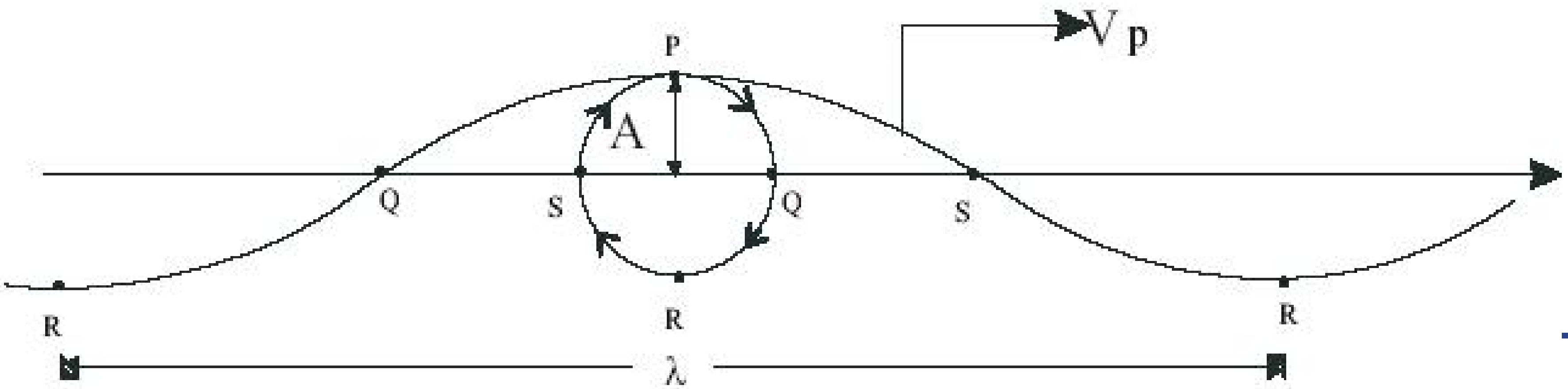
7.4 Airy Wave

For ***deep water wave***, $kh \gg 1$, major and minor axes become equal

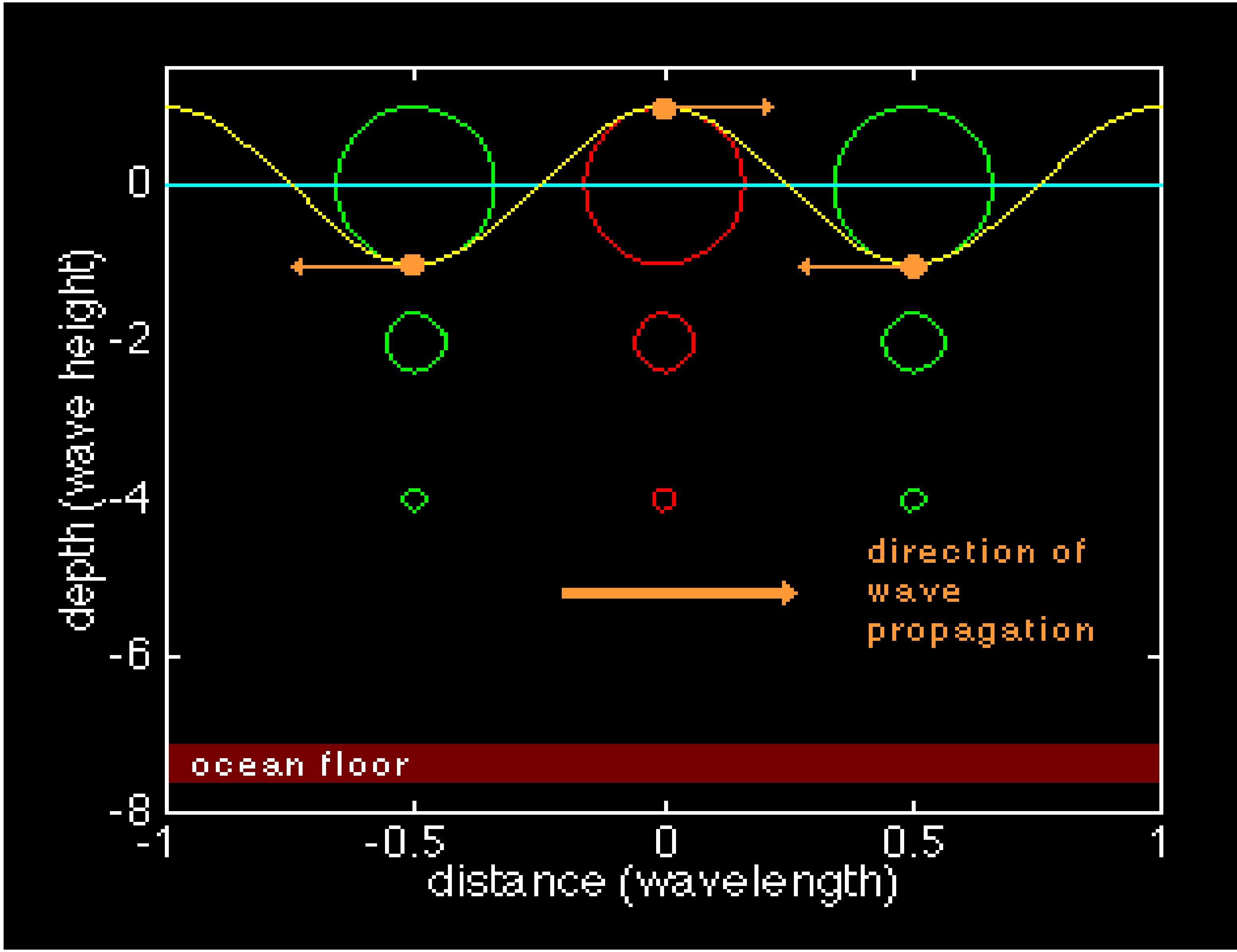
$$a = b = Ae^{k\bar{y}}$$

Accordingly *particle orbits* become *circles*. The radius is getting smaller with the