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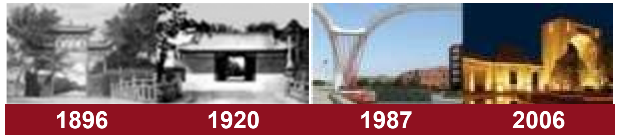


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Chapter 6
Potential Flow Theory

Sixth Assignment

- **The fifth assignment can be downloaded from following website:**

  Website： ftp://public.sjtu.edu.cn

  Username： dcwan

  Password： 2015mhydro

  Directory： IntroMHydro2015-Assignments

- **Seven problems**

- **Submit the assignment on May 14th (in English, written on paper)**

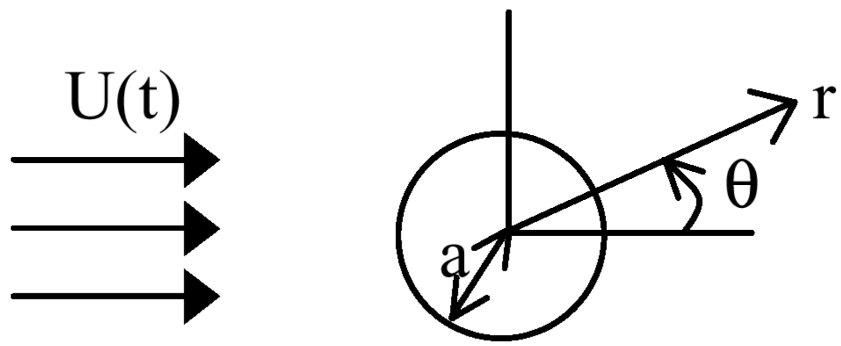
- Fluid mechanics
  - Ideal fluid mechanics
    - Rotational flows
    - Potential flows
    - Wave motion
  - Viscous fluid mechanics
    - Simple flows
    - Boundary layer flows
    - Laminar and turbulent flows

6.8 A Fixed Body in Unsteady Flow

Up to now, a body moving in calm water at an acceleration is considered and added inertia force is introduced. If the body is fixed on the earth and the flow is unsteady, whether a resultant force on the body is non-zero? whether an added mass is also derived?

Consider a *3d* sphere fixed on the earth, and the flow is unsteady. The solution can readily be obtained from the velocity potential of a steady uniform flow past a sphere, only need to change the velocity *U* to *U*(*t*), that is

$$\phi = U(t)\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$



6.8 A Fixed Body in Unsteady Flow

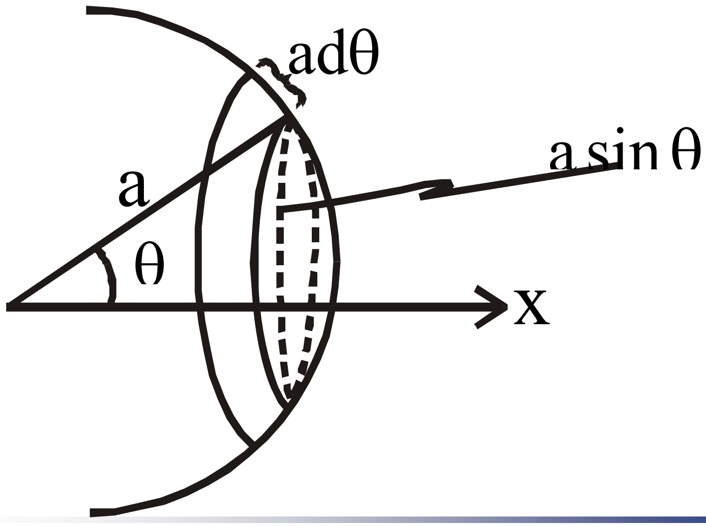
From the velocity potential, we prepare following terms in the evaluation of the hydrodynamic forces on the sphere

$$\left.\frac{\partial \phi}{\partial t}\right|_{r=a} = \dot{U}\left.\frac{3a^3\cos\theta}{2r^2}\right|_{r=a} = \frac{3}{2}\dot{U}a\cos\theta$$

$$\nabla\phi\big|_{r=a} = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) = \left(0,\ -\frac{3}{2}U\sin\theta,\ 0\right)$$

$$\left.\frac{1}{2}\left|\nabla\phi\right|^2\right|_{r=a} = \frac{9}{8}U^2\sin^2\theta$$

$$\iint_B dS = \int_0^{\pi}(a\,d\theta)(2\pi a\sin\theta)$$



6.8 A Fixed Body in Unsteady Flow

As we did in the previous section, according to *Bernoulli's equation*, the resultant horizontal force due to hydrodynamic pressures on the sphere is

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + \Pi - C(t) \right)_{r=a} n_x \, dS$$

$$= -\rho \int_0^\pi \left(\underbrace{\frac{3}{2}\dot{U} a \cos\theta}_{\frac{\partial \phi}{\partial t}} + \underbrace{\frac{9}{8} U^2 \sin^2\theta}_{\frac{1}{2}|\nabla \phi|^2} \right) \left(\underbrace{-\cos\theta}_{n_x} \right) \underbrace{\left(2\pi a^2 \sin\theta\right) d\theta}_{dS}$$

$$= 3\pi\rho\dot{U}a^3 \underbrace{\int_0^\pi \left(\sin\theta \cos^2\theta\right) d\theta}_{2/3} + \rho U^2 \frac{9\pi}{4} a^2 \underbrace{\int_0^\pi \left(\sin^3\theta \cos\theta\right) d\theta}_{0}$$

$$= \dot{U}(t)\left[2\rho\pi a^3\right] = \dot{U}(t)\left[\frac{4}{3}\rho\pi a^3 + \frac{2}{3}\rho\pi a^3\right] = \dot{U}(t)\left(\rho V\!\!\!/ + m_{(1)}\right)$$

6.8 A Fixed Body in Unsteady Flow

As we know in the previous section, a sphere moving in calm water is of added mass $m_{(1)}$, half the mass of the fluid displaced by the sphere. For a sphere fixed in an unsteady flow, it has another additional added mass, $\rho V\!\!\!/$, equal to the mass of the fluid displaced by the sphere.

