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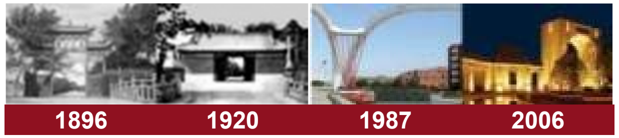



# Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

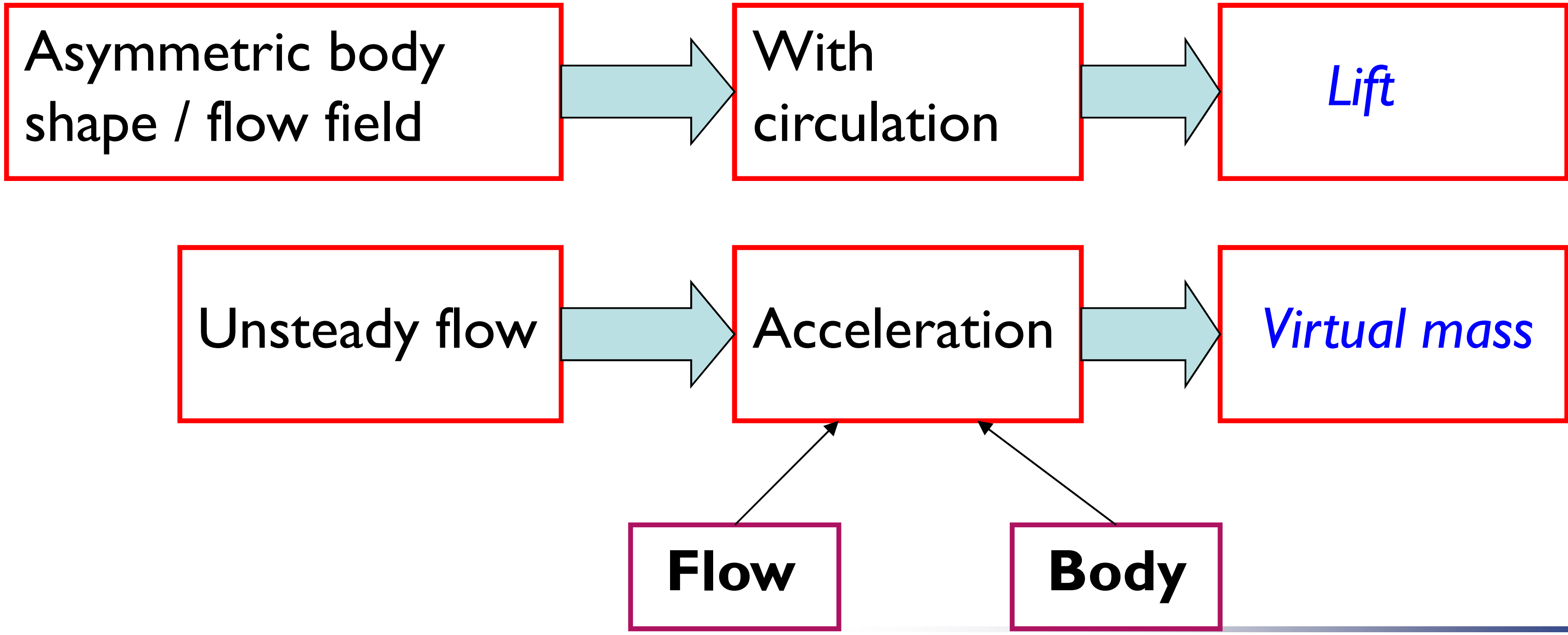
School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

# Chapter 6
# Potential Flow Theory

# 6.5 Potential Flow with Circulation

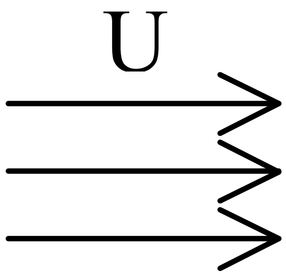
Now we know that whether from the point of view of a uniform flow flows past a fixed body, or from the point of view of a body moves in a calm water at constant velocity, the resultant forces are the same, all vanish (***d'Alembert's Paradox***). Then, **in what kind of potential flows, it will result a nonzero resultant force on the body?**

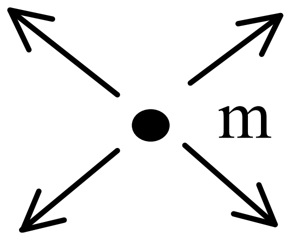


# 6.5 Potential Flow with Circulation

Let's look at flows with circulation. It results asymmetric flow fields.

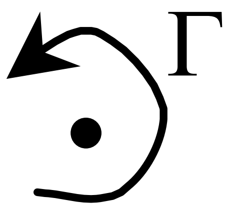
**Without circulation:** **A uniform flow + a point source**




**Symmetric flow fields**

**With circulation:** **A uniform flow + a point vortex**


**Asymmetric flow fields**

# 6.5 Potential Flow with Circulation

Consider a circular cylinder flow with circulation.

A uniform flow + a point dipole + a point vortex.

## (a) A circular flow without circulation

$$\phi_1 = U\left(1+\frac{a^2}{r^2}\right) r\cos\theta, \quad \psi_1 = U\left(1-\frac{a^2}{r^2}\right) r\sin\theta$$