increase of depth from the free surface. On the free surface, the radius is just equal to the wave amplitude.





7.4 Airy Wave

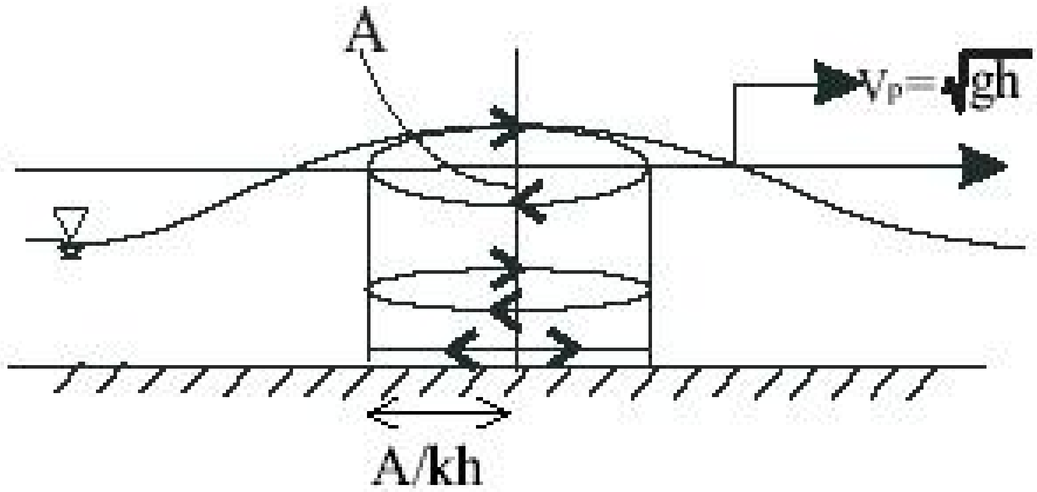

0
-2
-4
-6
-8
depth (wave height)
-1
-0.5
0
0.5
1
distance (wavelength)
direction of
wave
propagation
ocean floor