$$F_x = \dot{U}(t)\left(\rho V\!\!\!/ + m_{(1)}\right) = \dot{U}(t)\, m_{(2)}$$

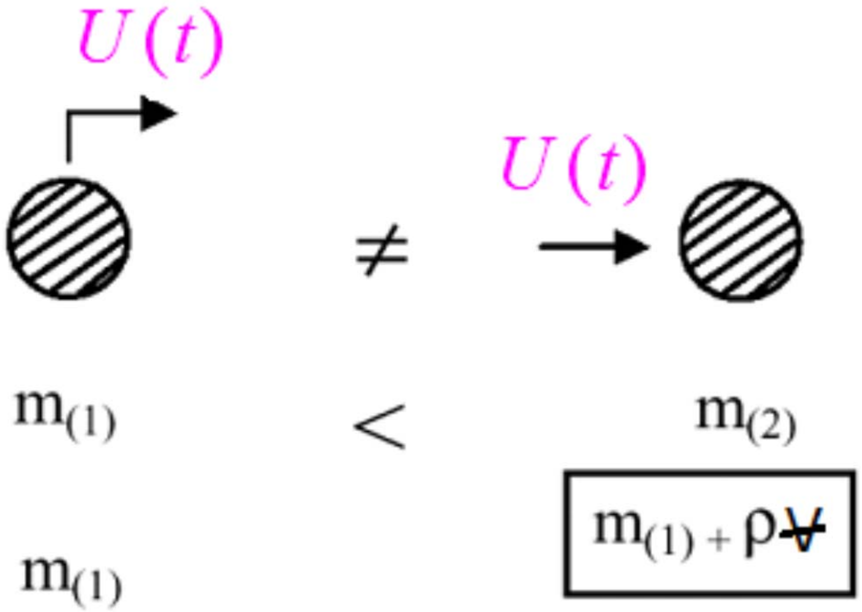
Generally, for a body fixed in an unsteady flow, we also derive an ***added mass (coefficient)***. It consists of two parts, except the added mass due to the body moving in calm water, an additional mass of the fluid occupied (or displaced) by the body.

$$m_{(2)} = \rho V\!\!\!/ + m_{(1)}$$

$$C_{m_{(2)}} = 1 + C_{m_{(1)}}$$

6.8 A Fixed Body in Unsteady Flow

Therefore, added mass of a body moving in calm water with acceleration is smaller than the added mass of the body fixed in an accelerated flow by an amount of the mass of the fluid displaced by the body. Accordingly, the resultant force is also smaller.



6.9 Kinetic Energy of Potential Flow

An unsteady flow will accompany an added inertia force. Magnitude of the added inertia force may be derived from its kinetic energy.

Green's formula

For vector $\mathbf{A}$

τ - singly-connected region

S - surface of τ

$\mathbf{n}$ - outward normal to S

Gauss' theorem

$$\iiint_\tau \nabla \cdot \mathbf{A}\, \mathrm{d}\tau = \oiint_S \mathbf{A} \cdot \mathbf{n}\, \mathrm{d}S$$

$$\mathbf{A} = \Phi \nabla \Psi$$

$$\mathbf{A} \cdot \mathbf{n} = \Phi \nabla \Psi \cdot \mathbf{n} = \Phi \frac{\partial \Psi}{\partial n}$$

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\Phi \nabla \Psi) = \Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi$$

$$\iiint_\tau (\Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi)\, \mathrm{d}\tau = \oiint_S \Phi \frac{\partial \Psi}{\partial n} \mathrm{d}S$$

Green's formula

6.9 Kinetic Energy of Potential Flow

$$\iiint_\tau (\Phi \nabla^2 \Psi + \nabla \Phi \cdot \nabla \Psi)\,\mathrm{d}\tau = \oiint_S \Phi \frac{\partial \Psi}{\partial n}\,\mathrm{d}S$$

Velocity potential of incompressible flow satisfies Laplace eq. $\nabla^2\phi=0$

$$\Phi = \Psi = \phi$$

$$\iiint_\tau (\nabla \phi)^2\,\mathrm{d}\tau = \oiint_S \phi \frac{\partial \phi}{\partial n}\,\mathrm{d}S$$

$$\nabla \phi = \mathbf{V}$$

$$(\nabla \phi)^2 = |\mathbf{V}|^2 = V^2$$

$$\times \frac{1}{2}\rho$$

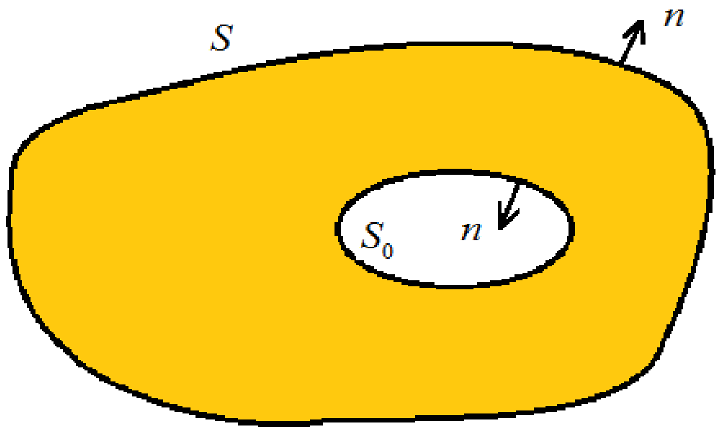
Kinetic Energy of the Fluid in t

$$T = \frac{1}{2}\rho \iiint_\tau V^2\,\mathrm{d}\tau = \frac{\rho}{2}\oiint_S \phi \frac{\partial \phi}{\partial n}\,\mathrm{d}S$$

Determined by ϕ and $\frac{\partial \phi}{\partial n}$ on S

6.9 Kinetic Energy of Potential Flow

If τ is not a singly-connected region, there exists an inner boundary S_0 as shown in the figure. Then,



$$T = \frac{1}{2}\rho\iint_S \phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S + \frac{1}{2}\rho\iint_{S_0}\phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S$$

At the far field, the flow is at rest

$$\iint_S \phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S \sim \frac{1}{r^3} \to 0$$

$$T = \frac{1}{2}\rho\iint_{S_0}\phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S \qquad (\mathbf{n}\text{ - inward normal to }S_0)$$

6.9 Kinetic Energy of Potential Flow

Eg. **A sphere of radius *a* moves in calm water along *x*-axis straight ahead at speed *U*. Calculate the kinetic energy of the flow field.**

Solution: We have obtained the velocity potential before,

$$\phi = U\frac{a^3}{2r^2}\cos\theta$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial n} = -\frac{\partial\phi}{\partial r} = U\frac{a^3}{r^3}\cos\theta$$

n is the normal inward to the sphere.

$$\Rightarrow \quad T = \frac{1}{2}\rho\iint_{S_0}\phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S$$

$$= \frac{1}{2}\rho\int_0^{\pi}(\frac{1}{2}Ua\cos\theta)(U\cos\theta)\cdot 2\pi a^2\sin\theta\ \mathrm{d}\theta$$

$$= \frac{1}{2}\pi\rho a^3U^2\int_0^{\pi}\cos^2\theta\sin\theta\ \mathrm{d}\theta = \frac{1}{3}\pi\rho a^3U^2 = \frac{1}{2}m_{(1)}U^2$$

6.10 Decomposition of Velocity Potential

Consider a body moving in calm water with an arbitrary motion.

