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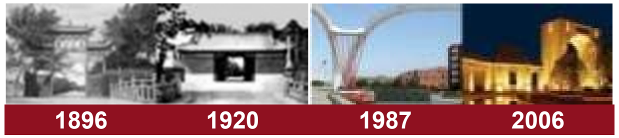


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

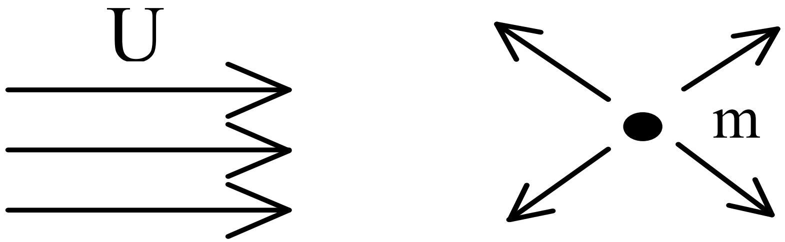
Shanghai Jiao Tong University

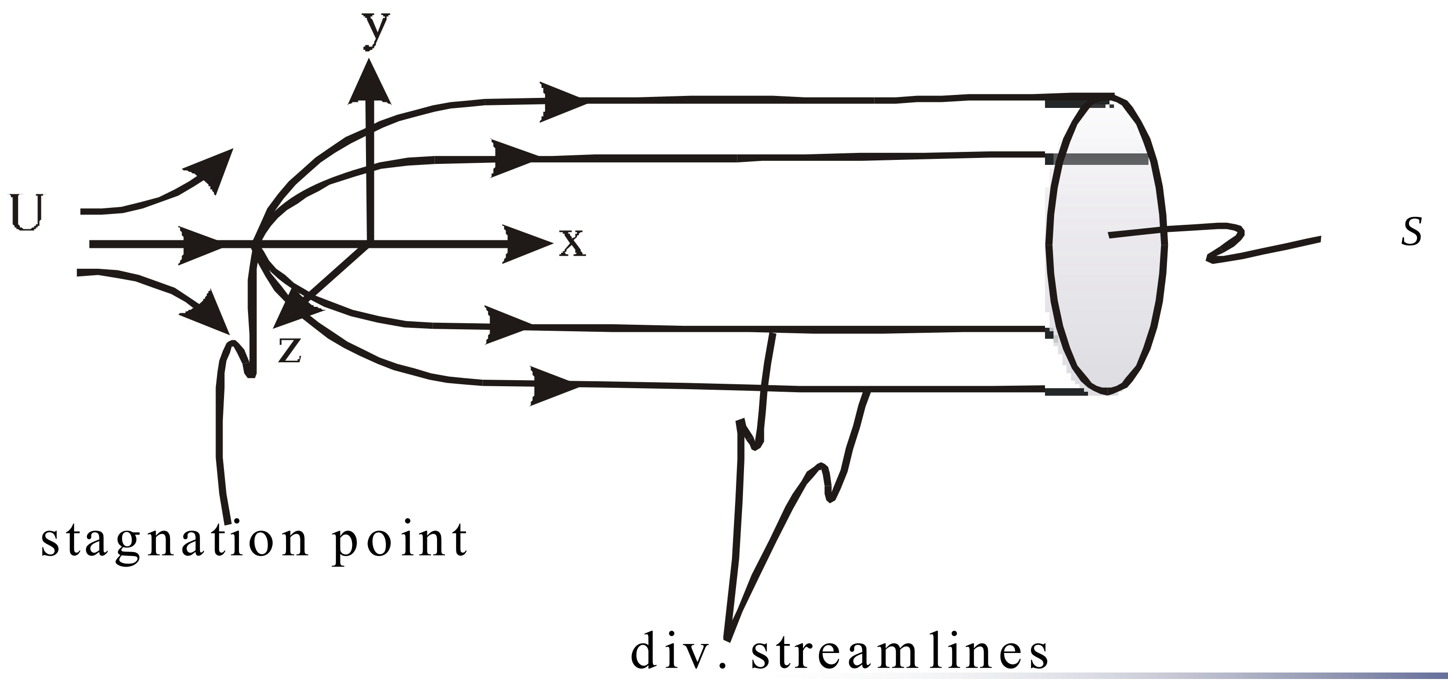
Chapter 6
Potential Flow Theory

6.3 Potential Flow around a Body

Eg.2: *Rankine Half-body Flow*

—— A superposition of *3d uniform flow* and a *point source* flow

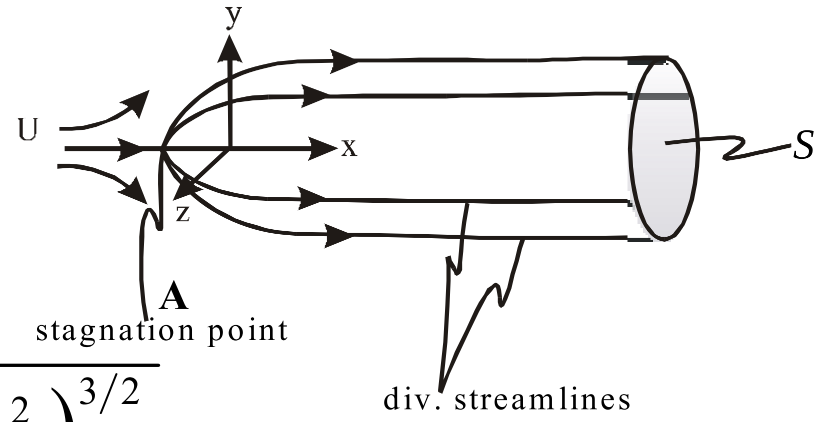




6.3 Potential Flow around a Body

In Cartesian coordinates, the superposed velocity potential is

$$\phi = Ux - \frac{m}{4\pi\sqrt{x^2+y^2+z^2}}$$



velocity:

$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}$$

stagnation point A:

$$u\Big|_{x=x_A,\ y=z=0} = U + \frac{m}{4\pi}\frac{x_A}{|x_A|^3} = 0 \quad \Rightarrow \quad x_A = -\sqrt{\frac{m}{4\pi U}}$$

As $x \to \infty$, we have $u \to U$, the body surface become a stream tube. The mass conservation law tells us that the flow rate equals the point source intensity m. Denote S the cross section area, we have

$$S \cdot U = m \quad \Rightarrow \quad S = \frac{m}{U}$$

6.3 Potential Flow around a Body

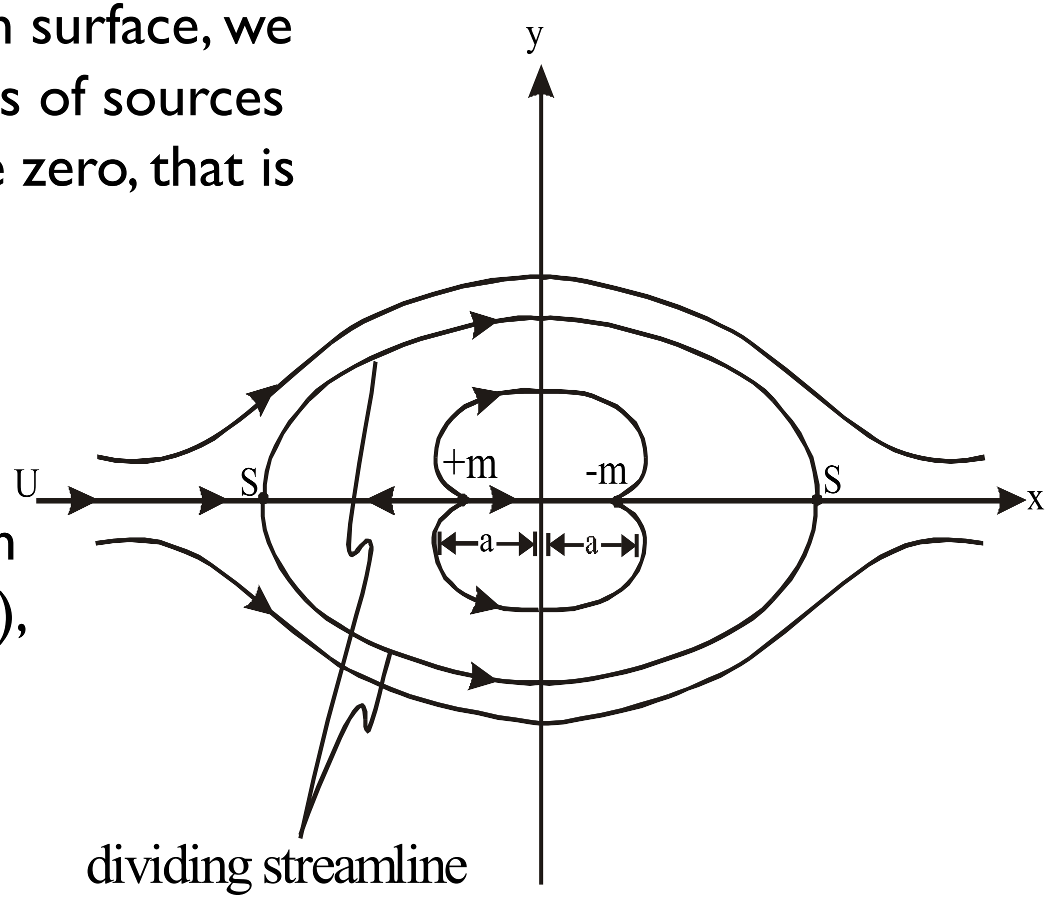
Eg.3： *Rankine Closed-body Flow*

—— A superposition of *3d uniform flow*, a *point source* and a *point sink*

As the body surface is a stream surface, we can guess that the sum of intensities of sources and sinks inside the body should be zero, that is

$$\sum_{\text{in body}} m = 0$$

Thus, we can get a closed stream surface (*Rankine Closed-body surface*), only if the sum of intensities of all point sources and the sum of the ones of all point sinks are of same magnitude.



6.3 Potential Flow around a Body

Denote m the source intensity, then velocity potential of the flow is expressed as

$$\phi = Ux - \frac{m}{2\pi}\left(\frac{1}{\sqrt{(x+a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2}}\right)$$

velocity:

$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\left[\frac{x+a}{\left((x+a)^2+y^2+z^2\right)^{3/2}} - \frac{x-a}{\left((x-a)^2+y^2+z^2\right)^{3/2}}\right]$$

stagnation point :

$$u\Big|_{x=x_S,\ y=z=0} = U + \frac{m}{4\pi}\left[\frac{1}{(x_S+a)^2} - \frac{1}{(x_S-a)^2}\right] = 0$$

$$\Rightarrow \qquad (x_S^2 - a^2)^2 = \left(\frac{m}{4\pi U}\right) 4ax_S$$

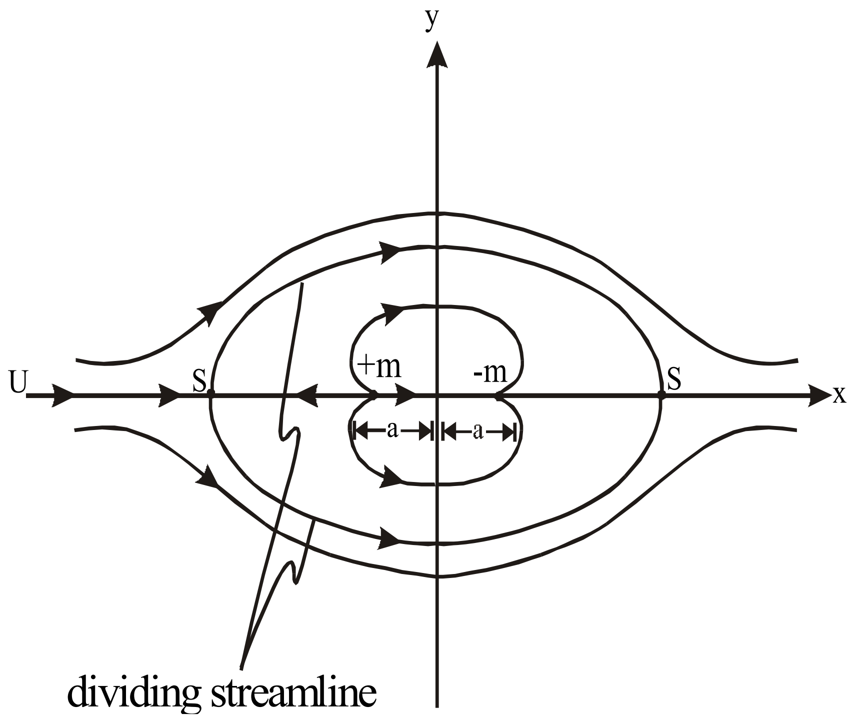
6.3 Potential Flow around a Body

Velocities on the cross section, x = 0, are

$$u\big|_{x=0} = U + \frac{m}{4\pi}\frac{2a}{\left(a^2 + R^2\right)^{3/2}}, \qquad \text{where } R^2 = y^2 + z^2$$

Radius R_0 of the body on the cross section, x = 0, is evaluated by solving the following equation

$$2\pi\int_0^{R_0} u\big|_{x=0} R\,dR = m$$



6.3 Potential Flow around a Body

Eg.4: *Circular Cylinder Flow*

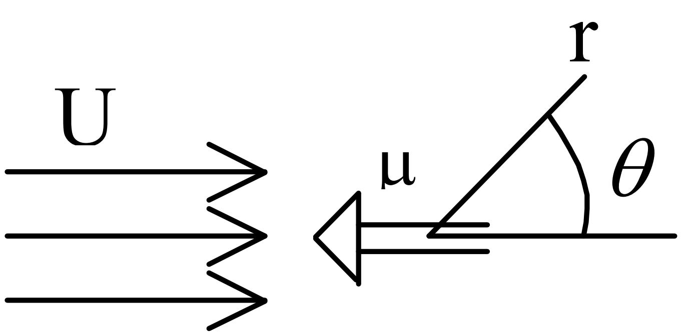
—— A superposition of *2d uniform flow* and a *2d dipole*

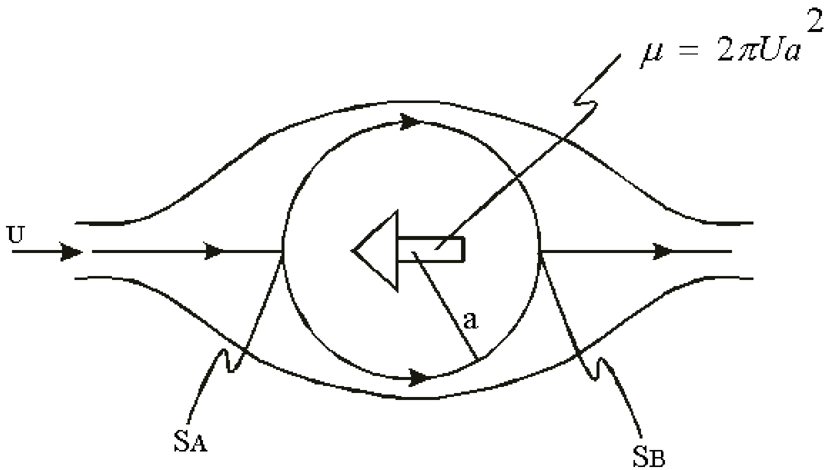
Velocity potential:

$$\phi = Ux + \frac{\mu x}{2\pi r^2} = \cos\theta \left(Ur + \frac{\mu}{2\pi r} \right)$$

Radial velocity:

$$V_r = \frac{\partial \phi}{\partial r} = \cos\theta \left(U - \frac{\mu}{2\pi r^2} \right)$$