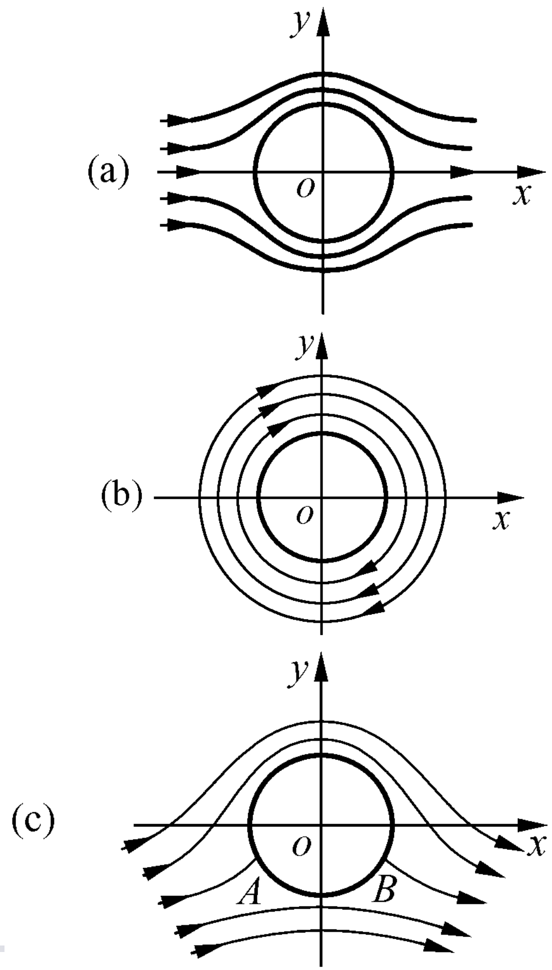
## (b) A point vortex

$$\phi_2 = \frac{\Gamma}{2\pi}\theta, \qquad \psi_2 = -\frac{\Gamma}{2\pi}\ln r$$

## (c) A circular flow with circulation

$$\phi = \phi_1 + \phi_2 = U\left(1+\frac{a^2}{r^2}\right) r\cos\theta + \frac{\Gamma}{2\pi}\theta$$

$$\psi = \psi_1 + \psi_2 = U\left(1-\frac{a^2}{r^2}\right) r\sin\theta - \frac{\Gamma}{2\pi}\ln r$$



# 6.5 Potential Flow with Circulation

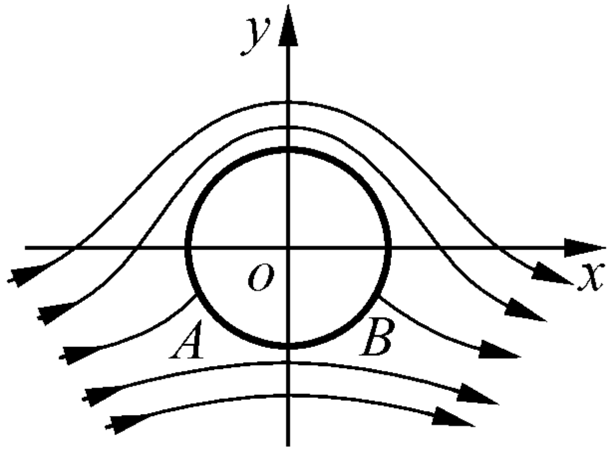
We can see that the superposed flow is asymmetric. On the upper side, the speed is getting higher, while on the lower side, getting slower, because the speed due to the point vortex coincides with or opposes to the ones of circular flow fields. The upper side pressure is reduced and the lower side one is increased. It results a resultant upward force, namely ***lift force***.

Velocities of the superposed flow are written as

$$V_r = \frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta,$$

$$V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

(c)



# 6.5 Potential Flow with Circulation

Now let's confirm the body surface condition and the far field condition.

1. The circle $r = a$ is a streamline, that is, $\psi = C$.

$$\psi = -\frac{\Gamma}{2\pi}\ln a = C$$

Or, on $r = a$ , $V_r = 0$. Fulfill kinematic body surface condition.

2. At far field, $r = \infty$ , $V = U$.

Fulfill the undisturbed condition.

Therefore, the body surface condition and far field condition are all fulfilled.

# 6.5 Potential Flow with Circulation

**Velocity distribution on the circle, *r* = *a*.**

$$\begin{cases} V_r = 0 \\ V_\theta = -2U\sin\theta + \dfrac{\Gamma}{2\pi a} \end{cases}$$

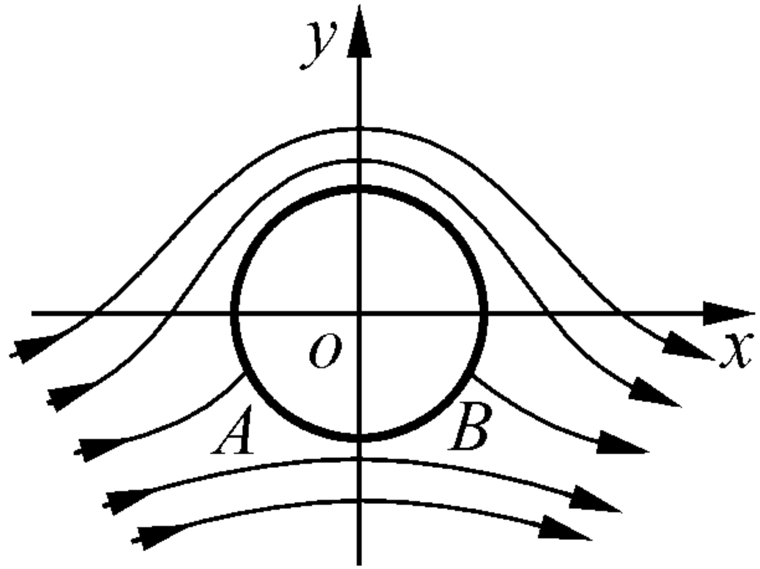
That is, the radial component vanishes, i.e. without separation from the surface, and the tangential component varies with a sine function of angle θ, which is the angle from the direction of the uniform flow to the radial line.

**Location of the *stagnation points*.**

$$V_\theta = 0 \quad \Rightarrow \quad \sin\theta = \frac{\Gamma}{4\pi a U}$$

# 6.5 Potential Flow with Circulation

How many ***stagnation points***?




$$V_\theta = 0 \quad \Rightarrow \quad \sin\theta = \frac{\Gamma}{4\pi a U}$$

(a) If $|\sin\theta| < 1$ and $|\Gamma| < 4\pi a U$ , there are 2 stagnation points.