In the earth-fixed coordinate system, velocity potential of the flow is governed by the following equations

$$\nabla^2 \phi(x, y, z, t) = 0 \qquad \text{in the flow field}$$

$$\frac{\partial \phi}{\partial n} = V_n \qquad \text{on the body surface}$$

$$\nabla \phi = 0 \qquad \text{at far field}$$

Giving velocity **V** of the body → velocity potential ϕ

6.10 Decomposition of Velocity Potential

Generally, motions of a body (a rigid body) can be resolved into a translation at velocity $\boldsymbol{V_O}$ at point *O*, and a rotation around *O*.

Set the origin of a coordinate system at *O*, and denote **r** as the position vector from *O*.

$$V = V_0 + \omega \times \mathbf{r}$$

$$V_0 = (u_0,\ v_0,\ w_0)$$
$$\vec{\omega} = (\omega_1,\ \omega_2,\ \omega_3)$$
$$\vec{r} = (x,\ y,\ z)$$

Kinetic body surface condition

outward unit normal vector **n**(n_1, n_2, n_3)

$$\frac{\partial \phi}{\partial n} = V_n = V_{0n} + (\vec{\omega} \times \vec{r})_n = u_0 n_1 + v_0 n_2 + w_0 n_3 + \omega_1 (y n_3 - z n_2) + \omega_2 (z n_1 - x n_3) + \omega_3 (x n_2 - y n_1)$$

6.10 Decomposition of Velocity Potential

Define a *general normal vector* and a *general velocity vector*.

$$\mathbf{n} = (n_1,\ n_2,\ n_3,\ yn_3 - zn_2,\ zn_1 - xn_3,\ xn_2 - yn_1)$$

$$\dot{\mathbf{x}} = (u_0,\ v_0,\ w_0,\ \omega_1,\ \omega_2,\ \omega_3)$$

In this way, the *kinetic body surface condition* is written as

$$\frac{\partial \phi}{\partial n} = \sum_{j=1}^{6} \dot{x}_j n_j$$

6.10 Decomposition of Velocity Potential

$$\nabla^2 \phi(x, y, z, t) = 0 \quad \text{in the flow field}$$

$$\frac{\partial \phi}{\partial n} = V_n \quad \text{on the body surface}$$

$$\nabla \phi = 0 \quad \text{at far field}$$

$$\phi = \sum_{j=1}^{6} \dot{x}_j \Phi_j$$

$\dot{x}_j$ is the general velocity

Φ_j is the velocity potential due to the motion of the body with unit velocity of the *j*-th mode.

$$\left.\begin{aligned} \nabla^2 \Phi_j &= 0 \\ \frac{\partial \Phi_j}{\partial n} &= n_j \\ \nabla \Phi_j &= 0 \end{aligned}\right\}_{j=1,2,\cdots,6}$$

$\dot{x}_j$ is dependent on t, but n_j and Φ_j are not.

Therefore, velocity potential, ϕ, is resolved into 6 independent components, Φ_j, which are velocity potentials corresponding to motion of the *j*-th mode of the body.

6.10 Decomposition of Velocity Potential

Kinetic energy of the flow field

$$T = -\frac{1}{2}\rho\iint_{S_0}\phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S$$

$$= -\frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}\dot{x}_i\dot{x}_j\left(\rho\iint_{S_0}\Phi_j\frac{\partial\Phi_i}{\partial n}\,\mathrm{d}S\right)$$

$$= \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6}m_{i,j}\dot{x}_i\dot{x}_j$$

n is the *outward normal* to the body surface

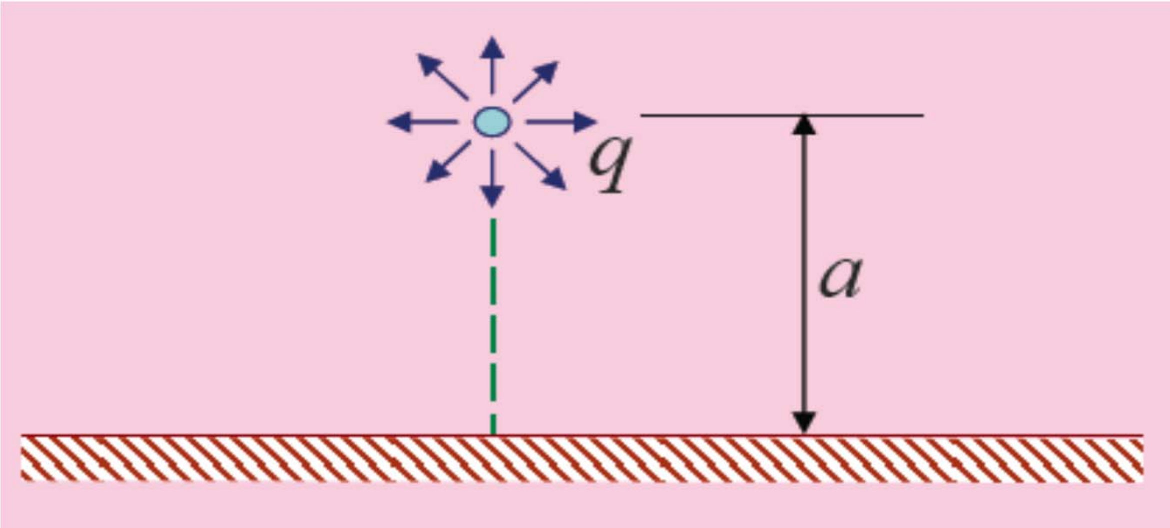
where

$$m_{ji} = -\rho\iint_{S_0}\Phi_j\frac{\partial\Phi_i}{\partial n}\,\mathrm{d}S$$

general added mass

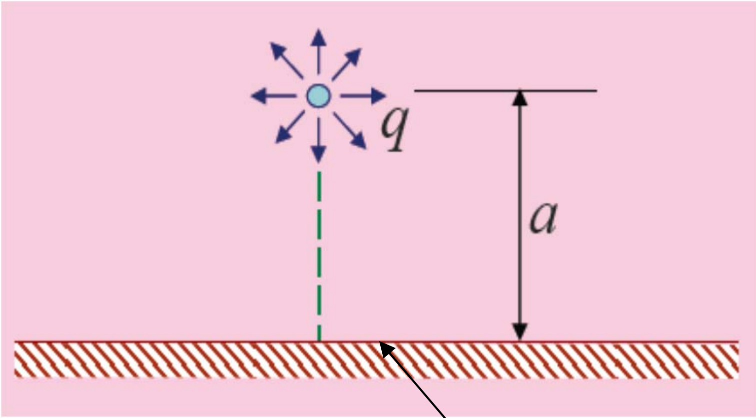
6.11 The Method of Images

It is found that for some simple body, when an elementary flow is introduced into the flow field, the *kinematic body surface condition* can be fulfilled by putting a corresponding elementary flow, such as due to a point source, a point sink, a doublet, a point vortex and so on, as well as a combination of these elementary flows, at the position(s) of the image point of the original elementary flow inside the body and taking the body away from the field. In this way, from the principle of superposition, the velocity potential will be equal to the sum of the original and the image elementary flows. This method is call the ***method of images***.