6.3 Potential Flow around a Body

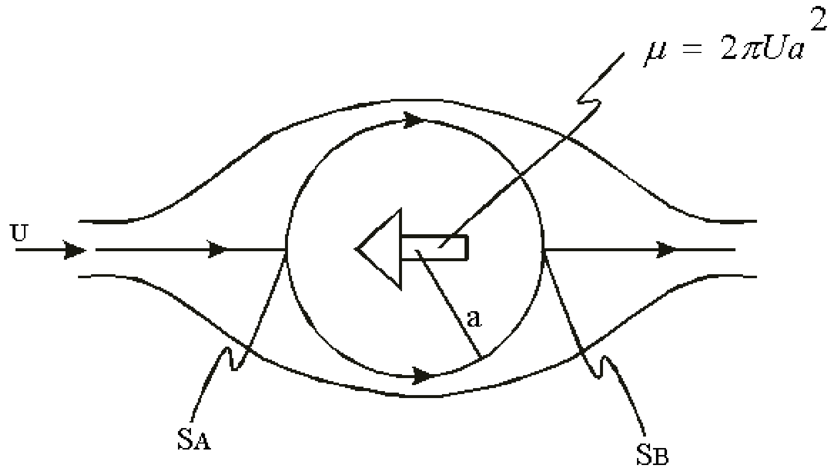
The *kinematic boundary condition* on the body surface, the circle $r = a$ (a is the radius of the circle), requires the radial velocity to be equal to zero, $V_r =$ 0, that is

$$V_r\big|_{r=a} = \cos\theta\left(U - \frac{\mu}{2\pi r^2}\right)_{r=a} = 0$$

$$\Rightarrow \quad a = \sqrt{\frac{\mu}{2\pi U}} \quad \text{or} \quad \mu = 2\pi U a^2$$

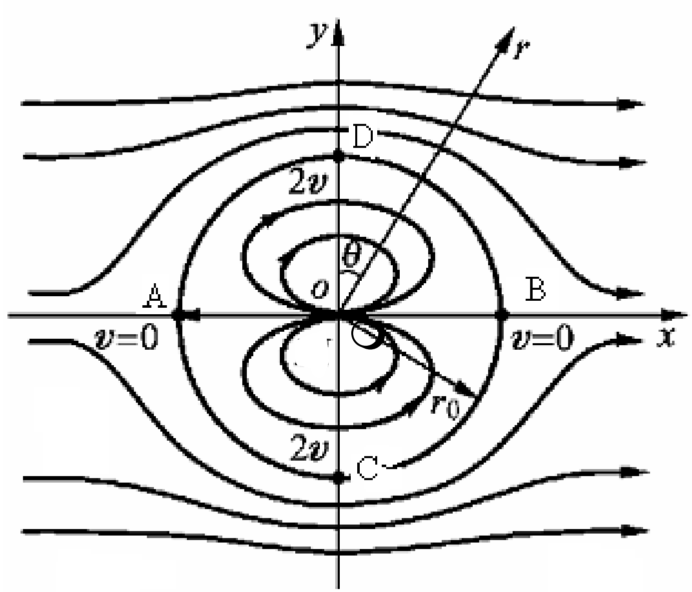
So, velocity potential of the *circular cylinder flow* is rewritten as

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)$$



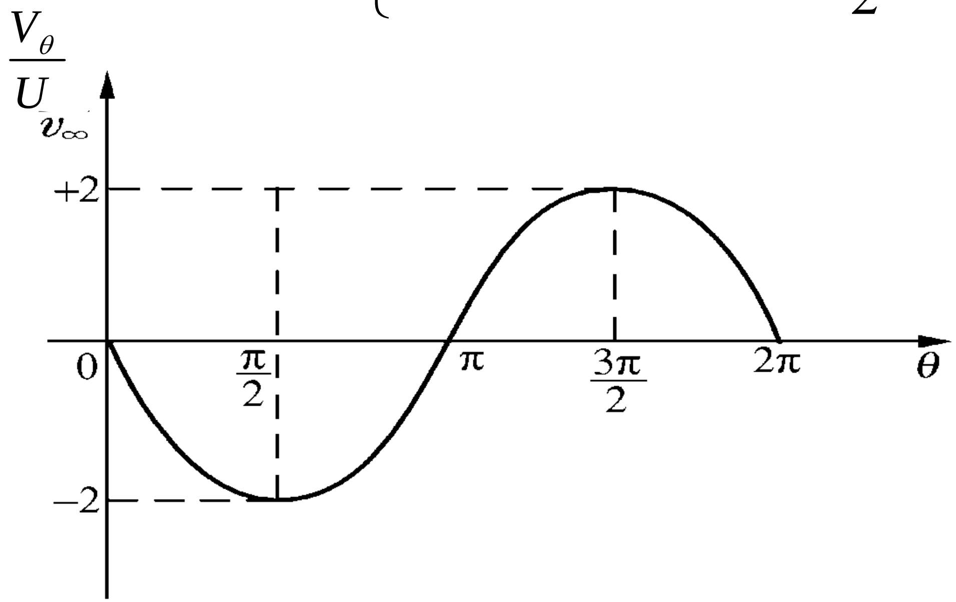
6.3 Potential Flow around a Body

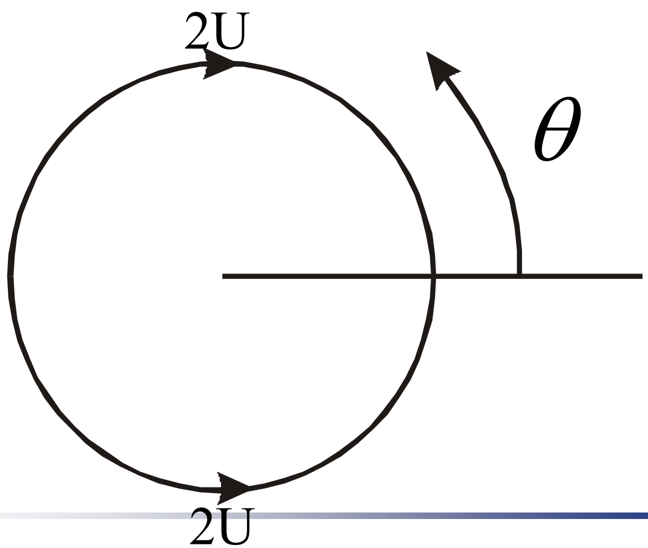
Velocities on the circle is a function of polar angle. Points *A* and *B* are *stagnation points.*