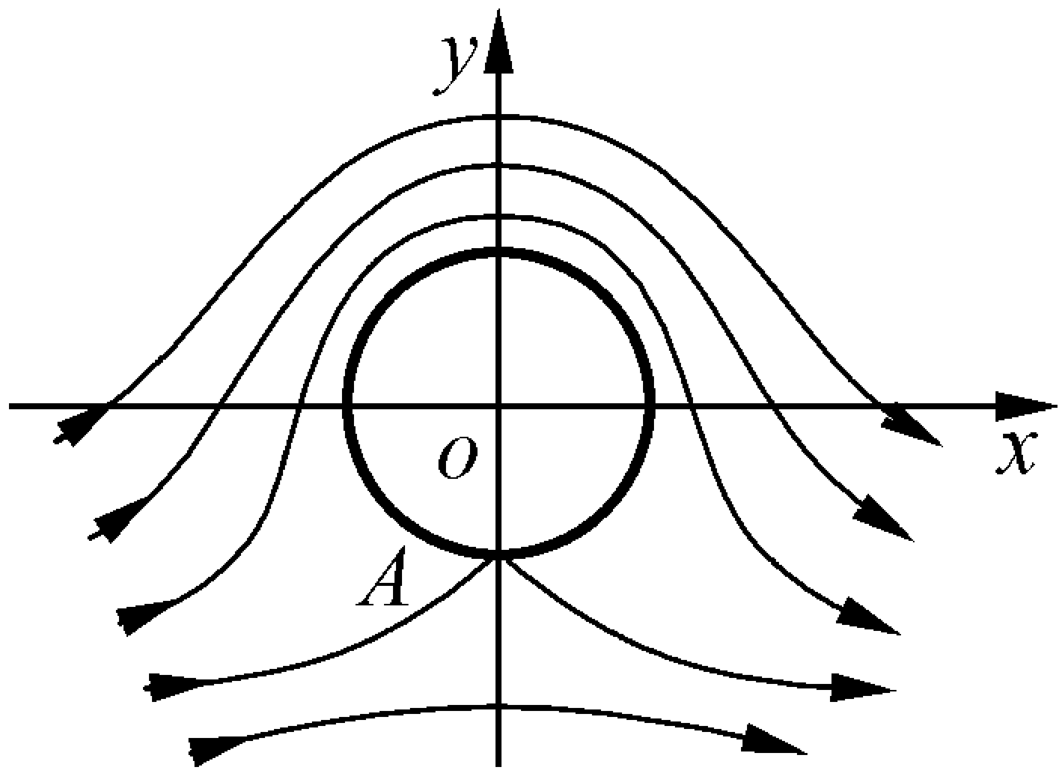
Since $\sin(-\theta) = \sin[-(\pi-\theta)]$ , then $-\theta$ and $-(\pi-\theta)$ are a pair of stagnation points. The larger the circulation, $\Gamma$, the larger the angle, $\theta$, that is, stagnation points are getting nearer to the bottom of the circle.

# 6.5 Potential Flow with Circulation

(b) If $|\Gamma| = 4\pi a U$ , we have $|\sin\theta| = 1 \quad \Rightarrow \quad \theta = -\frac{\pi}{2}$ .

The 2 stagnation points are overlapped.

They become a single stagnation point.

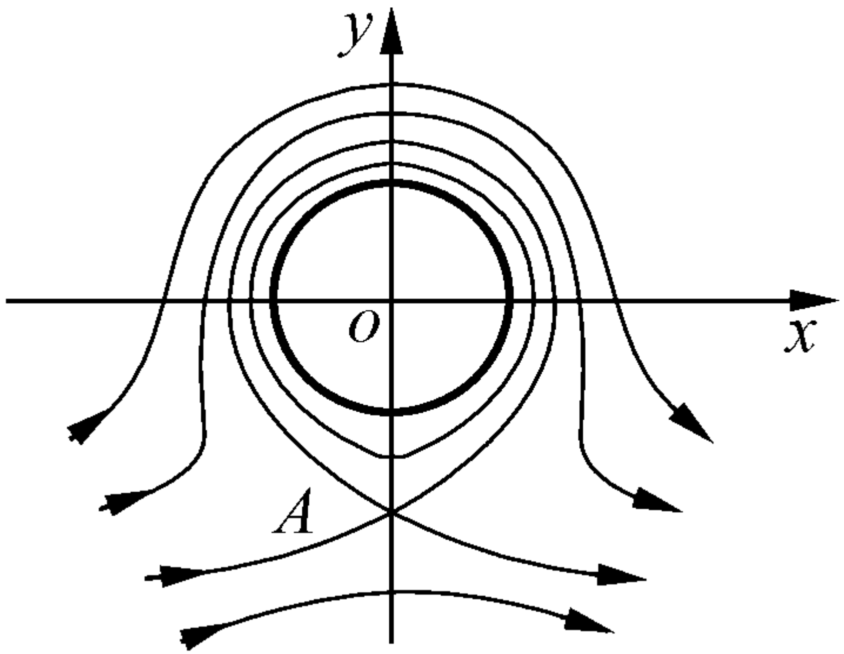


# 6.5 Potential Flow with Circulation

(c) If $|\Gamma| > 4\pi aU$, we have $|\sin\theta| > 1$, there will be no stagnation point on the circle. Solving the following equation, two stagnation points are obtained. One is located inside the circle, and the other is outside of it.

$$V_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta = 0,$$

$$V_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} = 0$$



# 6.5 Potential Flow with Circulation

## Pressure distribution on the circle, *r* = *a*

According to ***Bernoulli's equation***, we have

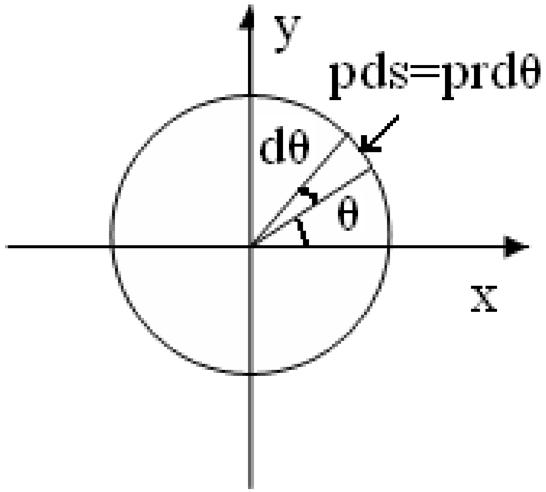
$$p + \frac{1}{2}\rho V_\theta^{\,2} = p_\infty + \frac{1}{2}\rho U^{\,2}$$