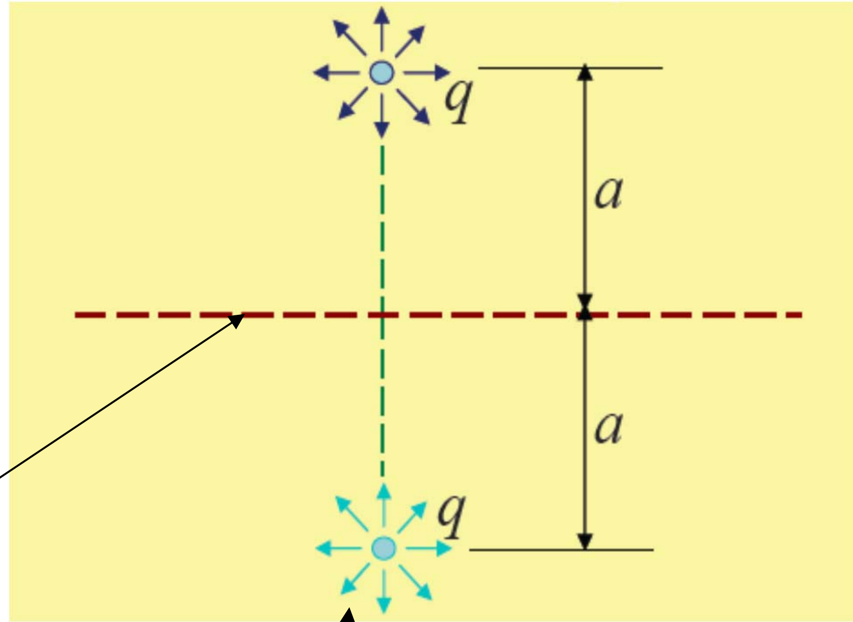


Example: In the upper half plane flow, there is a point source.

6.11 The Method of Images



$\Leftrightarrow$



$$\frac{\partial \phi}{\partial n} = v_n = 0$$

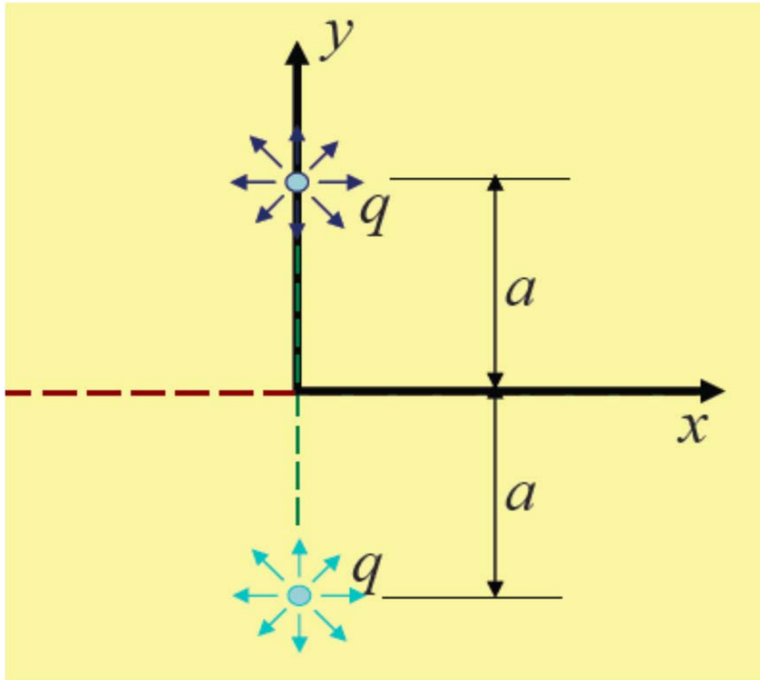
The boundary should be a streamline.

Put a new point source at the mirror image point

The boundary conditions for both flows are equivalent,
So both flows in the upper half plane are the same.

6.11 The Method of Images

Eg. Point source near a flat wall



$$\phi = \frac{q}{2\pi}\left(\ln\sqrt{x^2+(y-a)^2} + \ln\sqrt{x^2+(y+a)^2}\right)$$

$$u_x = \frac{\partial\phi}{\partial x} = \frac{q}{2\pi}\left(\frac{x}{\sqrt{x^2+(y-a)^2}} + \frac{x}{\sqrt{x^2+(y+a)^2}}\right)$$

$$u_y = \frac{\partial\phi}{\partial y} = \frac{q}{2\pi}\left(\frac{y-a}{\sqrt{x^2+(y-a)^2}} + \frac{y+a}{\sqrt{x^2+(y+a)^2}}\right)$$

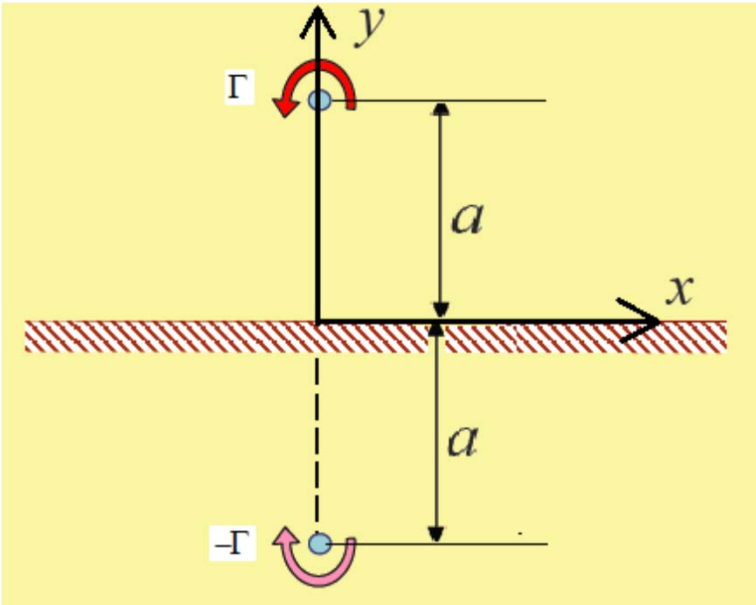
$$u_x\big|_{y=0} = \frac{q}{\pi}\frac{x}{\sqrt{x^2+a^2}}, \qquad u_y\big|_{y=0} = 0$$

Body surface condition on the horizontal flat wall is satisfied.