$$V_\theta\Big|_{r=a} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}\Big|_{r=a} = -U\sin\theta\left(1+\frac{a^2}{r^2}\right)\Big|_{r=a}$$

$$= -2U\sin\theta \begin{cases} = 0 & \text{for } \theta = 0, \pi \\ = \mp 2U & \text{for } \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \end{cases}$$





6.3 Potential Flow around a Body

Pressures on the circle can be evaluated by using *Bernoulli's equation*. At the *infinite far field*, velocity is *U, i.e.* the velocity of the uniform flow, and pressure there is denoted by p_∞. Then we have

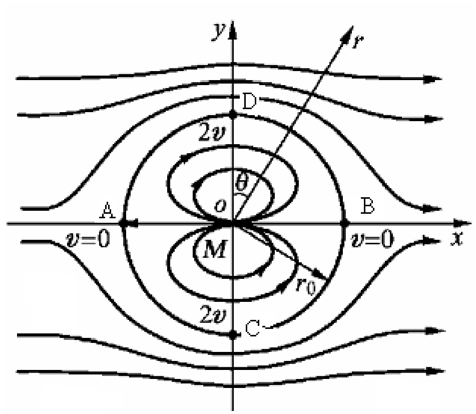
$$\frac{p}{\rho g}+\frac{V_\theta^{\ 2}}{2g}=\frac{p_\infty}{\rho g}+\frac{U^{\ 2}}{2g}$$

$$p=p_\infty+\frac{1}{2}\rho\left(U^2-V_\theta^{\ 2}\right)=p_\infty+\frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)$$

Generally a *pressure coefficient*, which is a dimensionless quantity, is used to express the pressure, it leads to

$$C_p=\frac{p-p_\infty}{\frac{1}{2}\rho U^2}=1-4\sin^2\theta$$

6.3 Potential Flow around a Body

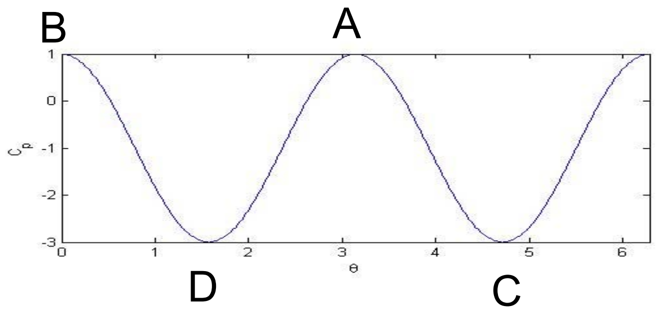


And pressure on the circle can be expressed with *pressure coefficient*

$$p = p_\infty + \frac{1}{2} C_p \rho U^2, \qquad C_p = 1 - 4\sin^2\theta$$

Discussions:

1. *Stagnation points* (point A and B)



$$V_r = V_\theta = 0, \quad C_p = 1, \quad p_{\max} = p_\infty + \frac{1}{2}\rho U^2$$

2. Point C and D takes the lowest pressure.

$$V_r = 0, \quad V_{\theta,\max} = 2U, \quad C_p = -3, \quad p_{\min} = p_\infty - \frac{3}{2}\rho U^2$$

3. Pressures on the circle are symmetry up ($0^\circ \le \theta \le 180^\circ$) and down ($180^\circ \le \theta \le 360^\circ$), that is, *vertically symmetry*, and will be vertically balanced.

6.3 Potential Flow around a Body

Eg.5： ***Sphere Flow*** (Flow around a sphere)

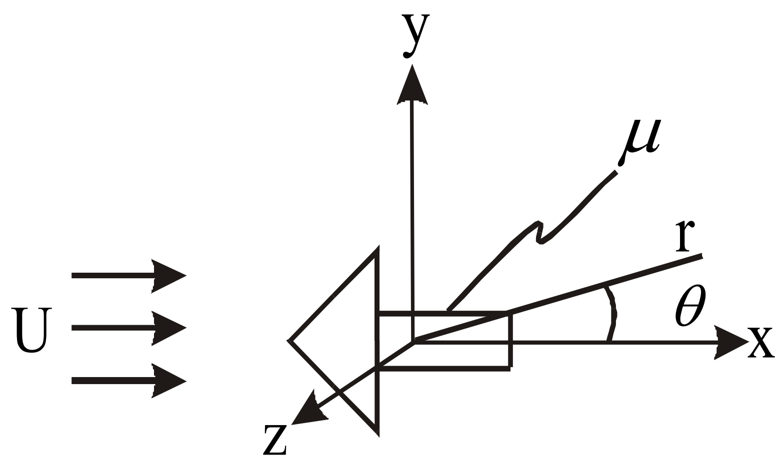
—— a superposition of a 3d uniform flow and a 3d dipole

Velocity potential:

$$\phi = Ux + \frac{\mu \cos\theta}{4\pi r^2}$$

Radial velocity:

$$V_r = \frac{\partial \phi}{\partial r} = \cos\theta \left(U - \frac{\mu}{2\pi r^3} \right)$$