It follows

$$p = p_\infty + \frac{1}{2}\rho\left[U^2 - \left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2\right]$$

# 6.5 Potential Flow with Circulation

## *Drag* and *lift* forces on the circular cylinder of unit length




$$F_D = F_x = -\int_0^{2\pi} pa d\theta \cos\theta$$

$$= -\int_0^{2\pi} \left\{ p_\infty + \frac{1}{2}\rho \left[ U^2 - \left( -2U\sin\theta + \frac{\Gamma}{2\pi a} \right)^2 \right] \right\} a\cos\theta d\theta$$

$$F_L = F_y = -\int_0^{2\pi} pa d\theta \sin\theta$$

$$= -\int_0^{2\pi} \left\{ p_\infty + \frac{1}{2}\rho \left[ U^2 - \left( -2U\sin\theta + \frac{\Gamma}{2\pi a} \right)^2 \right] \right\} a\sin\theta d\theta$$

# 6.5 Potential Flow with Circulation

$$F_D = -\int_0^{2\pi}\left\{p_\infty + \frac{1}{2}\rho\left[U^2 - (-2U\sin\theta + \frac{\Gamma}{2\pi a})^2\right]\right\}a\cos\theta d\theta$$

$$= -\int_0^{2\pi}\left\{\cancel{\left[p_\infty + \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\right]} + \frac{1}{2}\rho\left[\frac{2\Gamma U}{\pi a}\sin\theta - \cancel{\left(\frac{\Gamma}{2\pi a}\right)^2}\right]\right\}a\cos\theta d\theta$$

$$= -\frac{\rho\Gamma U}{\pi}\int_0^{2\pi}\sin\theta\cos\theta d\theta = 0$$

***As a result, there is no resultant drag force on the circle!***

**We shall see that the lift force does not vanish.**

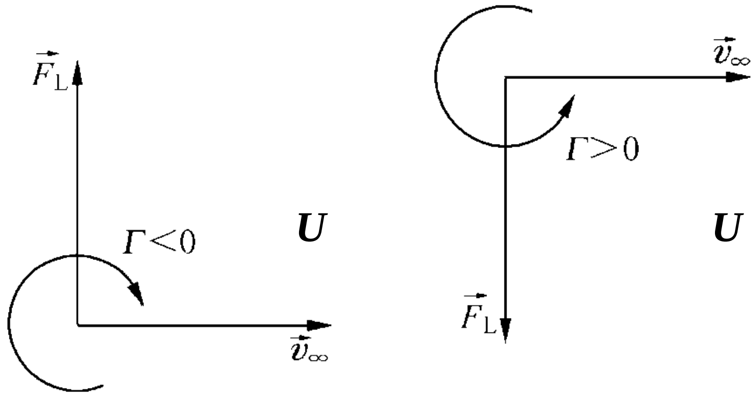
# 6.5 Potential Flow with Circulation

$$F_L = -\int_0^{2\pi} \left\{ p_\infty + \frac{1}{2}\rho\left[ U^2 - \left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2 \right] \right\} a\sin\theta d\theta$$

$$= -\int_0^{2\pi} \left\{ \cancel{\left[ p_\infty + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right) \right]} + \frac{1}{2}\rho\left[ \frac{2\Gamma U}{\pi a}\sin\theta - \cancel{\left(\frac{\Gamma}{2\pi a}\right)^2} \right] \right\} a\sin\theta d\theta$$

$$= -\frac{\rho\Gamma U}{\pi}\int_0^{2\pi}\sin^2\theta\, d\theta = -\rho\Gamma U$$

For the potential flow around a circle with circulation, a *lift force*, which is perpendicular to the uniform flow, is resulted.

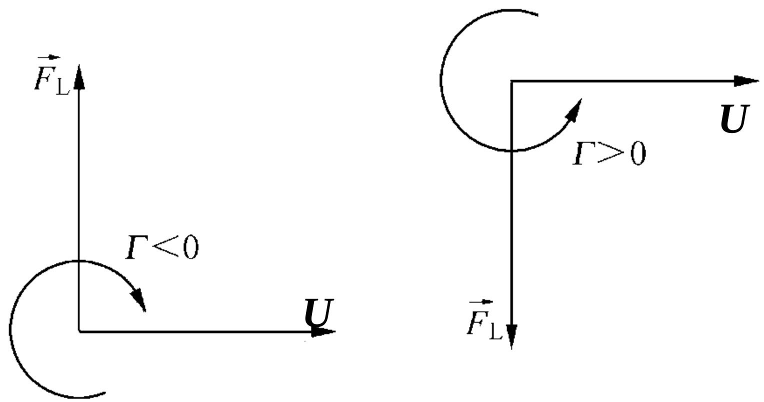



Its magnitude equals the product of **fluid density, speed of the uniform flow** and the **circulation**. Its direction is 90° turning from the direction of the uniform flow against the rounding direction of the circulation.

