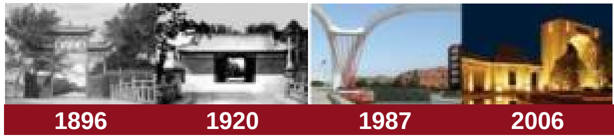


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Chapter 7
Water Waves

- Fluid mechanics
  - Ideal fluid mechanics
    - Rotational flows
    - Potential flows
    - Wave motion
  - Viscous fluid mechanics
    - Simple flows
    - Boundary layer flows
    - Laminar and turbulent flows

Water Waves

Free Surface Waves

Surface Gravity Waves

7.1 Water Wave Phenomenon

Main factors in water wave phenomenon:

Water waves

Medium → Water

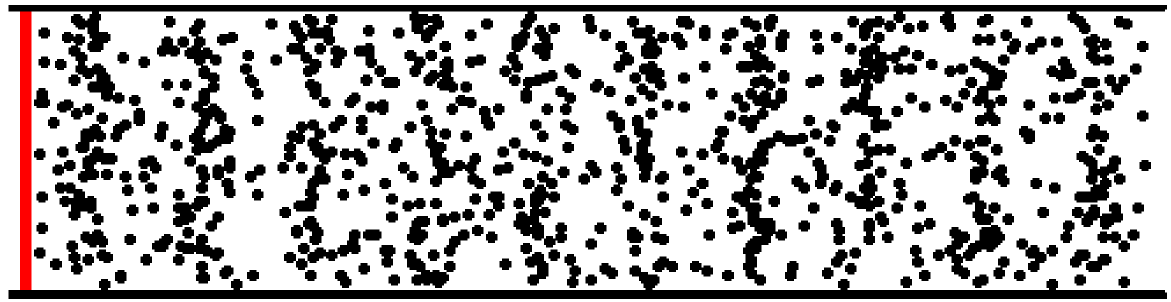
Disturbance → wind, tidal, earthquake, obstacles and moving bodies