7.4 Airy Wave

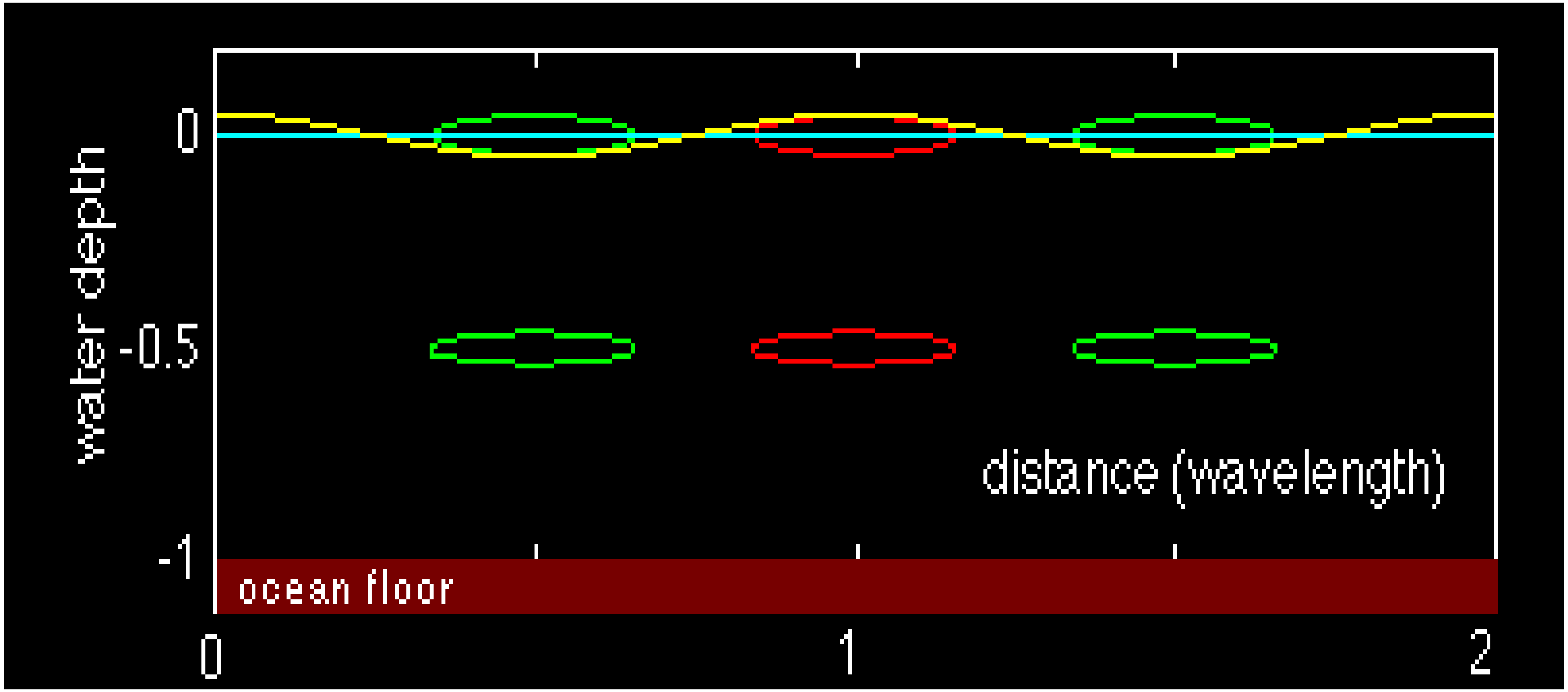
For ***shallow water waves***, $kh \ll 1$, the major and minor axes are

$$a = \frac{A}{kh} = \text{const}, \qquad b = A\left(1 + \frac{y}{h}\right)$$

The ***particle orbits*** are *ellipses*. Major axis is horizontal and keeps constant, while the minor axis is vertical and decreases linearly with the depth from the free surface: on free surface, $y = 0$, it equals the wave amplitude, and on sea bottom, $y = -h$, it reduces to zero.