# 6.5 Potential Flow with Circulation

It is concluded that a uniform flow, *U*, flows passing a body with circulation Γ, a ***lift*** force is generated. It is perpendicular to the uniform flow and its magnitude equals the product of fluid density, uniform flow speed and the circulation, which is named as ***Kutta-Joukowski formula***.

$$\boxed{\mathbf{F}_L = \rho \mathbf{U} \times \vec{\Gamma}}$$



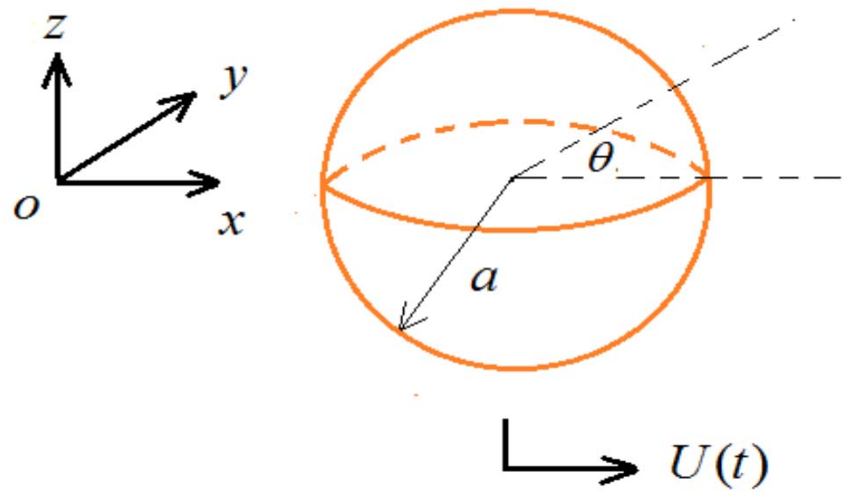

Generally, If a potential flow accompanies with *n* (>1) vortices, the circulation will be replaced by the sum of their circulations. Thus, we get the ***general Kutta-Joukowski formula*** of ***lift*** force.

$$\boxed{\mathbf{F}_L = \rho \mathbf{U} \times \left( \sum_{i=1}^{n} \vec{\Gamma}_i \right)}$$

# 6.6 Potential Flows due to A Body Moving at Varying Speeds

Now we consider a sphere moving in calm water at speed $U(t)$ varying with time along positive *x*-direction, and calculate resultant hydrodynamic force on it.

The reference frame $O(x, y, z)$ is fixed on the earth. Then, the flow is governed by

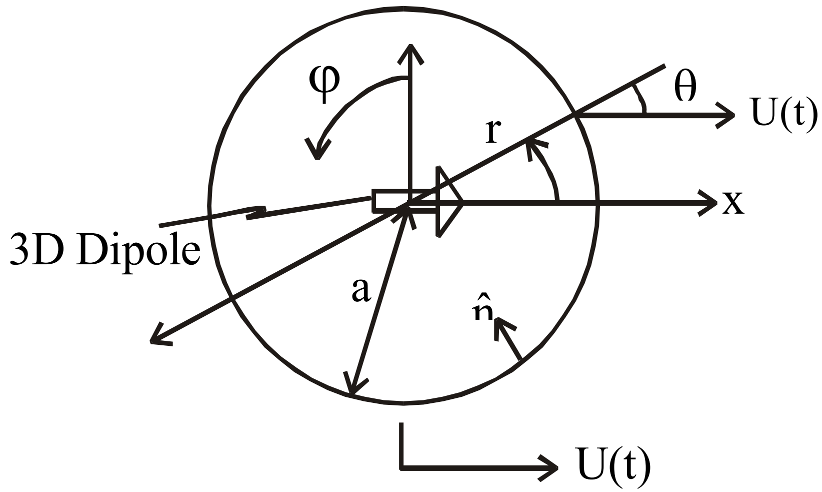



$$\begin{cases} \nabla^2 \phi = 0 \\ \left.\dfrac{\partial \phi}{\partial r}\right|_{r=a} = \mathbf{U} \cdot \mathbf{n} = U(t)\cos\theta, & \text{on sphere} \\ \nabla \phi \to 0, \quad \text{if } |\vec{x}| \to \infty, & \text{at far field} \\ \nabla \phi = \mathbf{U}_0, \quad \text{if } t = 0, & \text{initial cond.} \end{cases}$$

# 6.6 Potential Flows due to A Body Moving at Varying Speeds

This flow is equivalent to a *3d* dipole moving along *x*-axis with a velocity potential

$$\phi = \frac{M}{4\pi}\frac{x}{r^3} = \frac{M}{4\pi}\frac{\cos\theta}{r^2}$$




From ***impermeable*** condition on the sphere, moment *M* of the dipole is determined.

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \frac{\partial}{\partial r}\left.\left(\frac{M}{4\pi}\frac{\cos\theta}{r^2}\right)\right|_{r=a} = -\frac{M}{2\pi}\frac{\cos\theta}{a^3} = U(t)\cos\theta$$

$$\Longrightarrow \quad M = -2\pi a^3 U(t)$$

Thus, the velocity potential is explicitly expressed as

$$\phi = \frac{M}{4\pi}\frac{\cos\theta}{r^2} = -\frac{U(t)a^3}{2}\frac{\cos\theta}{r^2}$$

Substituting it in ***Bernoulli's equation*** (omitting body force), the dynamic pressure is obtained

$$p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2}\right) + C(t)$$

On $x$ direction, the resultant horizontal hydrodynamic force is expressed as

$$F_x = \iint_B p\big|_{r=a} n_x dS = -\rho\iint_B \left(\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2}\right)_{r=a} n_x\, dS$$

## 6.6 Potential Flows due to A Body Moving at Varying Speeds

From the derived expression of the velocity potential function, terms in the integral can be immediately given on the sphere.