Pressure can be evaluated from it

6.11 The Method of Images

Eg. Point vortex near a flat wall



$$\phi = \frac{\Gamma}{2\pi}\left(\tan^{-1}\frac{y-a}{x} - \tan^{-1}\frac{y+a}{x}\right)$$

It can be verified that

$$\phi(y) = \phi(-y)$$

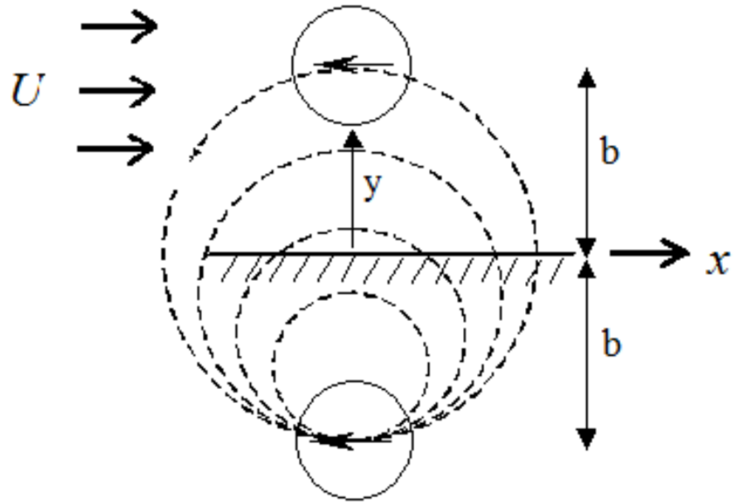
$$\frac{\partial \phi}{\partial y} = 0$$

That is the boundary condition on the flat wall.

Here both vortices will move.

6.11 The Method of Images

Eg. Circular cylinder near a flat wall



Put another circular cylinder, symmetric with the horizontal flat wall to the original one, outside the flow field. Then velocity potential will be a superposition of the ones of the two circular flows. It gives

$$\phi = Ux\left(1 + \frac{a^2}{x^2 + (y-b)^2} + \frac{a^2}{x^2 + (y+b)^2}\right)$$

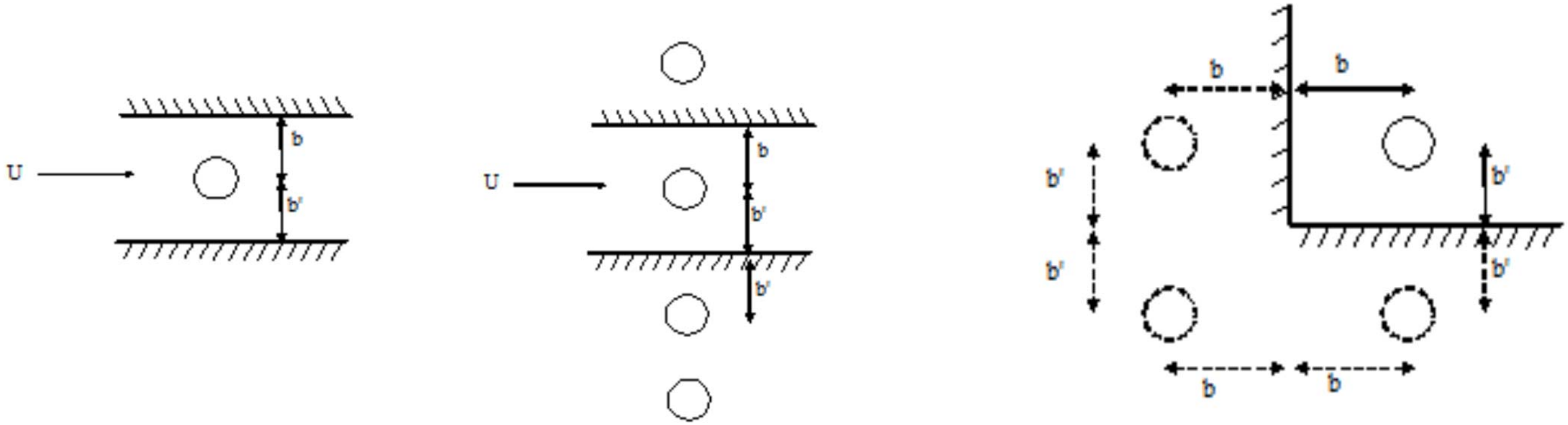
It can be verified (if $a \gg b$) that

$$\phi(y) = \phi(-y) \qquad \frac{\partial \phi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial n} = 0$$

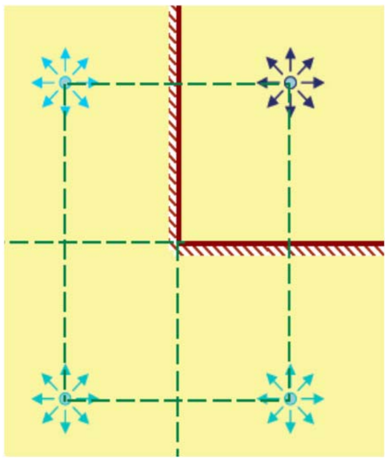
which is exactly the boundary condition on the horizontal flat wall the flow has to be satisfied.

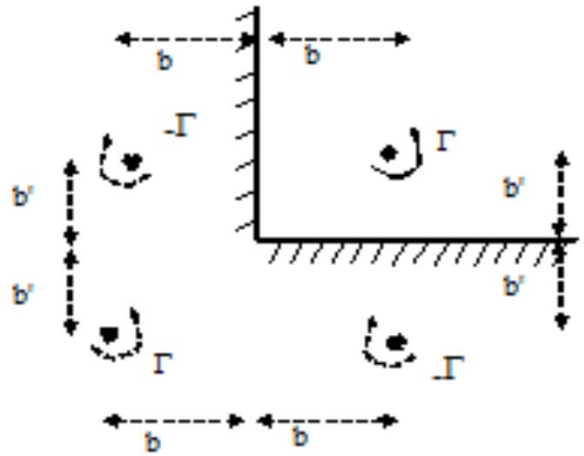
6.11 The Method of Images

Eg. A circular cylinder between two parallel flat walls



Eg. Point source or point vortex near two perpendicular walls





6.12 Corner Flows

If a function satisfies *Laplace's equation*, and fulfills the *kinematic body surface condition*, it will be a velocity potential or a stream function of the potential flow around the body, depending on which form the body surface condition is given, in velocity potential or in stream function. In this way, from a known ***harmonic function*** we can inversely find a new kind of potential flow.

Eg. Following is the 2d potential function and stream function constructed from triangular functions.

$$\phi = r^{\alpha} \cos \alpha\theta$$

$$\psi = r^{\alpha} \sin \alpha\theta$$

Note: ***Harmonic function*** means a function which is a solution of *Laplace's equation*.

6.12 Corner Flows

It can be verified that they satisfy Laplace's equation.