Restoring Forces → Gravity, Surface Tension

7.1 Water Wave Phenomenon

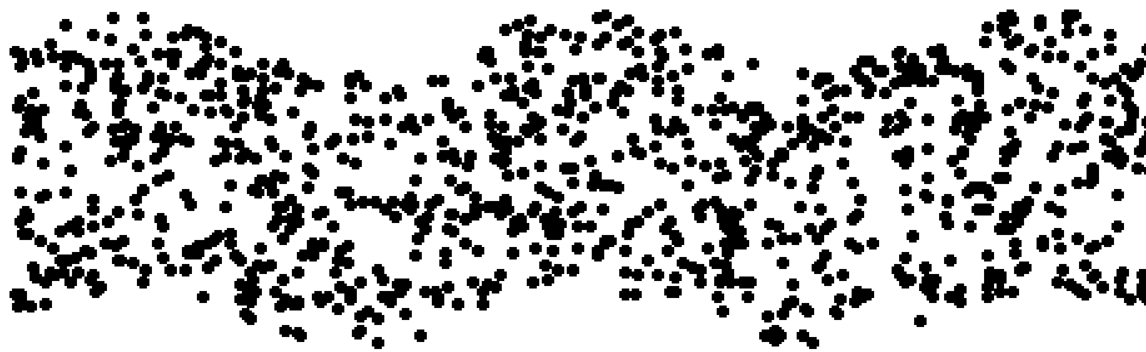
Longitudinal Waves

propagating Waves

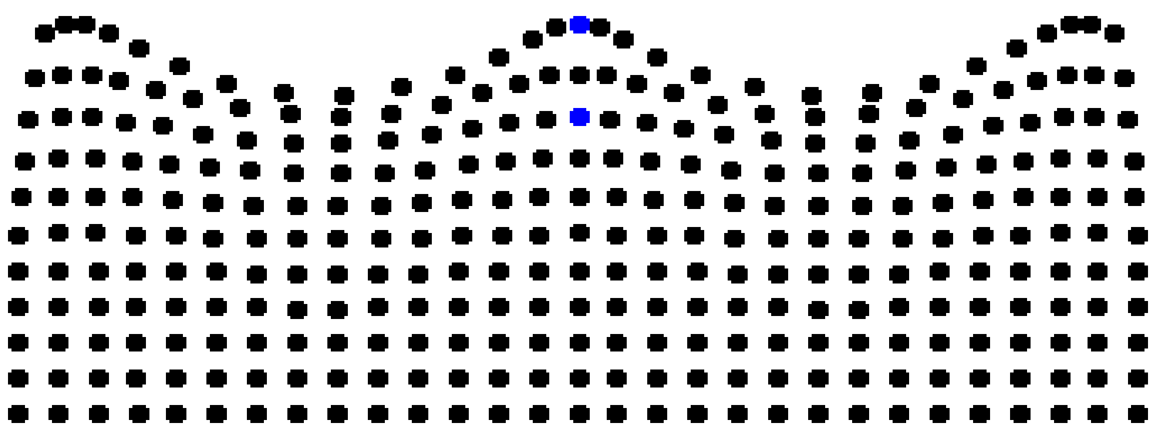


Transverse Waves

Standing Waves

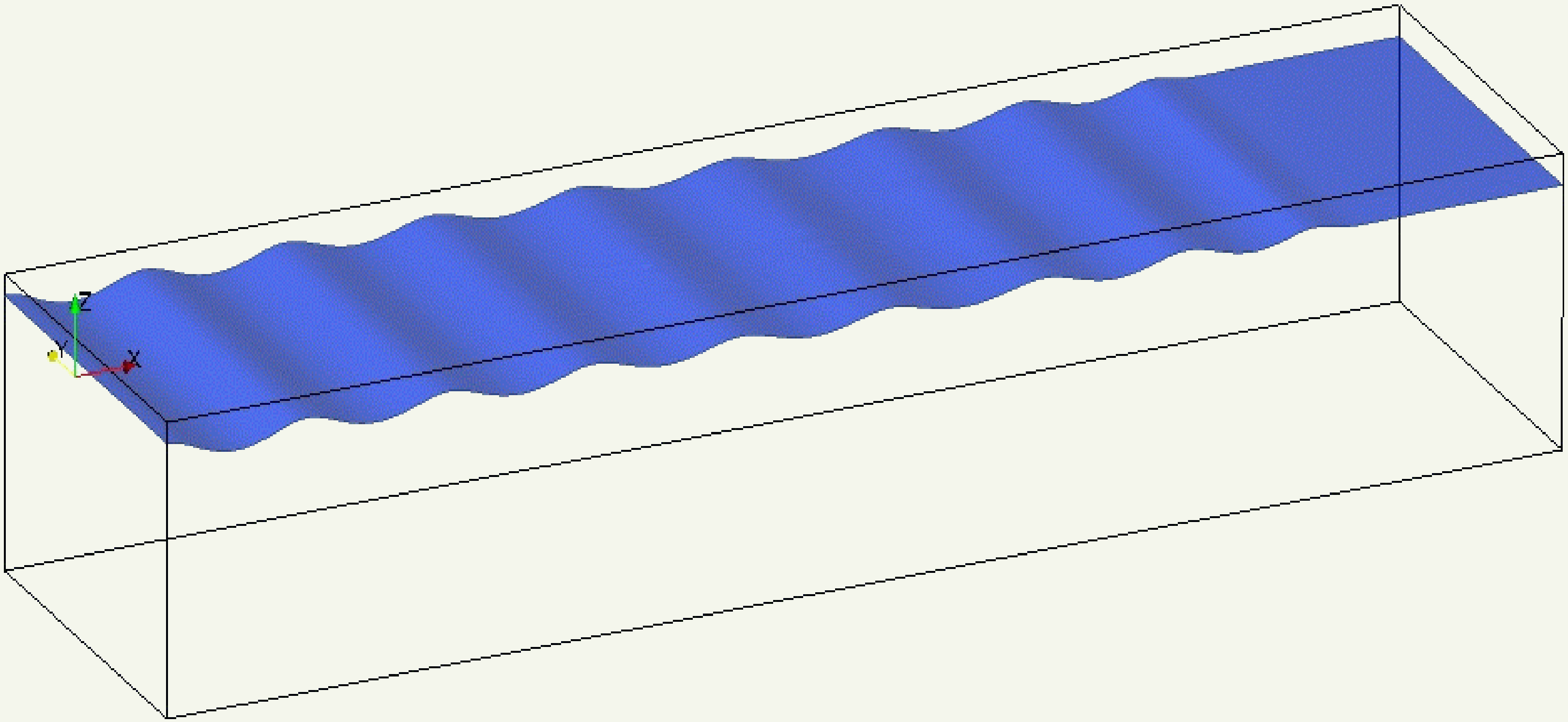


Water Waves

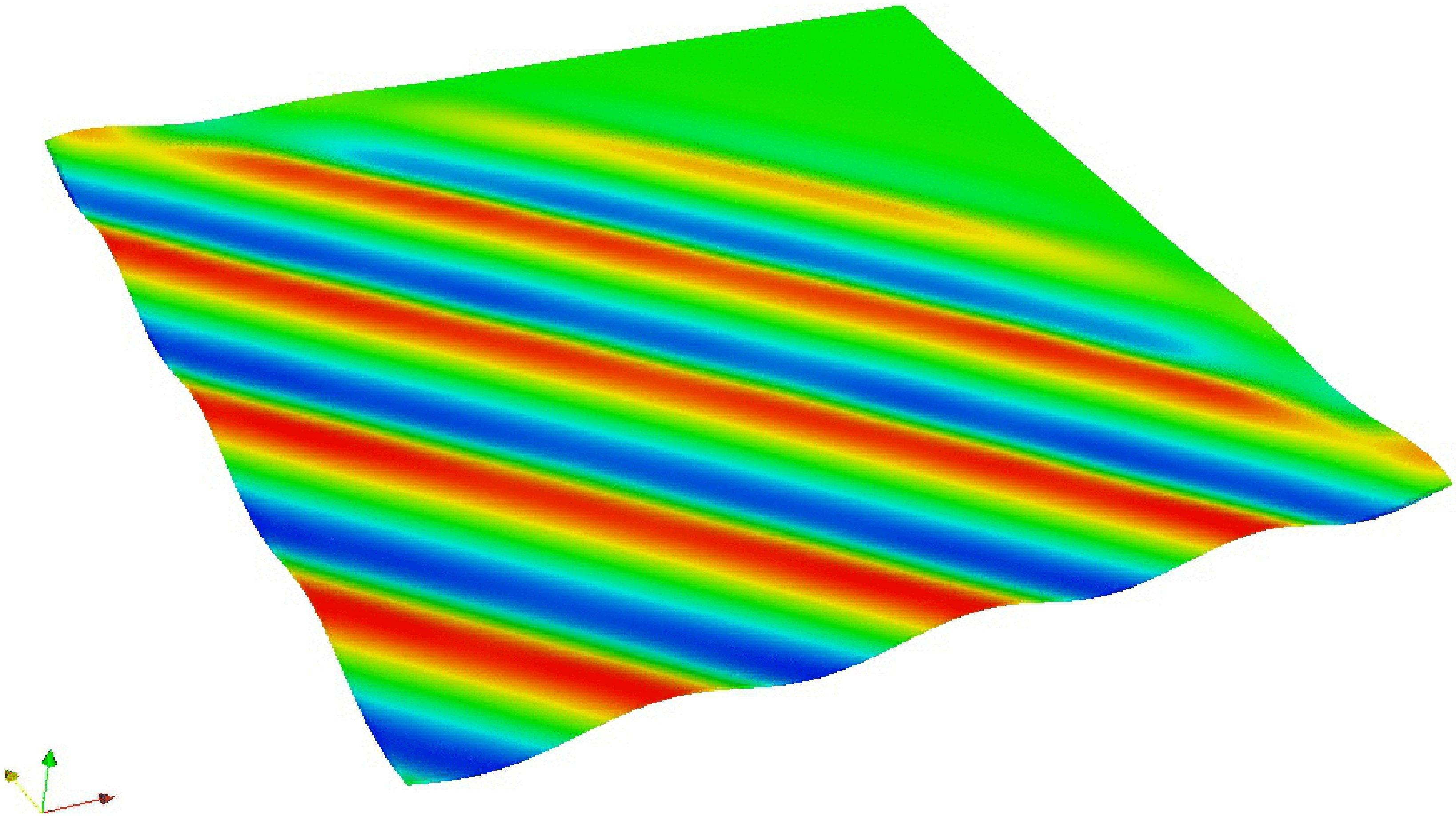


7.1 Water Wave Phenomenon

Periodic waves in time and space



Directional Waves