7.4 Airy Wave



8. Pressure field

According to the linearized dynamic condition (*Bernoulli's equation*), the pressure is evaluated as follows

$$p - p_a = -\rho \frac{\partial \phi}{\partial t} - \rho g y$$

$$= \rho g A \frac{\cosh k\left(y+h\right)}{\cosh kh} \cos\left(kx - \omega t\right) - \rho g y$$

$$= \rho g \frac{\cosh k\left(y+h\right)}{\cosh kh} \eta - \rho g y$$

7.4 Airy Wave

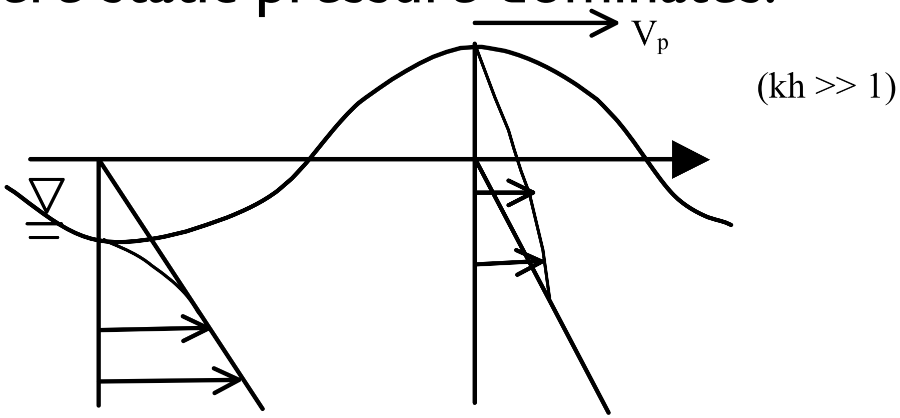
For ***deep water wave***, $kh \gg 1$, since we have approximation

$$\frac{\cosh k(y+h)}{\cosh kh} \approx e^{ky}$$

pressure is approximately written as

$$p - p_a = \rho g \left(\eta e^{ky} - y \right)$$

that is, the *dynamic pressure*, the exponential part, decays with the increase of depth from free surface. Below half wave length, it could be reasonably neglected, where static pressure dominates.



7.4 Airy Wave

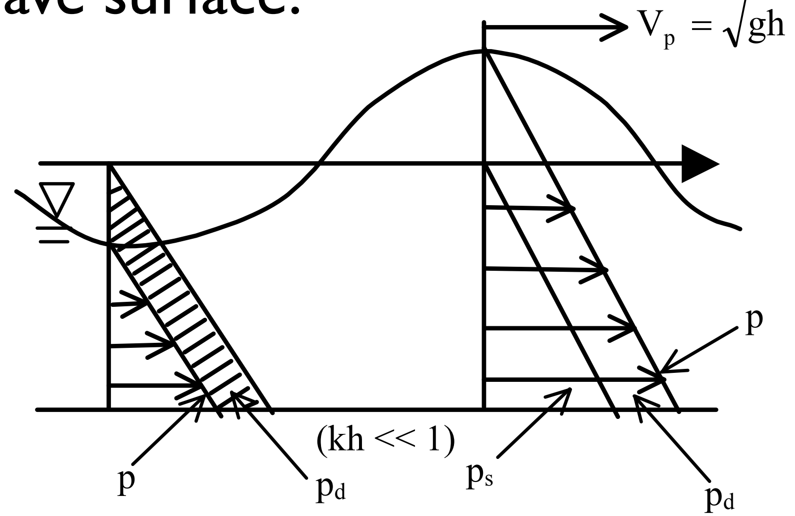
For ***shallow water wave***, $kh \ll 1$, since we have approximation on the left,