1. Laplace's equation

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0$$

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi = 0$$

2. Velocity field

$$u_r = \frac{\partial\phi}{\partial r} = \alpha r^{\alpha-1}\cos\alpha\theta$$

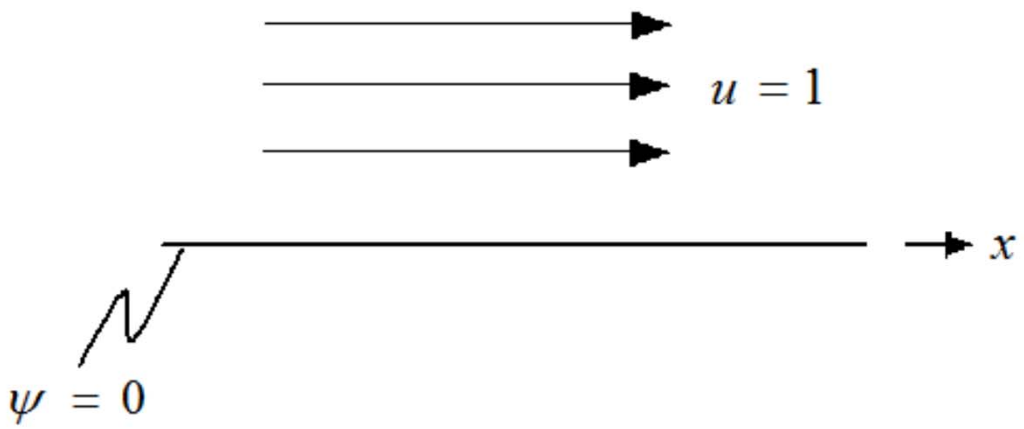
$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\alpha r^{\alpha-1}\sin\alpha\theta$$

6.12 Corner Flows

Now we give some potential flows corresponding to them.

1) **Uniform flow passing an infinitely long plate**

If $\alpha=1,\ \theta_0=0,\ \pi,\ 2\pi$ ⟹ $u=1,\ v=0$ (flat plate)

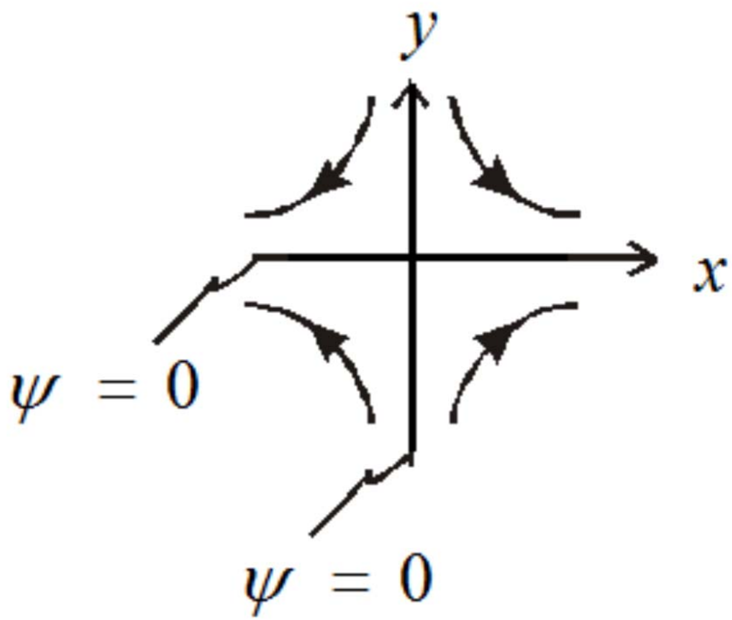


6.12 Corner Flows

2) 90° right angle corner flow

If $\alpha=2,\ \theta_0=0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$

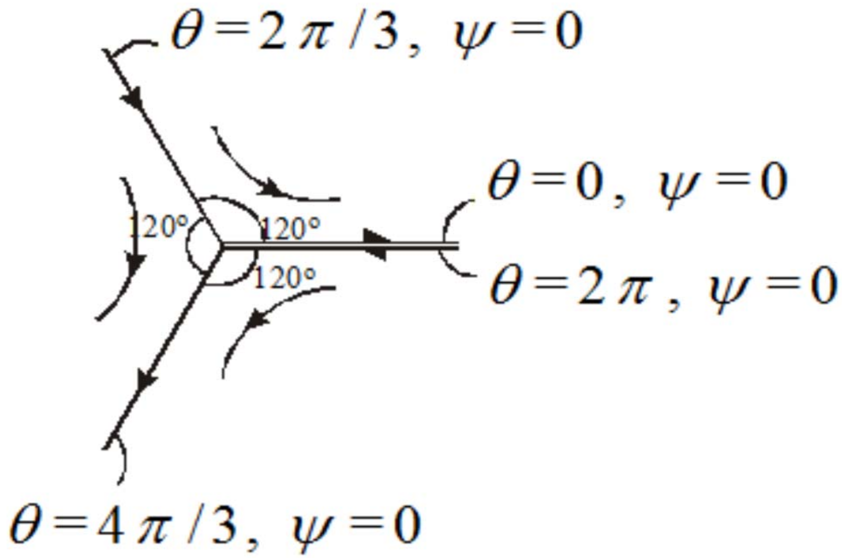
$$\Longrightarrow \quad u=2x,\ v=-2y$$



6.12 Corner Flows

3) 120° obtuse angle corner flow

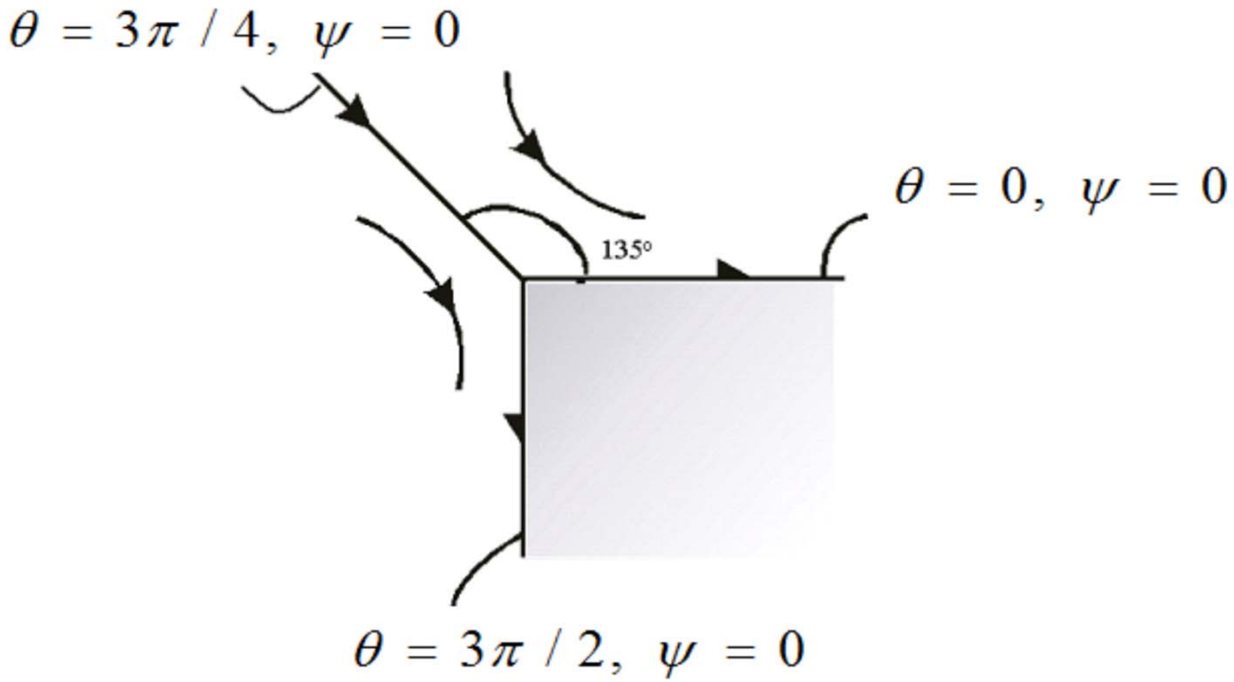
In this case, $\alpha = \dfrac{3}{2}$, $\theta_0 = 0,\ \dfrac{2\pi}{3},\ \dfrac{4\pi}{3},\ 2\pi$