7.1 Water Wave Phenomenon

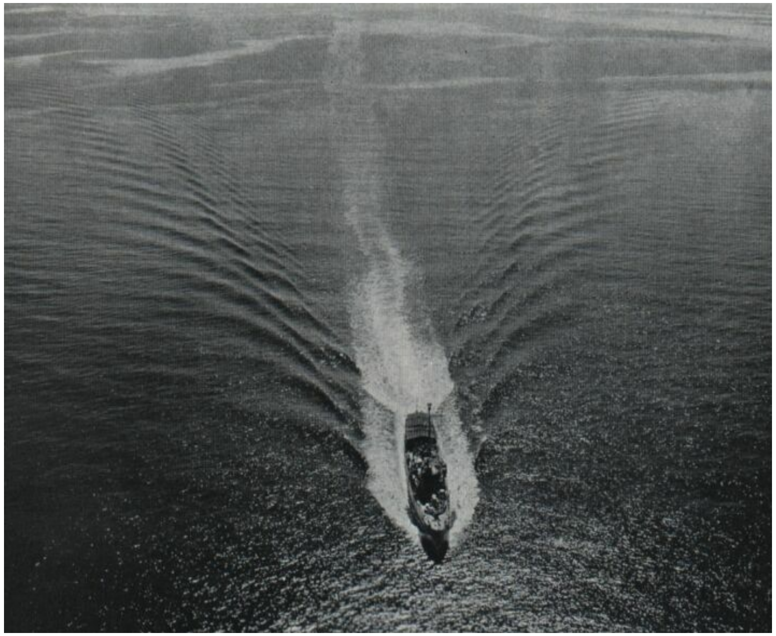
Water waves generated by a moving ship

7.1 Water Wave Phenomenon

Water waves generated by a moving ship

Shanghai Jiao Tong University

7.1 Water Wave Phenomenon



Water waves generated by a moving ship

Figure 1 Sub-critical wash

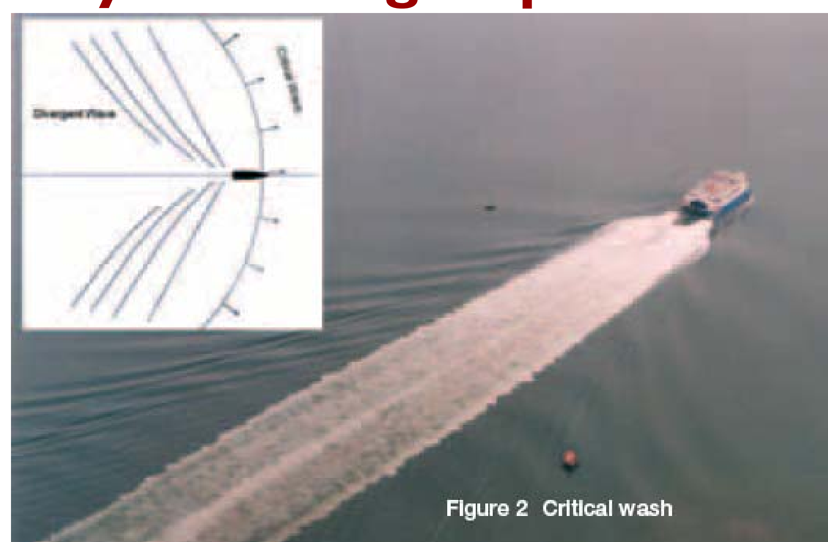
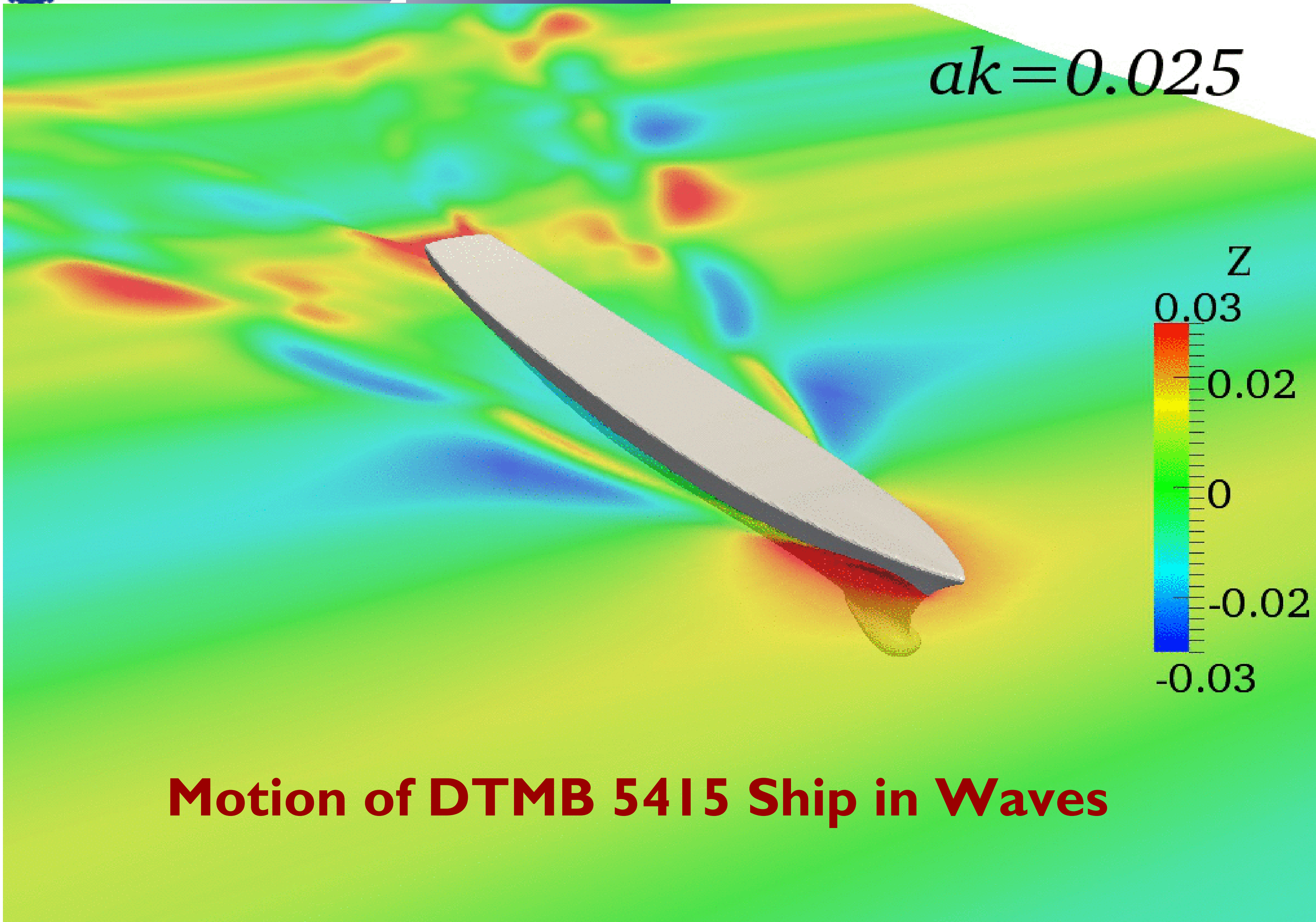
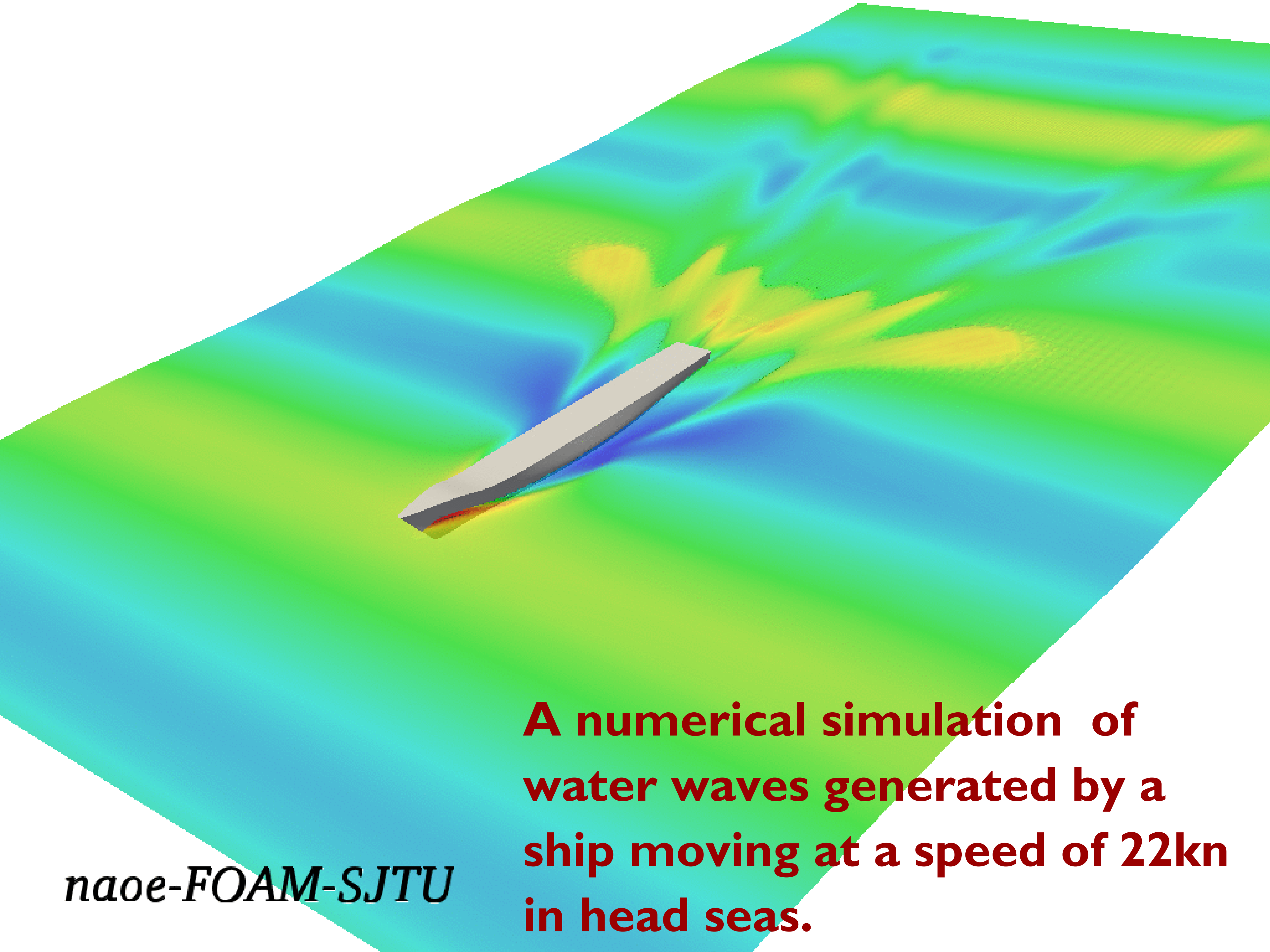