6.3 Potential Flow around a Body

On the sphere, ***r* = *a* (*a*** is the radius **)** , the ***impermeable*** body surface condition requires radial velocity to be zero, $V_r = 0$, that is

$$V_r \big|_{r=a} = \cos\theta \left(U - \frac{\mu}{2\pi r^3} \right)_{r=a} = 0$$

$$\Rightarrow \quad a = \sqrt[3]{\frac{\mu}{2\pi U}} \quad \text{or} \quad \mu = 2\pi U a^3$$

In terms of this result, the velocity potential of the sphere flow can be written as

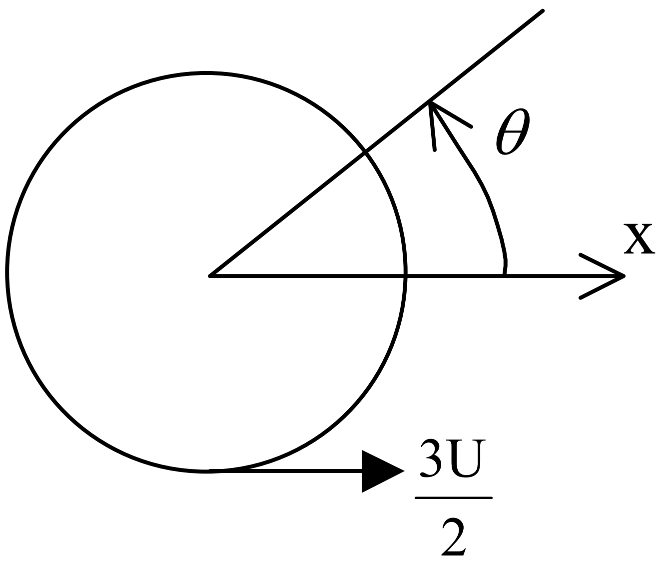
$$\phi = U \cos\theta \left(r + \frac{a^3}{2r^2} \right)$$

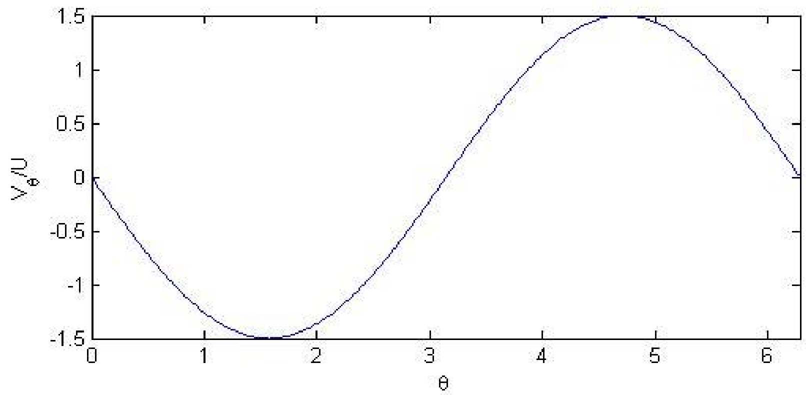
6.3 Potential Flow around a Body

Velocities on the sphere:

$$V_\theta\Big|_{r=a} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r=a} = -U\sin\theta\left(1+\frac{a^3}{2r^3}\right)\Big|_{r=a}$$

$$= -\frac{3U}{2}\sin\theta = \begin{cases} 0 & \text{for } \theta = 0, \pi \\ -\dfrac{3U}{2} & \text{for } \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \end{cases}$$





6.3 Potential Flow around a Body

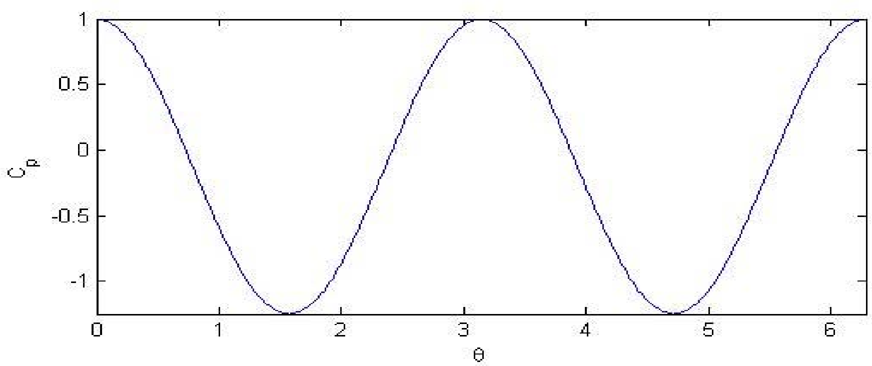
Pressure distribution on the sphere can be evaluated by using *Bernoulli's equation*. At the infinitely far place, velocity is ***U***, i.e. velocity of the uniform flow, pressure there is denoted as p_∞ , then

$$\frac{p}{\rho g}+\frac{V_\theta^{\;2}}{2g}=\frac{p_\infty}{\rho g}+\frac{U^2}{2g}$$

$$p=p_\infty+\frac{1}{2}\rho\left(U^2-V_\theta^{\;2}\right)=p_\infty+\frac{1}{2}\rho U^2\left(1-\frac{9}{4}\sin^2\theta\right)$$