$$\frac{\cosh k\left(y+h\right)}{\cosh kh} \approx 1$$

pressure distribution is approximated as

$$p - p_a = \rho g\left(\eta - y\right)$$

that is, the pressure is very similar to static pressure distribution. The only difference is that the depth is measured from the instantaneous wave surface.



7.4 Airy Wave

Eg. 1 For an Airy wave, it is given that amplitude $A = 0.3\ m$, wave period $T = 2\ s$, calculate its circular frequency, wave number, phase velocity, wave length and the maximum wave slope.

Solution:

Circular frequency: $$\omega = \frac{2\pi}{T} = \frac{2\times 3.1415927}{2} = 3.142\ \mathrm{s}^{-1}$$

Wave number: $$\omega^2 = gk$$

$$\Rightarrow \quad k = \frac{\omega^2}{g} = \frac{(3.142)^2}{9.81} = 1.011\,\mathrm{m}^{-1}$$

Phase velocity: $$c = \frac{\omega}{k} = \frac{3.142}{1.011} = 3.1078\ \mathrm{m/s}$$

7.4 Airy Wave

Wave length: $$\lambda = \frac{2\pi}{k} = \frac{2\times 3.1415927}{1.011} = 6.2148 \text{ m}$$

Wave elevation: $$\eta = A\cos(kx - \omega t) = 0.3\cos(1.01x - 3.142t)$$

Wave slope: $$\mathrm{tg}\,\alpha = \frac{\partial \eta}{\partial x} = Ak\sin(kx - \omega t) = 0.3\times 1.01\cos(1.01x - 3.142t)$$

The maximum wave slope angle:

$$\mathrm{tg}\,\alpha_{\max} = 0.3\times 1.01 = 0.303$$

$$\Rightarrow \quad \alpha_{\max} = 16.86^{\circ}$$

7.4 Airy Wave

Eg. 2 A boat on a water wave is rolling at a rate of 30 cycles per minute. The sea is assumed deep enough. Calculate the wave length L, circular frequency ω, wave number k and the phase velocity c.

Solution: Since the boat is not traveling, its rolling is assumed to be caused by an incident water waves. Then,

wave period: $T = \frac{1}{30}\min = 2(s)$

circular frequency:

$$\omega = \frac{2\pi}{T} = 3.14(1/s)$$

wave number:

$$k = \frac{\omega^2}{g} = 1.006$$

wave length:

$$L = \frac{2\pi}{k} = 6.26(m)$$

phase velocity:

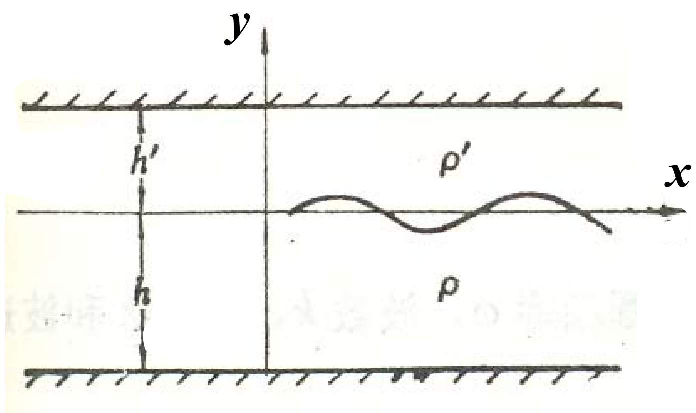
$$C = \frac{L}{T} = 3.13(m/s)$$

7.4 Airy Wave

Eg. 3 Consider two fluid layers, the upper fluid with density ρ_1, depth h_1, the lower fluid with density ρ, depth h. Two fluids are bounded by uppermost and lowest horizontal rigid walls. Determine phase velocity for the wave at the separating surface of the two layers with wave number k.