Figure 2 Critical wash

7.1 Water Wave Phenomenon

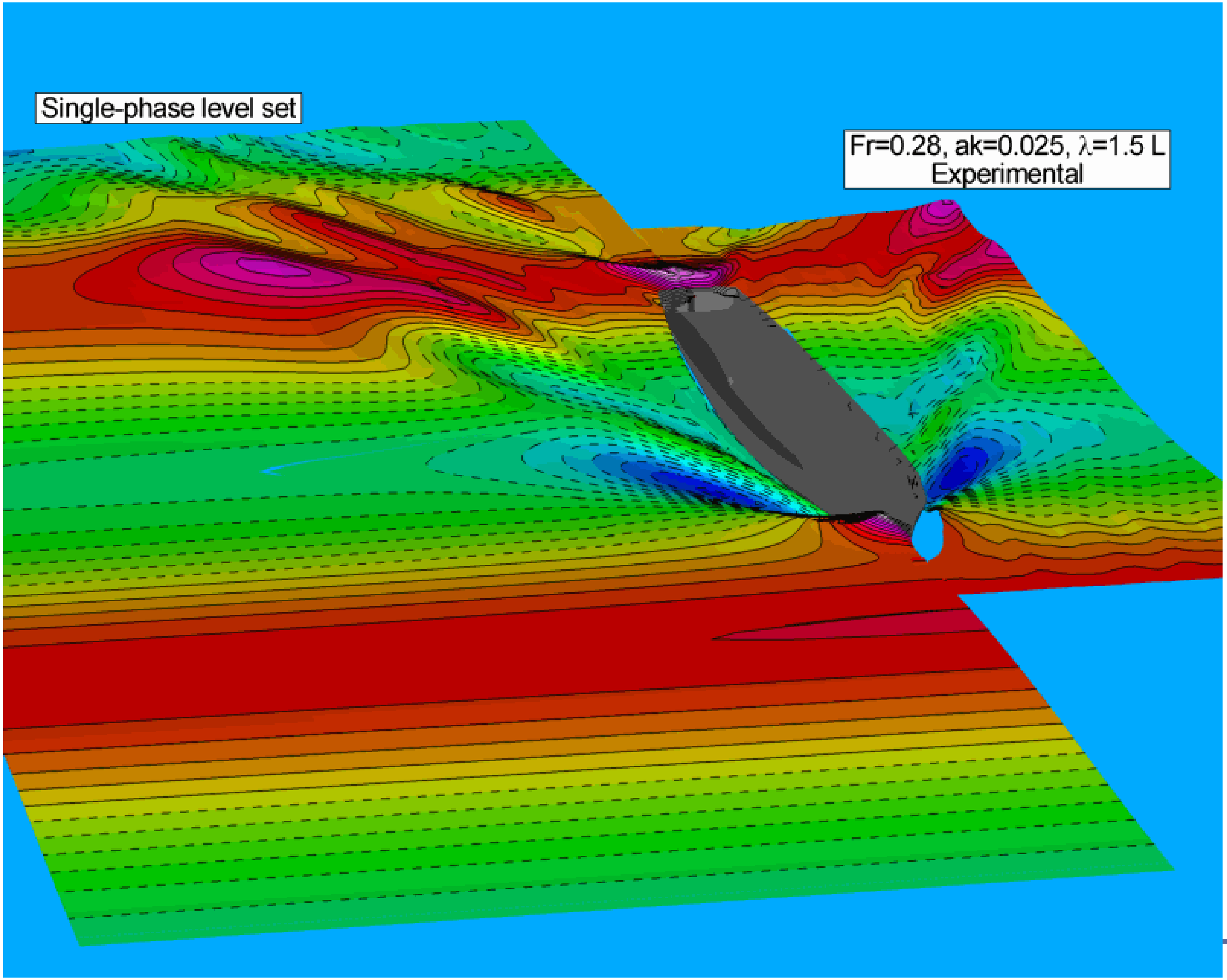


Motion of DTMB 5415 Ship in Waves

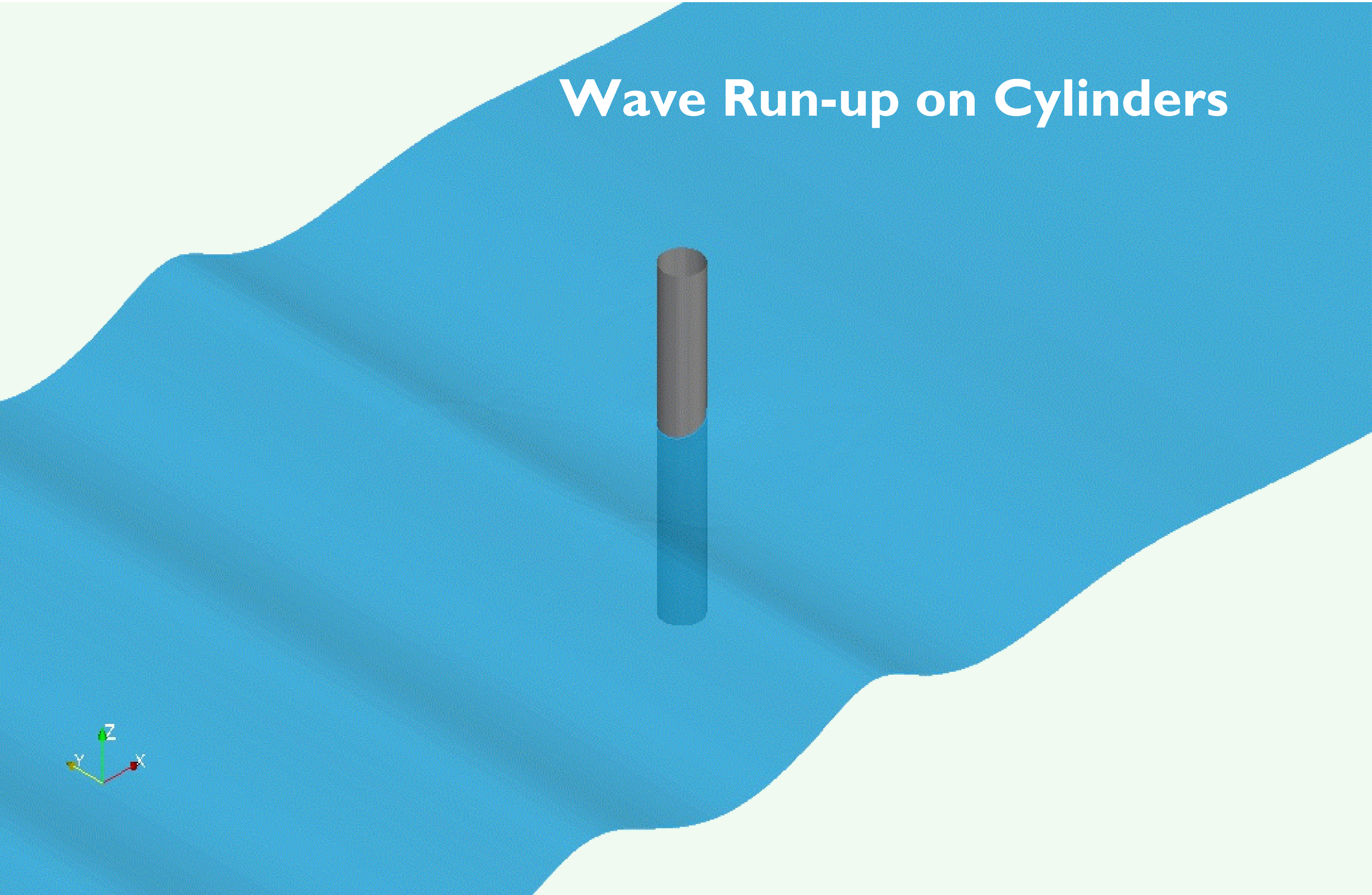


A numerical simulation of water waves generated by a ship moving at a speed of 22kn in head seas.