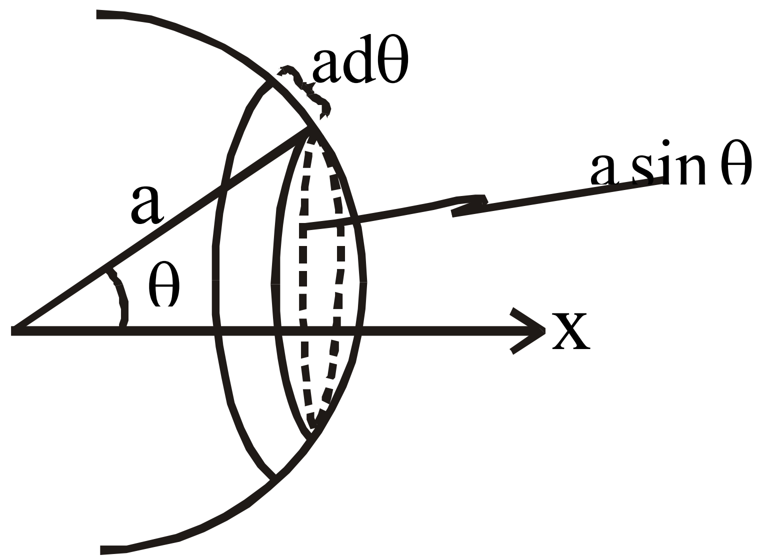
$$\left.\frac{\partial \phi}{\partial t}\right|_{r=a} = -\frac{\mathrm{d}U}{\mathrm{d}t}\left.\frac{a^3\cos\theta}{2r^2}\right|_{r=a} = -\frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}t}a\cos\theta$$

$$\left.\nabla\phi\right|_{r=a} = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

$$= \left(U\cos\theta, \frac{1}{2}U\sin\theta, 0\right)$$

$$\left.\left|\nabla\phi\right|^2\right|_{r=a} = U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta$$

$$\iint_B dS = \int_0^{\pi}(a\,d\theta)(2\pi a\sin\theta)$$



Filling above terms in the integral for horizontal force on the sphere, it gives

$$F_x = -\rho \iint_B \left[ \frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} \right]_{r=a} n_x \, dS$$

$$= -\rho \int_0^\pi \left[ \underbrace{-\frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}t} a\cos\theta}_{\frac{\partial \phi}{\partial t}} + \frac{1}{2}\left( \underbrace{U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta}_{|\nabla\phi|^2} \right) \right] \left( \underbrace{-\cos\theta}_{n_x} \right) \underbrace{\left(2\pi a^2 \sin\theta\right) d\theta}_{dS}$$

$$= -\pi\rho \frac{\mathrm{d}U}{\mathrm{d}t} a^3 \underbrace{\int_0^\pi \left(\sin\theta\cos^2\theta\right) d\theta}_{2/3} + \rho U^2 \pi a^2 \underbrace{\int_0^\pi (\sin\theta\cos\theta)\left( \cos^2\theta + \frac{1}{4}\sin^2\theta \right) d\theta}_{0}$$

$$= -\dot{U}(t)\left[ \rho \frac{2}{3}\pi a^3 \right]$$

That is, the horizontal force is proportional to the sphere's acceleration

$$F_x = -\dot{U}(t)\left[\rho\frac{2}{3}\pi a^3\right]$$

Mass dimension

**Case 1:** If the speed is constant, *i.e.* $\mathrm{d}U(t)/\mathrm{d}t = 0$, again we get $F_x = 0$, the same result as the fore mentioned ***d'Alembert's paradox***.

**Case 2:** If the sphere moves with an acceleration, the resultant horizontal force will be

$$F_x = -\dot{U}(t)\left[\rho\frac{2}{3}\pi a^3\right] = -\dot{U}(t)\cdot m_A$$

Denote $m_A = \rho\frac{2}{3}\pi a^3 = \frac{1}{2}(\rho V)$, called ***added mass*** / ***virtual mass***.

Volume of the sphere

# 6.7 Added Mass

Now we know that when a body moves in calm water with acceleration, it will result a hydrodynamic force **F**, namely *added inertia force*, on it due to the hydrodynamic pressure on the body surface from the surrounded water. It is proportional to the acceleration, and the coefficient has the dimensional of mass, namely ***added mass***, denoted in $m_A$. If an external force **P** is applied to the body of mass $m$, it obeys *Newton's 2nd law*, that is

$$m\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{P} - \mathbf{F} = \mathbf{P} - \dot{\mathbf{U}} \cdot m_A \Longrightarrow \boxed{(m + m_A)\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{P}}$$

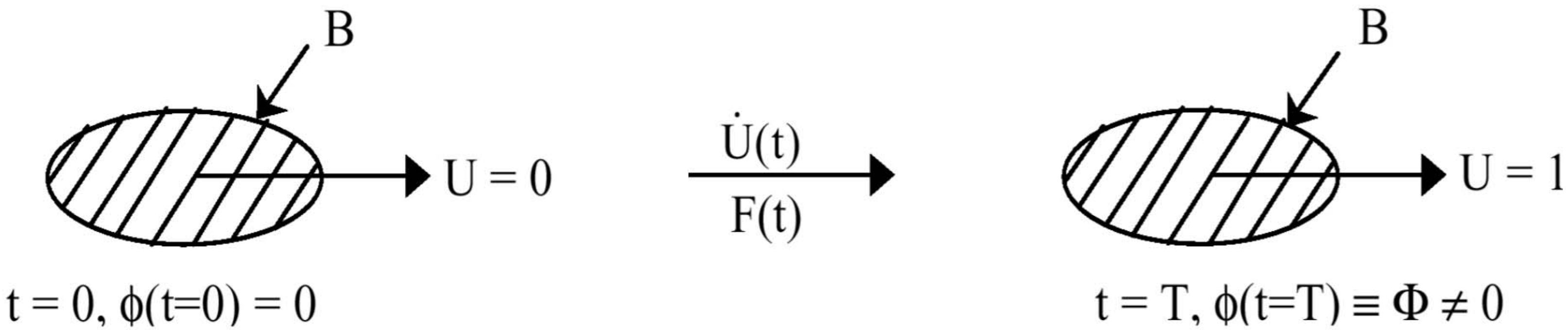
That is, when a body moves in calm water with acceleration, as if it moves in vacuum with an additional mass, *virtual mass* or *added mass*, $m_A$, added to the original mass $m$. In fact, movement of the body will take part of the fluid moving partially together with the body. In some sense, the added mass is a measure of that part of fluid.