6.12 Corner Flows

4) 135° obtuse angle corner flow

$$\text{Let } \alpha = \frac{4}{3},\ \theta_0 = 0,\ \frac{3\pi}{4},\ \frac{3\pi}{2}$$

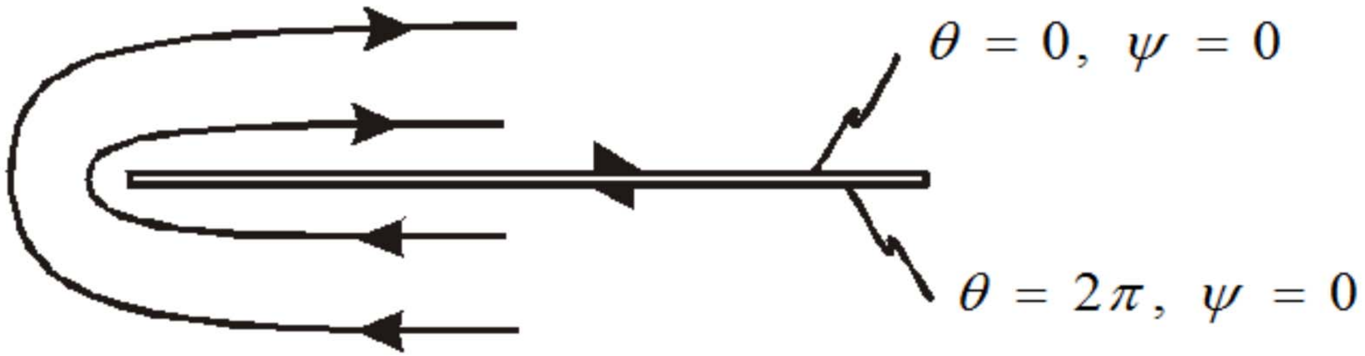


6.12 Corner Flows

5) 180° corner flow

(Half infinitely long flat plate without thickness)

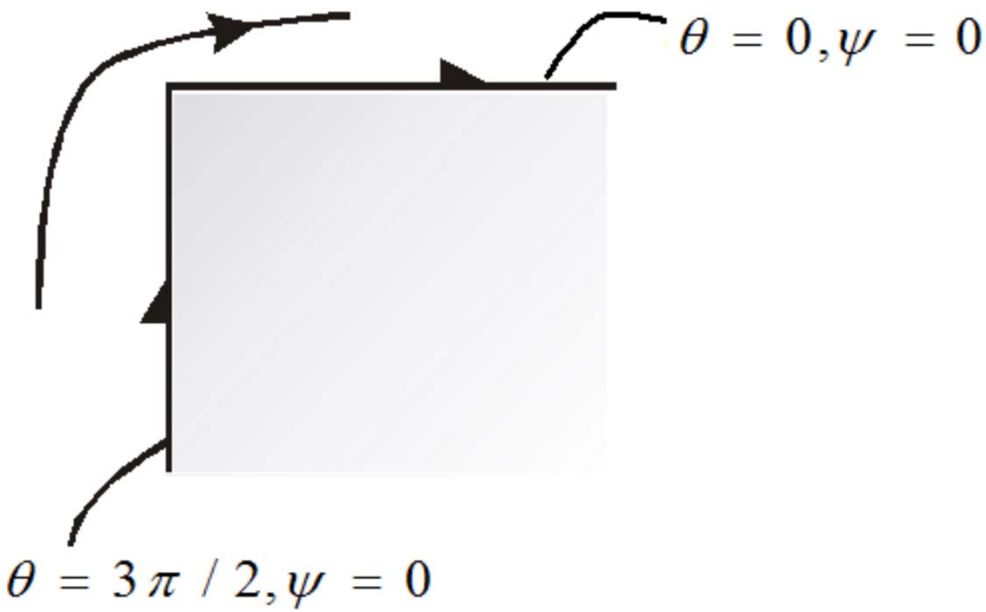
$$\text{Let } \alpha = \frac{1}{2},\ \theta_0 = 0,\ 2\pi$$



6.12 Corner Flows

6) 270° corner flow

In this case, $\alpha = \dfrac{2}{3}$, $\theta_0 = 0$, $\dfrac{3\pi}{2}$



Review

- **Governing equations of incompressible potential flows**

Laplace's eq.

Dynamic cond. (Bernoulli eq.)

Kinematic cond.

Far field cond.

Initial condition

$$
\begin{cases}
\nabla^2 \phi = 0 \\
\dfrac{\partial \phi}{\partial t} + \dfrac{|\nabla \phi|^2}{2} + \dfrac{p}{\rho} + \Pi = C(t) \\
\dfrac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U}_n \qquad \text{(on body surface)} \\
\nabla \phi = \mathbf{U}_\infty, \qquad p = p_\infty \\
\nabla \phi \big|_{t=0} = \mathbf{U}_0(\mathbf{x}), \qquad p \big|_{t=0} = p_0(\mathbf{x})
\end{cases}
$$

Review

- **Some elementary flows:**

Name	2D	3D
Point Source Point Sink	$\phi = \frac{m}{2\pi}\ln r,\quad \psi = \frac{m}{2\pi}\theta$	$\phi = \frac{m}{4\pi r}$
Doublet Flow (Dipole)	$\phi = -\frac{M}{2\pi}\frac{x}{r^2},\quad \psi = \frac{M}{2\pi}\frac{y}{r^2}$	$\phi = \frac{M}{4\pi}\frac{x}{r^3}$
Point Vortex	$\phi = \frac{\Gamma}{2\pi}\theta,\quad \psi = -\frac{\Gamma}{2\pi}\ln r$	——
Uniform Flow	$\phi = Ux,\quad \psi = Uy$	$\phi = Ux$

Review

Superposition Principle: A superposition of elementary potential flows gives another new complex potential flow.