7.1 Water Wave Phenomenon

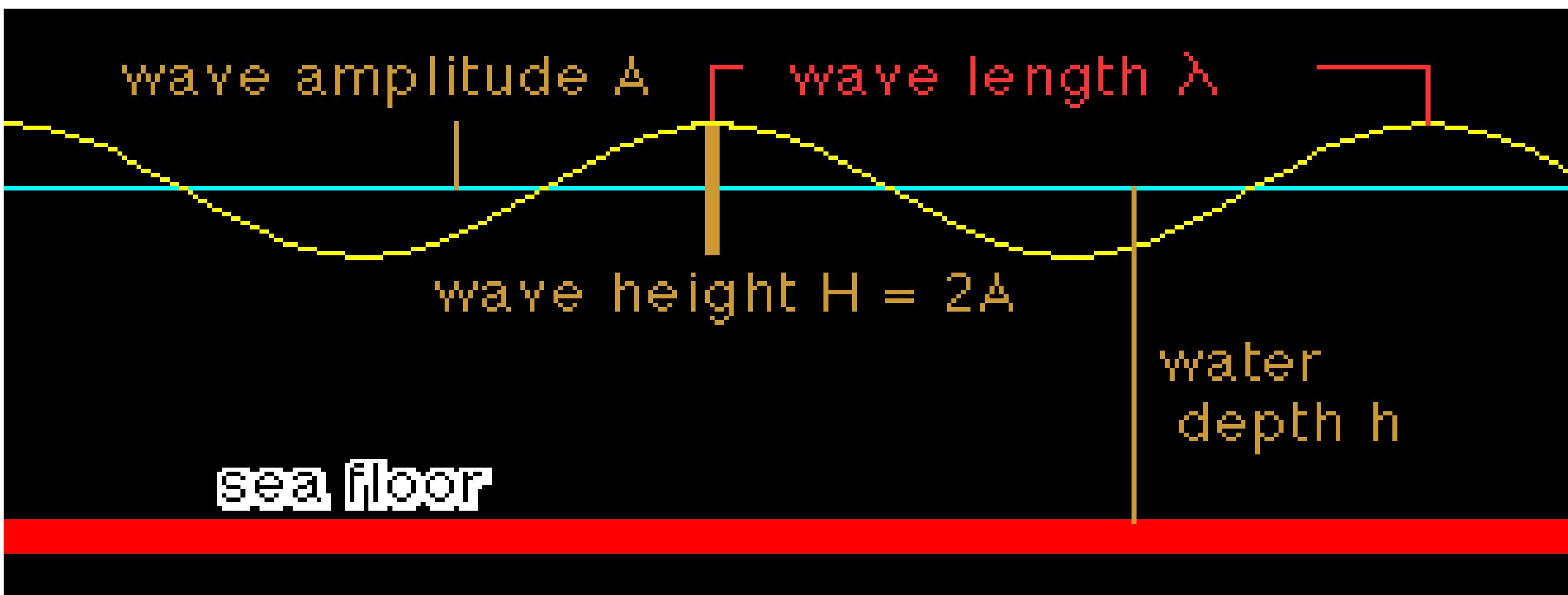


7.1 Water Wave Phenomenon



7.2 Water Wave Description

In calm water, water surface is horizontally flat. When it is disturbed, fluid particles will leave their equilibrium positions and start to move up or down. When a fluid particle moves up, the downward gravitational force tends to pull it downward back to its equilibrium position. On the moment it is passing through the equilibrium position, it will continue to move downward due to its inertia until its downward velocity component decreases to zero. At that position, the fluid particle will reversely move up toward its undisturbed equilibrium position, due to the buoyancy effect. As viscous frictional force is very small and can be harmlessly neglected, fluid particles in wave motion will periodically move up and down. In this way, a water wave is perceived on the upper water surface.



7.2 Water Wave Description

A: **wave amplitude**

$H = 2A$: **wave height**

λ: **wave length**

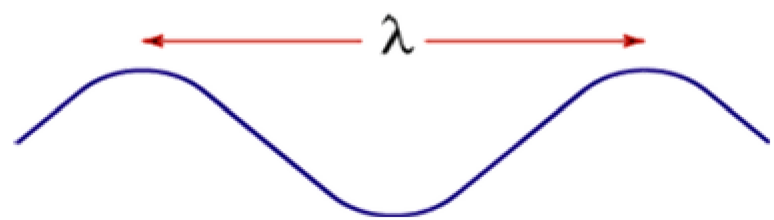
crest

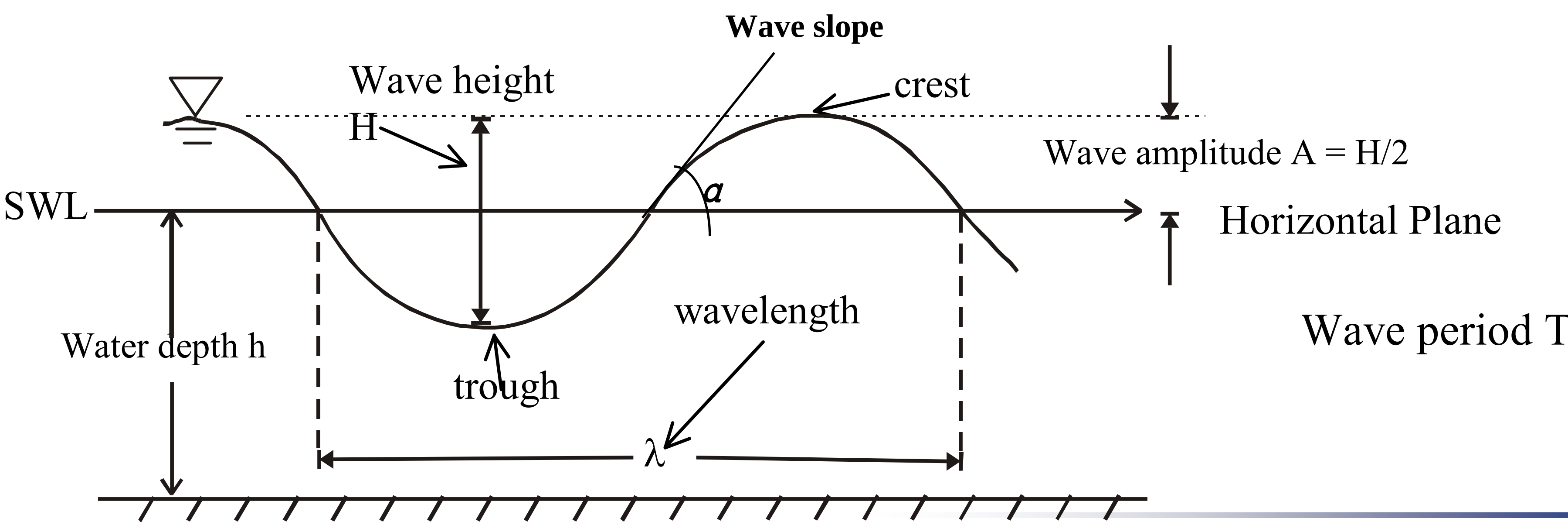
trough

h: **water depth**

α : **wave slope**

$k = 2\pi/\lambda$: **wave number**





7.3 Governing Equations of Water Waves

Three fundamental assumptions for water waves:

1. **Water can be regarded as inviscid**
2. **Water can be treated as incompressible**
3. **Flow of water waves is irrotational**

Water waves are a particular case of *irrotational* flow of a *perfect incompressible* fluid

Water waves are a particular case of *incompressible potential flow*

7.3 Governing Equations of Water Waves

In Chapter 6, governing equations of incompressible potential flows are given, and we write them again as follows