Expressed in a form of *pressure coefficient*

$$C_p=\frac{p-p_\infty}{\frac{1}{2}\rho U^2}=1-\frac{9}{4}\sin^2\theta$$

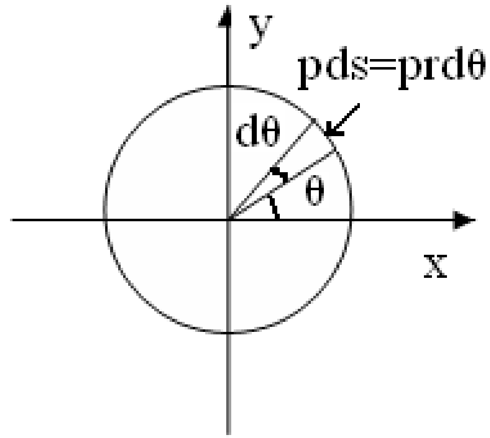


6.3 Potential Flow around a Body

We can see that in 2d circular flow and 3d sphere flow, pressure distribution are symmetric, both horizontally and vertically, so neither a *drag* force nor a *lift* force can be resulted, as following integrals show.

$$F_D = F_x = -\int_0^{2\pi} a\left[p_\infty + \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\right]\cos\theta d\theta = 0$$

$$F_L = F_y = -\int_0^{2\pi} a\left[p_\infty + \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\right]\sin\theta d\theta = 0$$



This result is historically concluded as ***d'Alembert's paradox***: When a uniform flow flows past a rigid body, it generates neither ***drag*** force nor ***lift*** force, provided the flow is in the regime of potential flow and without a circulation around the body. This is apparently contradict with our daily experiences. The reason is possibly due to the omission of fluid viscosity in our consideration, while in reality, a fluid is more or less with some viscosity.

6.4 Potential Flow Generated by a Moving Body

In practice, a body moving in calm water (or fluid), such as a ship travelling in sea, an airplane flying on sky, is more common rather than the whole fluid flows passing a fixed body. Now questions arise

1) Whether those two flows (flow fields) are the same? In other words, whether a uniform flow passing a fixed (steady) body is equivalent to the flow generated by a body moving in calm water (fluid) at a constant velocity?

2) Whether the flow generated by a body moving in calm water (fluid) is steady? Or unsteady?

3) Whether a *drag* or a *lift* is resulted in the moving body flow? Whether the ***d'Alembert's paradox*** is still derived?

6.4 Potential Flow Generated by a Moving Body

At first, let's consider the 2nd question: ***Is the moving body flow steady or unsteady?*** We can conclude that it depends on the choice of reference frame. Generally, we consider a body moving in calm water at a constant speed *U* along *x*-axis.

In an earth-fixed system, say (O, x, y, z), the flow is unsteady. It is governed by

$$
\begin{cases}
\nabla^2 \phi = 0 \\
\mathbf{V} \cdot \mathbf{n} = \dfrac{\partial \phi}{\partial n} = \mathbf{U} \cdot \mathbf{n} = (U, 0, 0) \cdot (n_x, n_y, n_z) = U n_x, \quad \text{on body surf.} \\
\mathbf{V} \to 0, \quad \phi \to 0, \qquad \text{if } |\mathbf{x}| \to \infty, \qquad \text{at far field} \\
\mathbf{V} = \nabla \phi = \mathbf{U}_0, \qquad \text{if } t = 0, \qquad \text{at initial instant}
\end{cases}
$$

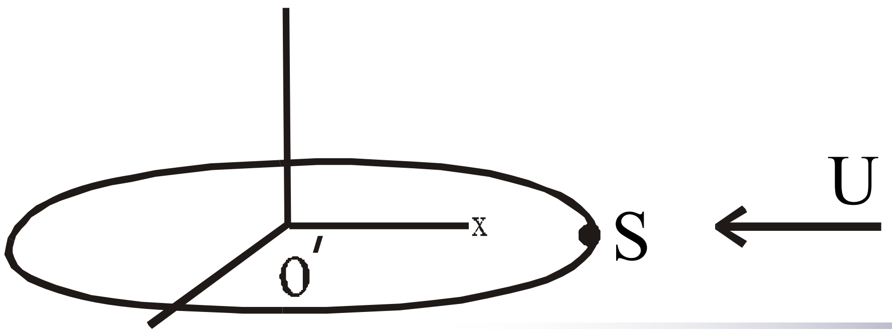
6.4 Potential Flow Generated by a Moving Body

In a body-fixed system, say (O', x', y', z'), the flow will be steady.

It is governed by

$$\begin{cases} \nabla^2 \phi' = 0 \\ \mathbf{V}' \cdot \mathbf{n}' = \dfrac{\partial \phi'}{\partial n'} = 0, \qquad \text{on body surface} \\ \mathbf{V}' \to (-U, 0, 0), \quad \phi' \to -Ux, \quad \text{if } |\mathbf{x}'| \to \infty, \quad \text{at far field} \end{cases}$$

It is the same as a uniform flow passing a fixed body.