a) A point source + a point sink → a dipole

b) A point source + a point vortex → a spiral flow

c) A uniform flow + a point source → a Rankine half-body flow

d) A uniform flow + a point source + a point sink → a flow around a closed body

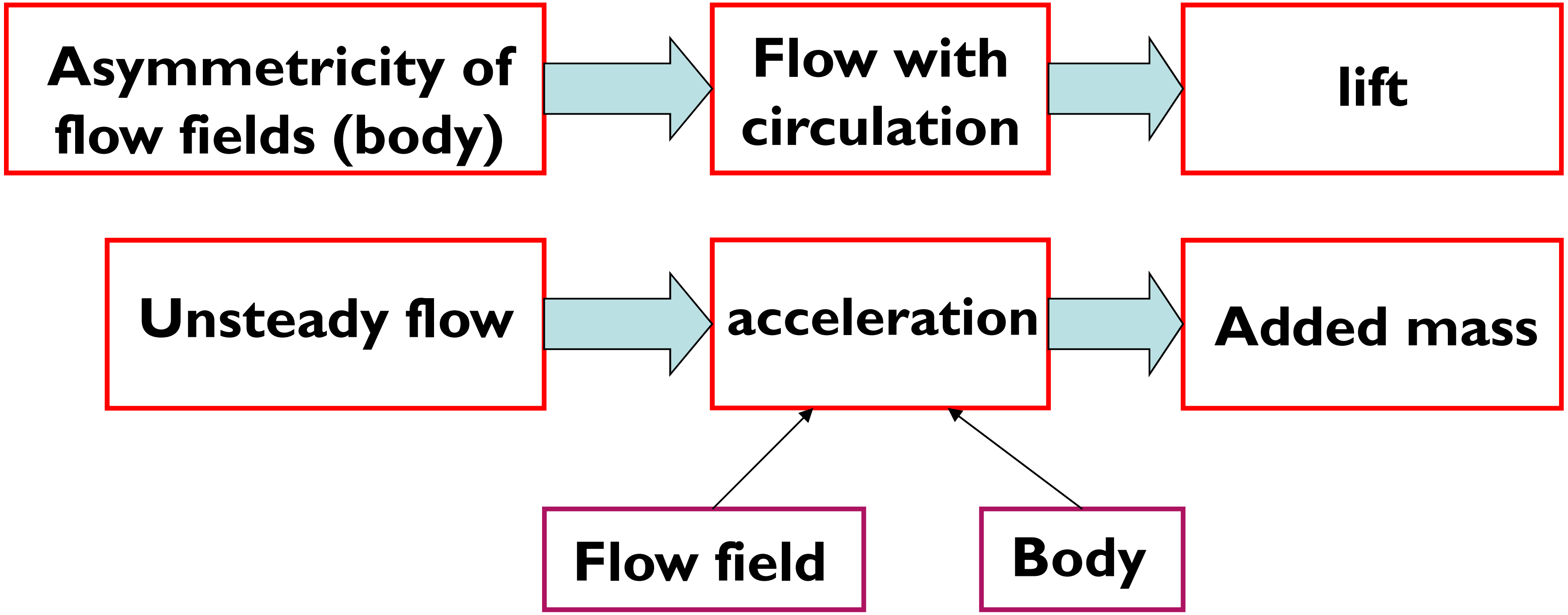
d'Alembert's paradox

For the following two cases, the resultant hydrodynamic forces on a body vanish.

1) A body moving in calm water at constant velocity

2) A body fixed in a uniform flow field

Review

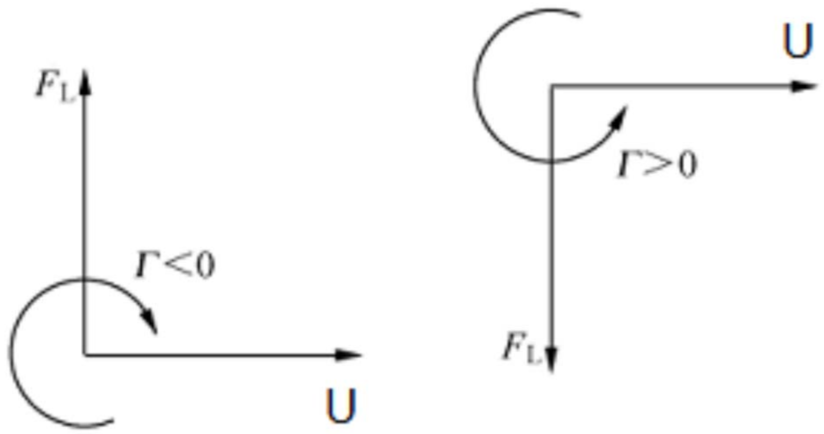

Asymmetricity of flow fields (body)
Flow with circulation
lift
Unsteady flow
acceleration
Added mass
Flow field
Body

Review

Lift -- *Kutta-Joukowski* formula

$$\mathbf{F}_L = \rho \mathbf{U} \times \vec{\Gamma}$$

$$\mathbf{F}_L = \rho \mathbf{U} \times \left(\sum_{i=1}^{n} \vec{\Gamma}_i \right)$$



Direction of a *lift* is the one turning 90° from the direction of the uniform flow against the direction of circulation.

Review

Name	components	Velocity potential	Pressure coefficient	Drag & Lift
Circular Flow	2d Uniform Flow+2d Dipole	$U\cos\theta(r+\frac{a^2}{r})$	$1-4\sin^2\theta$	none
Sphere Flow	3d Uniform Flow+3d Dipole	$U\cos\theta(r+\frac{a^3}{2r^2})$	$1-\frac{9}{4}\sin^2\theta$	none
Circular Flow with Circulation	2d Uniform Flow+2d Dipole + Point Vortex	$U(1+\frac{a^2}{r^2})r\cos\theta+\frac{\Gamma}{2\pi}\theta$	$1-\left(2\sin\theta-\frac{\Gamma}{2\pi aU}\right)^2$	lift
Unsteady Circular Flow	Moving 2d Dipole	$U(t)a^2\frac{\cos\theta}{r}$	---	drag
Unsteady Sphere Flow	Moving 3d Dipole	$\frac{U(t)a^3}{2}\frac{\cos\theta}{r^2}$	---	drag

Review

Added Mass

$$m_{ji} = \rho \iint_B \Phi_i n_j \, dS = \rho \iint_B \Phi_i \frac{\partial \Phi_j}{\partial n} dS = \rho \iiint_{\forall} \nabla \Phi_i \cdot \nabla \Phi_j d\forall$$

Added mass depends on the **shape of the body**, **mode of the motion** and **density of the fluid**. Generally it is a 6×6 symmetric matrix with 36 components, among which only 21 components are independent. If the body has some symmetricity, the independent components will be further reduced.

Review

Added mass coefficient

$$C_m = \frac{m_A}{\rho \forall}$$

	Sphere	Circular Cylinder	Square Cylinder
C_m	$\frac{1}{2}$	1	$\frac{3\pi}{8}$

Review

Added mass **due to a body moving in calm water with acceleration is smaller than the added mass due to an unsteady flow passing the body.**

$$C_{m_{(2)}} = 1 + C_{m_{(1)}}$$