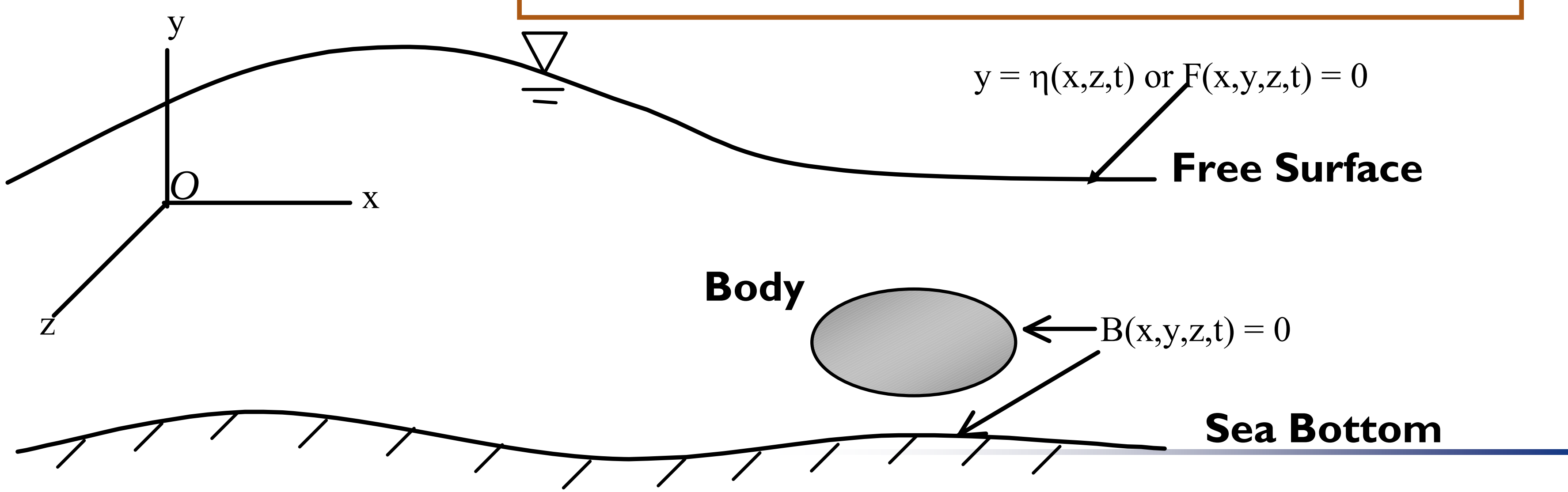
$$
\begin{cases}
\nabla^2 \phi = 0 & \text{Lapace's Eq.} \\
\dfrac{\partial \phi}{\partial t} + \dfrac{|\nabla \phi|^2}{2} + \dfrac{p}{\rho} + \Pi = C(t) & \text{Dynamic cond.} \\
\dfrac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U}_n \quad \text{On the body surface} & \text{Kinematic cond.} \\
\nabla \phi = \mathbf{U}_\infty, \quad p = p_\infty & \text{Far field cond.} \\
\nabla \phi \big|_{t=0} = \mathbf{U}_0(\mathbf{x}), \quad p \big|_{t=0} = p_0(\mathbf{x}) & \text{Initial cond.}
\end{cases}
$$

7.3 Governing Equations of Water Waves

In addition to velocity and pressure, shape of the free surface is also unknown and needs to be determined. Take coordinate plane *xOz* to coincide with the undisturbed calm water surface, axis *Oy* upwards.

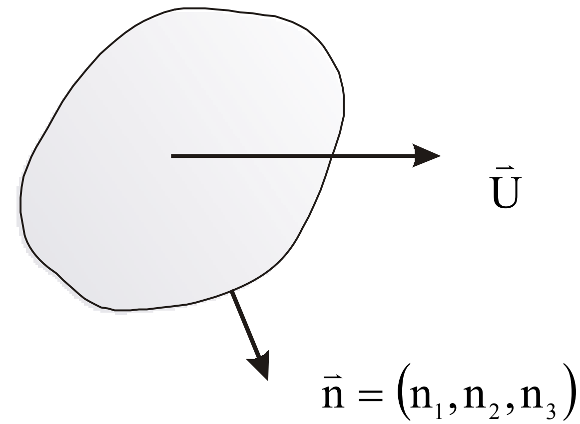
Variables to be determined

1. **Velocity** $\mathbf{V} = \nabla \phi$
2. **Free surface shape** $y = \eta\left(x, z, t\right)$
3. **Pressure** $p\left(x, y, z, t\right)$



1. Kinematic condition on rigid body surface: *impermeable*

$$\mathbf{V}\cdot\mathbf{n}=\mathbf{U}\cdot\mathbf{n}\quad\Rightarrow\quad\nabla\phi\cdot\mathbf{n}=\mathbf{U}_n$$



$$\boxed{\frac{\partial\phi}{\partial\mathbf{n}}=\mathbf{U}_n}$$

On body surface **B**

Another form of the body surface condition can be derived from material derivative of body surface equation.

$$\frac{DB}{Dt}=\frac{\partial B}{\partial t}+(\mathbf{V}\cdot\nabla)B=\boxed{\frac{\partial B}{\partial t}+(\nabla\phi\cdot\nabla)B=0}$$

On body surface **B:** $\quad B(x,y,z,t)=0$

2. Kinematic condition on sea bottom: *impermeable*

$$\mathbf{V}\cdot\mathbf{n}=\mathbf{0}\cdot\mathbf{n} \quad\Rightarrow\quad \nabla\phi\cdot\mathbf{n}=0$$

$$\boxed{\frac{\partial\phi}{\partial\mathbf{n}}=0}$$

On the bottom $y = -h$

For horizontal sea bottom, $B(x,y,z,t)=y+h=0$

$$\frac{DB}{Dt}=\frac{\partial B}{\partial t}+(\nabla\phi\cdot\nabla)B=\frac{\partial\phi}{\partial y}=0$$

$$\boxed{\frac{\partial\phi}{\partial y}=0}$$

On the sea bottom $y = -h$

7.3 Governing Equations of Water Waves

3. Kinematic condition on the free surface: *impermeable* --
Fluid particles on free surface will stay on it; that is, normal velocity of fluid particle should be equal to that of the free surface.

The free surface can be generally written as $y = \eta(x, z, t)$, or as

$$F(x, y, z, t) = y - \eta(x, z, t) = 0$$

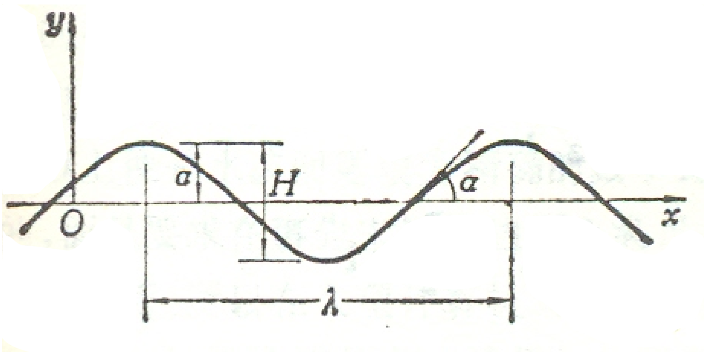
then $$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\nabla\phi \cdot \nabla)F = 0$$ on the free surface $y = \eta$

$$\frac{\partial(y-\eta)}{\partial t} + \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \cdot \left(\frac{\partial(y-\eta)}{\partial x}, \frac{\partial(y-\eta)}{\partial y}, \frac{\partial(y-\eta)}{\partial z}\right) = 0$$

7.3 Governing Equations of Water Waves

$$-\frac{\partial \eta}{\partial t}-\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial y}-\frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z}=0$$

$$\boxed{\frac{\partial \phi}{\partial y}=\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z}}$$



Fluid velocity

Wave slope

on the free surface $y = \eta$

That is the **kinematic condition on the free surface,** a nonlinear equation, where both the shape of free surface **y =** η and the velocity potential are unknown and vary with time. Besides, this condition should be satisfied on the instantaneous free surface, which itself is an unknown.

4. Far field condition: *undisturbed calm water*

$$
\begin{cases}
\dfrac{\partial \phi}{\partial t} \rightarrow 0 \\[2ex]
\mathbf{V} = \nabla \phi \rightarrow 0 \\[2ex]
p = p_a - \rho g y
\end{cases}
$$

Atmospheric pressure

Static pressure

5. Dynamic condition: ***Gravity*** is the only body force.

$$\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + \frac{p}{\rho} + gy = C(t)$$

In far field, free surface is the calm horizontal plane, $p = p_a,\ y = 0$

$$C(t) = \frac{p_a}{\rho}$$

Result is the dynamic condition of water waves for $y \le \eta(x, z, t)$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + \frac{p - p_a}{\rho} + gy = 0}$$

Pressure distributions will be calculated from this equation.

7.3 Governing Equations of Water Waves

On the free surface, $y=\eta(x,z,t)$, $p=p_a$, then the dynamic condition becomes

$$\boxed{\frac{\partial \phi}{\partial t}+\frac{|\nabla \phi|^2}{2}+g\eta = 0}$$

This is the **dynamic condition on the free surface**, a non-linear equation. It yields the shape of the free surface, **y =** η**,** provided the velocity potential has been determined.

6. Initial condition:

Unknowns

1. **Velocity** $\mathbf{V} = \nabla \phi$
2. **Free surface** $y = \eta(x, z, t)$
3. **Pressure** $p(x, y, z, t)$

Velocity potential and the shape of free surface are individual unknowns. And pressure is determined from the velocity potential. Because boundary value problem is Laplacian, initial values on the boundary are sufficient.

$$\boxed{\phi(x, \eta, z, 0) = f(x, z), \quad \eta(x, z, 0) = g(x, z)}$$

There are 2 kind of disturbances: initial velocity initial elevation

7.3 Governing Equations of Water Waves

A summary of the governing equations for water waves.