Solution:

Take origin at the mean separating surface, x-axis horizontal and y-axis vertical, upward positive. Then velocity potentials of the wave in each layers are respectively as follows,



in the upper layer

$$\phi' = \frac{gA}{\omega}\frac{\cosh k\left(y-h'\right)}{\cosh kh'}\sin\left(kx-\omega t\right) = C\cosh k\left(y-h'\right)\sin\left(kx-\omega t\right)$$

in the lower layer

$$\phi = \frac{gA}{\omega}\frac{\cosh k\left(y+h\right)}{\cosh kh}\sin\left(kx-\omega t\right) = D\cosh k\left(y+h\right)\sin\left(kx-\omega t\right)$$

7.4 Airy Wave

On the interface of the upper and lower fluid layers, pressure and velocity should coincide with each other.

According to *Bernoulli's equation*, we have

$$\frac{p'}{\rho'} + gy + \frac{\partial \phi'}{\partial t} = 0, \quad \frac{p}{\rho} + gy + \frac{\partial \phi}{\partial t} = 0$$

On the interface, $y = \eta$, pressure coincidence requires, $p = p'$, i.e.

$$\rho' g \eta + \rho' \frac{\partial \phi'}{\partial t} = \rho g \eta + \rho \frac{\partial \phi}{\partial t} \quad (y = 0)$$

$$\eta = \frac{1}{g(\rho - \rho')}\left(\rho' \frac{\partial \phi'}{\partial t} - \rho \frac{\partial \phi}{\partial t}\right)\bigg|_{y=0}$$

7.4 Airy Wave

Next, velocity coincidence. On the interface, y-components of the upper and lower velocities should be equal, that is

$$\frac{\partial \phi'}{\partial y} = \frac{\partial \phi}{\partial y} \quad (y = 0)$$

$$kC \sinh k(-h')\sin(kx - \omega t) = kD \sinh kh \sin(kx - \omega t)$$

Thus, $$C \sinh kh' = -D \sinh kh$$

From the *linear kinematic condition* of Airy wave, we have

$$\frac{\partial \phi'}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad (y = 0)$$

7.4 Airy Wave

Therefore,

$$\frac{\partial \phi}{\partial y} = \frac{1}{g(\rho - \rho')}\left(\rho' \frac{\partial^2 \phi'}{\partial t^2} - \rho \frac{\partial^2 \phi}{\partial t^2}\right) \quad (y = 0)$$

$$kD \sinh kh \sin(kx - \omega t)$$

$$= \frac{\left[-\rho'\omega^2 C \cosh k(-h') + \rho\omega^2 D \cosh kh\right]}{g(\rho - \rho')} \sin(kx - \omega t)$$

$$\Rightarrow \quad \omega^2 = \frac{kDg(\rho - \rho')\sinh kh}{\rho D \cosh kh - \rho' C \cosh kh'}$$

7.4 Airy Wave

From above expression, finally we can get ***phase velocity*** formula.

$$c=\frac{\omega}{k}=\sqrt{\frac{\omega^2}{k^2}}$$

$$=\sqrt{\frac{Dg\left(\rho-\rho'\right)\sinh kh}{k\left(\rho D\cosh kh-\rho' C\cosh kh'\right)}}$$

$$=\sqrt{\frac{g\left(\rho-\rho'\right)}{k\left(\rho\frac{\cosh kh}{\sinh kh}-\rho'\frac{C\cosh kh'}{D\sinh kh}\right)}}$$

$$=\sqrt{\frac{g\left(\rho-\rho'\right)}{k\left(\rho/\tanh kh+\rho'/\tanh kh'\right)}}$$

End of Eg. 3.