# 6.7 Added Mass

## Derivation of the expression of added mass

Consider a body at rest in calm water at $t$ = 0 starting to speed up along $x$-axis, and its speed at $t$ = $T$ increased up to $U$ = 1. Following the *kinetic momentum law*, the change of the kinetic momentum during this course is definitely due to the whole effects of the actions on it, that is

$$\int_0^T \left[-\mathbf{F}(t)\right]dt = \mathbf{M}(t=T) - \mathbf{M}(t=0)$$



# 6.7 Added Mass

Kinetic momentum of the fluid can be expressed in velocity potential

$$\mathbf{M}(t)=\iiint_{\forall}\rho\mathbf{V}d\forall=\iiint_{\forall}\rho\nabla\phi d\forall=\iint_{B}\rho\phi\mathbf{n}dS$$

The component on *x*-axis is simply

$$\mathbf{M}_x(t)=\iint_{B}\rho\phi(t)n_x dS$$

Since at *t* = 0 the fluid is calm water, the velocity potential is a constant, could take the value of 0 without loss of generality, that is

$$\phi(t)=\begin{cases}0 & \text{at } t=0\\ \Phi & \text{at } t=T\end{cases}$$

so

$$\mathbf{M}_x(t)=\begin{cases}0 & \text{at } t=0\\ \iint_{B}\rho\Phi n_x dS & \text{at } t=T\end{cases}$$

# 6.7 Added Mass

We have derived in the last section that for a body moves in calm water at acceleration $\dot{\mathbf{U}}$, the resulted hydrodynamic force on the body is equivalent to an inertial force due to a *virtual mass* (*added mass*), $m_A$. Based on ***Newton's 3rd law***, the body will apply a reaction -**F**, of the same magnitude but opposite direction, to the water.

$$-\mathbf{F}(t) = -\left(-m_A\dot{\mathbf{U}}\right) = m_A\dot{\mathbf{U}}$$

The *x*-component

$$-F_x(t) = m_A\dot{U}$$

$$\int_0^T \left[-F_x(t)\right]dt = \int_0^T m_A\dot{U}dt = m_A U\Big|_{t=0}^{t=T} = m_A$$

# 6.7 Added Mass

Following the fore mentioned *kinetic moment law*

$$\int_0^T \left[-F_x(t)\right]dt = \mathbf{M}_x(t=T) - \mathbf{M}_x(t=0)$$

It results the expression for added mass in velocity potential

$$m_A = \iint_B \rho \Phi n_x dS$$

From the body surface condition at *t* = *T* , we have

$$\frac{\partial \Phi}{\partial n} = \nabla \Phi \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n} = \underbrace{U}_{1} n_x = n_x$$

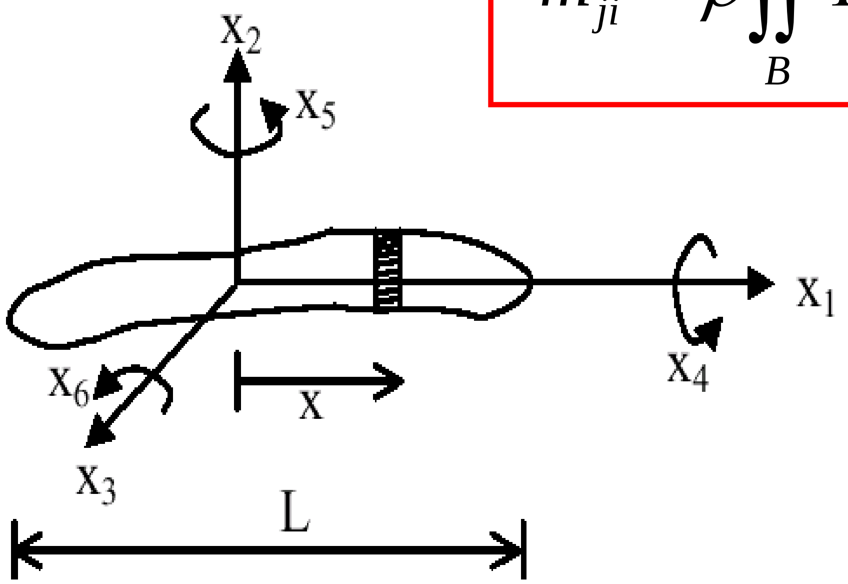
then finally the ***added mass*** is expressed as

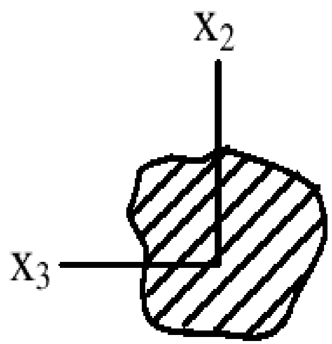
$$\boxed{m_A = \rho \iint_B \Phi \frac{\partial \Phi}{\partial n} dS}$$

# 6.7 Added Mass

The last expression is a special case of *added mass*, *i.e.* the *x*-component of reaction due to the body moving along *x*-axis. Generally, if a body moves in *i*-th degree of freedom among *6 degrees of freedom* (6 DOF), the *j*-th component of *added mass* is expressed as

$$\boxed{m_{ji} = \rho \iint_B \Phi_i n_j \, dS = \rho \iint_B \Phi_i \frac{\partial \Phi_j}{\partial n} dS}$$



# 6.7 Added Mass

## Some properties of added mass

**1. *Added mass* is related to the *shape of the body*, *mode of the motion* and the *fluid density***

From the above expression, added mass is related to the shape of the body, $n_j$ and $B$, mode of motion, $\Phi_i$, and the fluid density, ρ. Combination of $i$ and $j$, gives totally 6×6=36 kinds of *added mass*. For body moves at $i$-th mode with $U_i = 1$, the $j$-th component of the *virtual inertial force* leads to *added mass* component $m_{ji}$