Field equations: on $y \le \eta\left(x, z, t\right)$

Laplace's Eq. $\nabla^2\phi = 0$

Dynamic cond. $$\frac{\partial\phi}{\partial t} + \frac{\left|\nabla\phi\right|^2}{2} + \frac{p - p_a}{\rho} + gy = 0$$

Far field cond. $$\frac{\partial\phi}{\partial t} \to 0, \quad \mathbf{V} = \nabla\phi \to 0, \quad p = p_a - \rho g y$$

Initial cond. $$\begin{cases} \phi\left(x, \eta, z, 0\right) = f\left(x, z\right) \\ \eta\left(x, z, 0\right) = g\left(x, z\right) \end{cases}$$ **on** $y = \eta$

7.3 Governing Equations of Water Waves

Boundary conditions:

On body surf., B

$$\frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U}_n \qquad \frac{\partial B}{\partial t} + (\nabla \phi \cdot \nabla) B = 0$$

On the bottom, *y* = -*h*

$$\frac{\partial \phi}{\partial y} = 0$$

On free surface, *y* = *η*
(kinematic cond.)

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z}$$

On free surface, *y* = *η*
(dynamic cond.)

$$\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + g\eta = 0$$

7.4 Airy Wave

In last section, governing equations of water waves are given. It is very difficult to solve these equations because the free surface conditions, both kinematic one and dynamic one, are

1. **nonlinear**,
2. to be satisfied at the **unknown free surface**

Airy wave (*i.e.* small amplitude wave, that is, $\dfrac{A}{\lambda} \ll 1$)

1. Free surface conditions is linearized.
2. Free surface conditions is satisfied at the mean free surface.

7.4 Airy Wave

In the case of Airy wave, the vertical elevation η is a small quantity, and velocity and acceleration of fluid particles are also small quantities. Their products are higher-order small quantities in comparison with the first-order quantities and can be neglected. As η is a small quantity, the free surface conditions on $y = \eta$ can be changed to the mean free surface $y = 0$. Therefore,

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \cancel{\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}} + \cancel{\frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z}} \qquad \text{on } y = \eta$$

$$\boxed{\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \qquad \text{on } y = 0}$$

7.4 Airy Wave

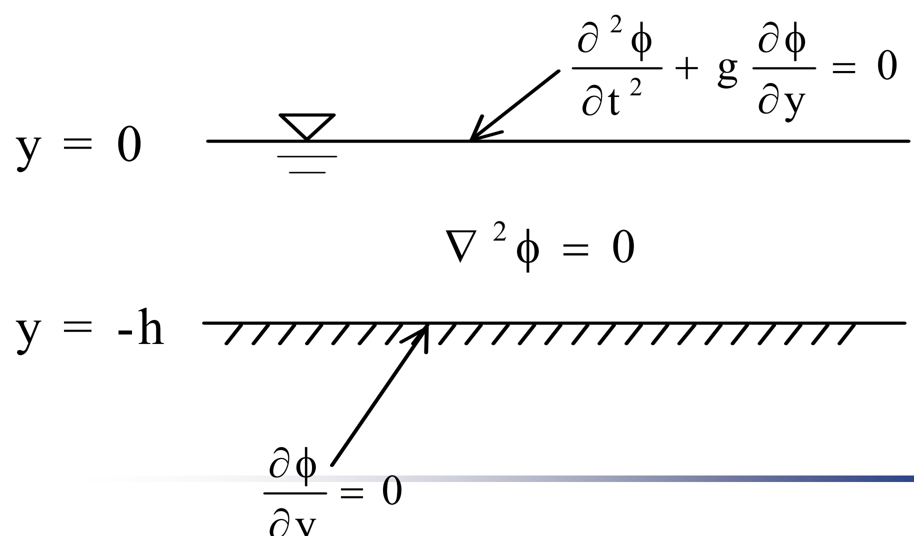
Dynamic condition on $y = \eta$

$$\frac{\partial \phi}{\partial t} + \cancel{\frac{|\nabla \phi|^2}{2}} + g\eta = 0$$

$$\boxed{\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t} \qquad \text{on } y = 0}$$

Kinematic and dynamic conditions on $y = \mathbf{0}$ lead to

$$\boxed{g\frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial t^2} = 0}$$



7.4 Airy Wave

Dynamic condition in the flow field：

$$\frac{\partial \phi}{\partial t} + \cancel{\frac{|\nabla \phi|^2}{2}} + \frac{p - p_a}{\rho} + gy = 0$$

$$\boxed{p - p_a = -\rho \frac{\partial \phi}{\partial t} - \rho g y}$$

dynamic pressure

hydrostatic pressure

7.4 Airy Wave

After linearization, the governing equations of Airy waves are as follows.

Field equations: in $y \le \eta(x, z, t)$

1. Laplace's Eq. $\nabla^2 \phi = 0$

2. Dynamic cond. $p - p_a = -\rho \dfrac{\partial \phi}{\partial t} - \rho g y$

3. Far field cond. $\dfrac{\partial \phi}{\partial t} \to 0, \quad \mathbf{V} = \nabla \phi \to 0, \quad p = p_a - \rho g y$

4. Initial cond.
$$\begin{cases} \phi(x, 0, z, 0) = f(x, z) \\ \eta(x, z, 0) = g(x, z) \end{cases}$$

7.4 Airy Wave

Boundary conditions:

5. On body surface, B

$$\frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U}_n \qquad \frac{\partial B}{\partial t} + (\nabla \phi \cdot \nabla) B = 0$$

6. On the bottom, y = -*h*

$$\frac{\partial \phi}{\partial y} = 0$$

7. On the mean free surface, y = 0

$$g \frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial t^2} = 0$$

$$\eta = -\frac{1}{g} \frac{\partial \phi}{\partial t}$$

7.4 Airy Wave

Now we assume that there is no body in the wave field and the initial conditions are ignored, then potential of Airy wave is governed by the following three equations :

1. Laplace's Eq.

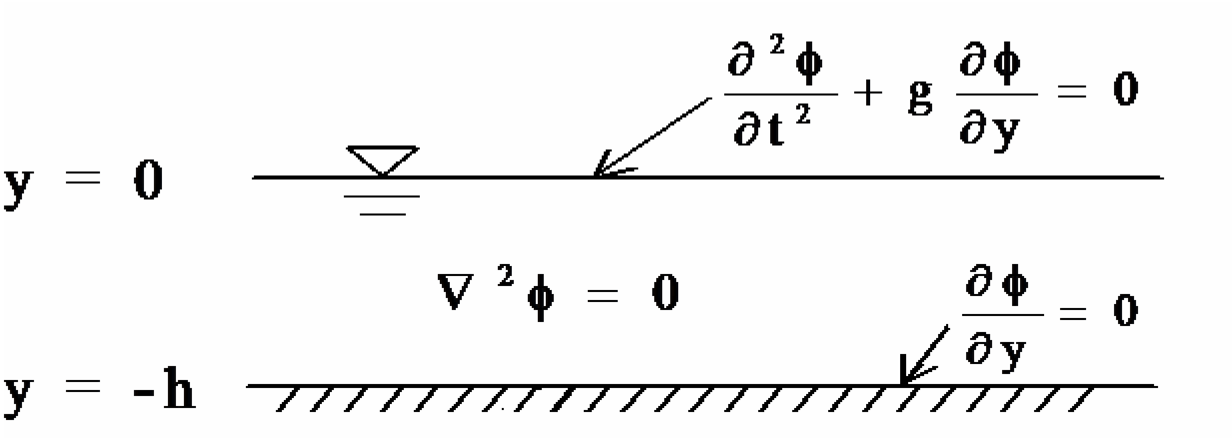
6. On the bottom, *y* = -*h*

7. On the mean free surface, *y* = 0

$$\nabla^2\phi = 0$$

$$\frac{\partial\phi}{\partial y} = 0$$

$$g\frac{\partial\phi}{\partial y} + \frac{\partial^2\phi}{\partial t^2} = 0$$



7.4 Airy Wave

By variable separation method, the velocity potential function of Airy wave can be immediately derived.