$$m_{ji} = \rho \iint_B \Phi_i \frac{\partial \Phi_j}{\partial n} dS$$

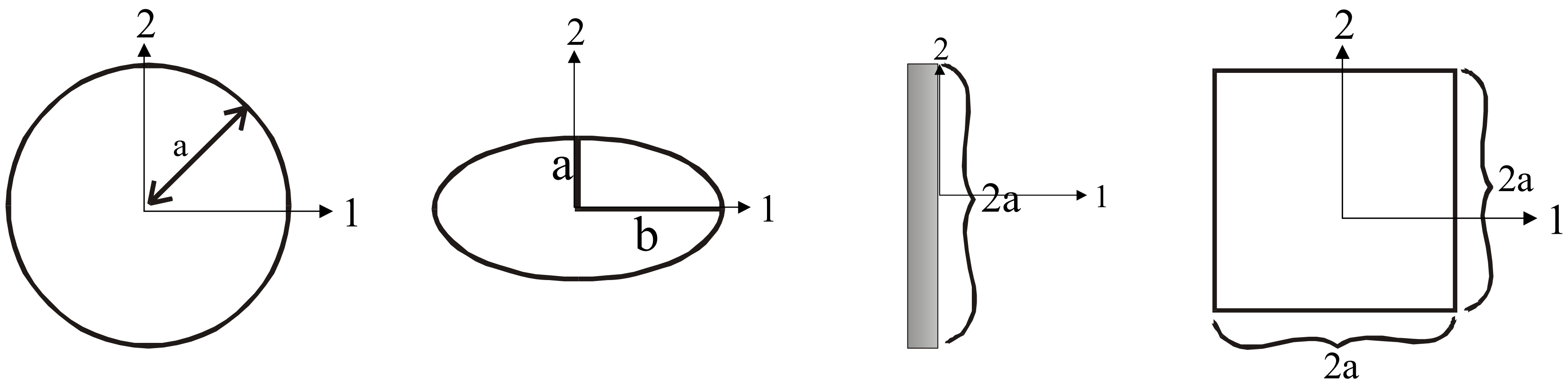
The fluid density

Motion mode

Shape of the body

# 6.7 Added Mass

## Shape of the body and the motion mode




$$m_{11} = m_{22} = \rho \pi a^2$$

$$m_{11} = \rho \pi a^2$$
$$m_{22} = \rho \pi b^2$$

$$m_{11} = \rho \pi a^2$$
$$m_{22} = 0$$

$$m_{11} = m_{22} = \frac{3}{2} \rho \pi a^2$$

## Fluid density

Since the ***added mass*** is proportional to the fluid density, the larger the fluid density is, the greater the added mass is. Thus, the added mass in air is much smaller than the one of the same shaped body in water, so it becomes negligible comparing with its mass. Therefore, ***the added mass in air is generally neglected, while in naval architecture and ocean engineering, the added mass is usually comparable with its mass, and more often is a key factor.*** For example, in maneuvering and seakeeping, ship motions are generally unsteady and added mass always appears in motion equations.

## 2. *Added mass* is a symmetric matrix of order 6

$$m_{ji} = \rho \iint_B \Phi_i \frac{\partial \Phi_j}{\partial n} dS = \rho \iint_B \Phi_i \left( \nabla \Phi_j \cdot \mathbf{n} \right) dS = \rho \oiint_{B+\infty} \Phi_i \left( \nabla \Phi_j \cdot \mathbf{n} \right) dS$$

$$= \rho \iiint_{\mathcal{V}} \nabla \cdot \left( \Phi_i \cdot \nabla \Phi_j \right) d\mathcal{V} = \rho \iiint_{\mathcal{V}} \left( \nabla \Phi_i \cdot \nabla \Phi_j + \cancel{\Phi_i \nabla^2 \Phi_j} \right) d\mathcal{V}$$

$$= \rho \iiint_{\mathcal{V}} \left( \nabla \Phi_i \cdot \nabla \Phi_j \right) d\mathcal{V} = m_{ij}$$

therefore

$$m_{ji} = \rho \iiint_{\mathcal{V}} \left( \nabla \Phi_i \cdot \nabla \Phi_j \right) d\mathcal{V} = m_{ij}$$

and added mass is a symmetric matrix. Because of its symmetricity, ***among 36 components only 21 of them are independent***.

# 6.7 Added Mass

**It can be proved (the proof is omitted here) that**

(a) If a body has a symmetric plane which is chosen as a coordinate plane, in the 21 independent components, 9 of them will vanish, and only 21 - 9 = 12 of them are non zero.

(b) If the body has two symmetric planes which are all coordinate planes, in the 21 independent components, 13 of them will vanish, and only 21 - 13 = 8 of them are non zero.

(c) If the body has three symmetric planes which are all coordinate planes, in the 21 independent components, 15 of them will vanish, and only 21 - 15 = 6 of them are non zero.

## 3. *Added mass* and kinetic energy of the fluid

Suppose a body moving at all 6 motion modes (3 translations, 3 rotations), with velocity (translational and angular) $U_i$ ( $i$ = 1, …, 6), corresponds velocity potential $U_i\Phi_i$ ($\Phi_i$ is denoted as the velocity potential at $U_i$ = 1). The total velocity potential is expressed as a superposition of these velocity potentials $\phi = U_i \Phi_i$ , and the *kinetic energy* of fluid can be written in the total velocity potential as

$$E_F = \iiint_{\forall} \frac{1}{2}\rho V^2 d\forall = \frac{1}{2}\rho \iiint_{\forall} \left(\nabla\phi \cdot \nabla\phi\right) d\forall = \frac{1}{2}\rho \iiint_{\forall} \left(U_i \nabla\Phi_i \cdot U_j \nabla\Phi_j\right) d\forall$$

$$= \frac{1}{2}\rho U_i U_j \iiint_{\forall} \left(\nabla\Phi_i \cdot \nabla\Phi_j\right) d\forall = \frac{1}{2} U_i U_j m_{ij}$$

$$\boxed{E_F = \frac{1}{2} U_i U_j m_{ij}}$$

# 6.7 Added Mass

## 4. *Added mass* coefficient

Added mass coefficient is defined as the ratio of added mass to the mass of fluid displaced by the body, that is

$$\boxed{C_m = \frac{m_A}{\rho \forall}}$$

where $m_A$ is the added mass, $\rho$ is the fluid density and $\forall$ is the volume of the body immersed in the fluid, or displaced by the fluid.