$$\boxed{\phi = \frac{gA}{\omega}\sin\left(kx - \omega t\right)\frac{\cosh k\left(y+h\right)}{\cosh kh}}$$

If the depth of water tends to infinity, $h \to \infty$, it becomes

$$\boxed{\phi = \frac{gA}{\omega}\sin\left(kx - \omega t\right)e^{ky}}$$

7.4 Airy Wave

Wave elevation of Airy wave is obtained from the fulfillment of free surface dynamic condition.

$$\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t}\bigg|_{y=0} = A\cos(kx - \omega t)$$

From the dynamic condition, pressure distribution in Airy wave flow field can also be derived.

$$\begin{aligned} p - p_a &= -\rho\frac{\partial \phi}{\partial t} - \rho g y \\ &= -\rho g y - \rho A g \cos(kx - \omega t)\frac{\cosh k(y+h)}{\cosh kh} \end{aligned}$$

7.4 Airy Wave

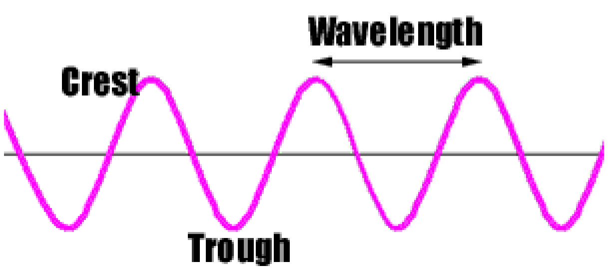
For Airy wave, we need to determine three parameters, **(A, k, ω)**. In fact, only two of them, **(A, k)**, need to be determined, due to the dispersion relation.

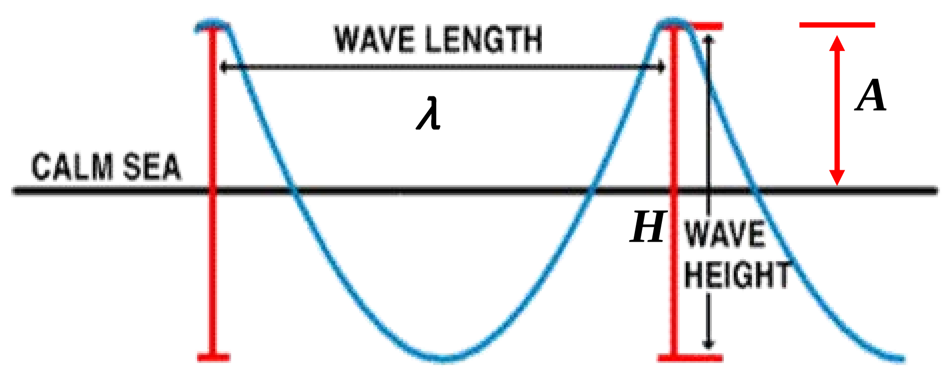
$$\eta = A\cos\left(kx - \omega t\right)$$

1. Airy wave is a 2-dimensional cosine function, known as cosine wave, sine wave or linear sinusoidal wave.

A -- **wave amplitude**; $H = 2A$ -- **wave height**.

λ -- **wave length**, the distance between two adjacent crests or troughs.





7.4 Airy Wave

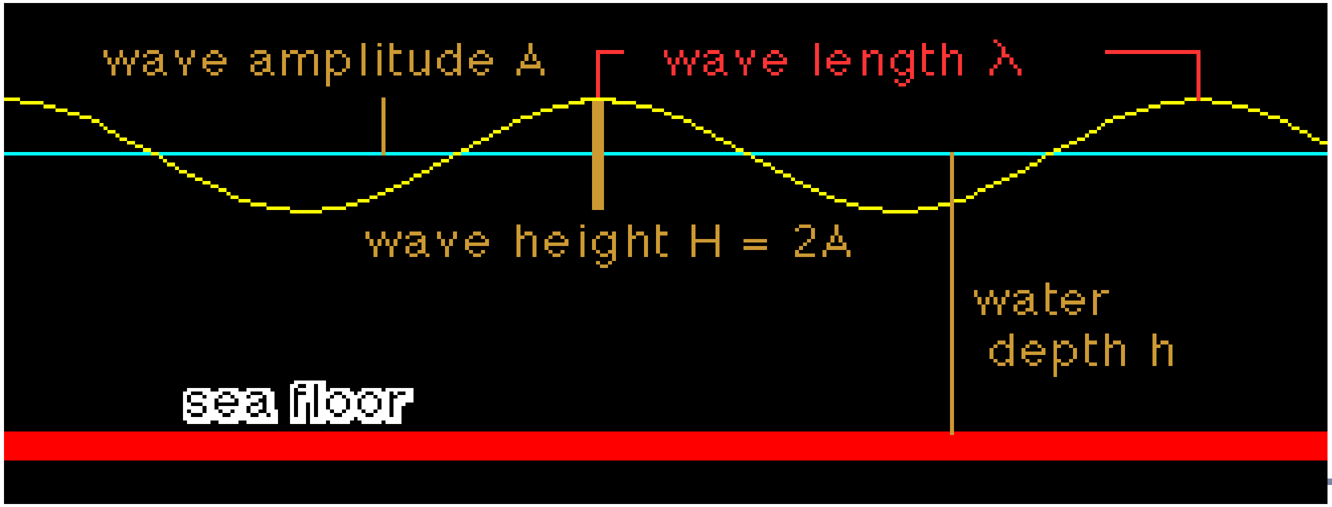
For Airy wave, we need to determine three parameters, **(A, k, ω)**. In fact, only two of them, **(A, k)**, need to be determined, since a dispersion relation exists, that will be introduced soon later in this section.

$$\eta = A\cos\left(kx - \omega t\right)$$

1. Airy wave is a 2-dimensional cosine function, known as cosine wave, sine wave or linear sinusoidal wave.

A -- **wave amplitude**; $H = 2A$ -- **wave height**.

λ -- **wave length**, the distance between two adjacent crests or troughs.



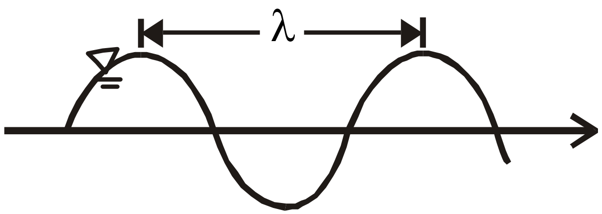
2. k : the wave number

Let $t = 0$, Airy wave becomes simply a cosine function of x.

$$\eta = A\cos(kx) \qquad kx = 2\pi n$$ (**n** is an integer)

For $n = 1$, it corresponds one wave form, that is, $x = \lambda$, therefore

$$\boxed{k = \frac{2\pi}{\lambda}}$$



k = wavenumber = $2\pi/\lambda$ $[L^{-1}]$

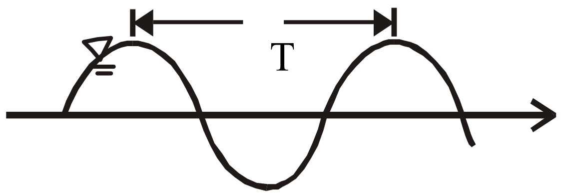
3. ω : circular frequency

Let $x = 0$, Airy wave becomes a cosine function of t.

$$\eta = A\cos(\omega t) \qquad \omega t = 2\pi n \quad (n \text{ is an integer})$$

For $n = 1$, it corresponds one period, $t = T = 1/f$, therefore

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f} \qquad \text{where} \quad T = \frac{1}{f}$$



ω = frequency = $2\pi/T$ $\quad [T^{-1}]$, e.g. rad/sec

7.4 Airy Wave

4. Phase velocity / Celerity (c or V_p): wave form moving velocity

Let's look at a fixed position in space where at an instant a crest located at. Then the crest moves forward. The duration until another crest arrives at that point is just one wave period. During this period, we can see the wave form moves forward just one wave length. So, velocity the wave form advances is

$$\boxed{c = V_p = \frac{\lambda}{T}}$$