6.4 Potential Flow Generated by a Moving Body

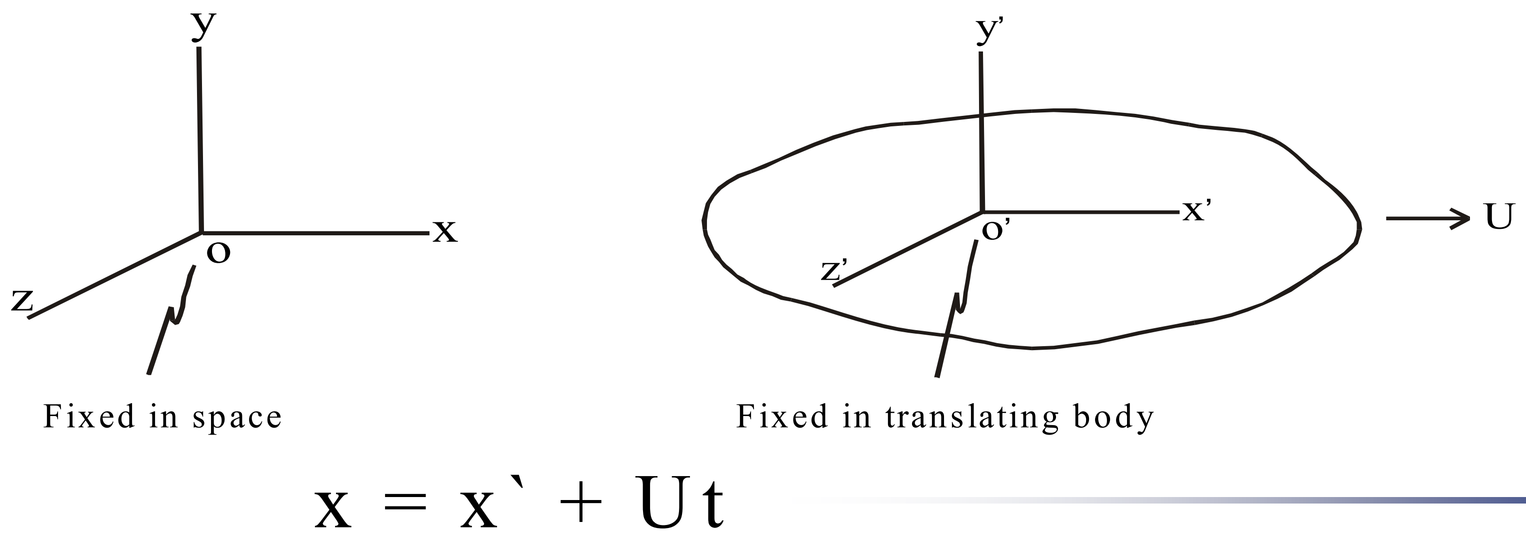
The 2 different descriptions may be transformed with each other. Following is the coordinate system transformation.

$$\mathbf{x} = \mathbf{x}' + \mathbf{U} \cdot t$$

$$\mathbf{V}(x, y, z, t) = \mathbf{V}'(x', y', z') + \mathbf{U} = \mathbf{V}'(x - Ut, y, z) + \mathbf{U}$$

$$\phi(x, y, z, t) = \phi'(x', y', z') + Ux' = \phi'(x - Ut, y, z) + Ux'$$

$$-Ux' + \phi(x' + Ut, y', z', t) = \phi'(x', y', z')$$



6.4 Potential Flow Generated by a Moving Body

Following the dynamic condition (ignoring body forces), at far field, the difference between the two reference frames is

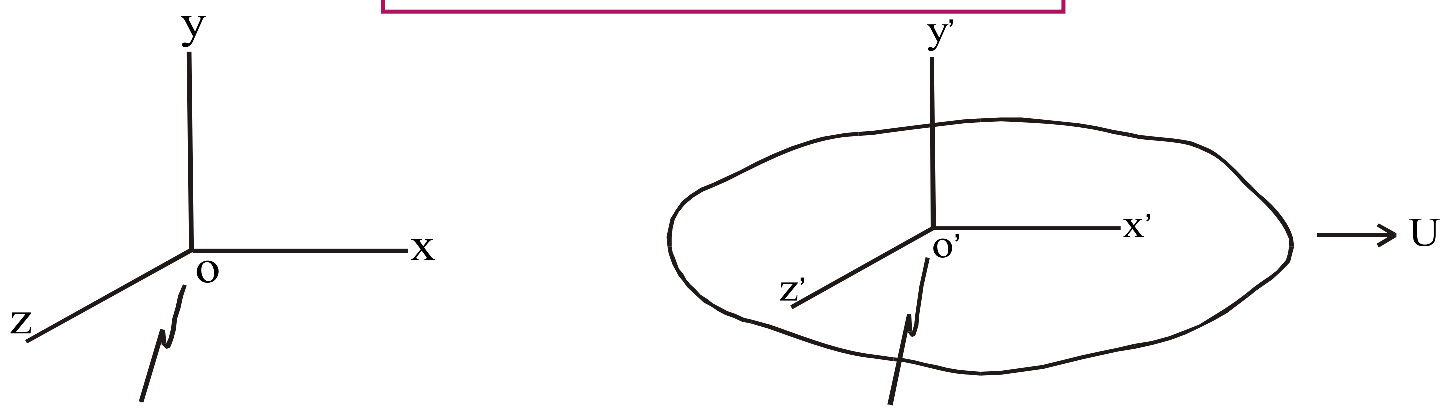
$$p_\infty + \frac{1}{2}\rho V_\infty^2 = C_O(t)$$

$$p_\infty = C_O(t)$$

$$p_\infty + \frac{1}{2}\rho V_\infty'^2 = C_O'$$

$$p_\infty = C_O' - \frac{1}{2}\rho U^2$$

$$C_O(t) = C_O' - \frac{1}{2}\rho U^2$$



Fixed in space

Fixed in translating body

$$x = x` + Ut$$

6.4 Potential Flow Generated by a Moving Body

According to the dynamic condition (ignoring body force), ***Bernoulli equation***, at the ***stagnation point***, *S*, in the earth-fixed system **O**

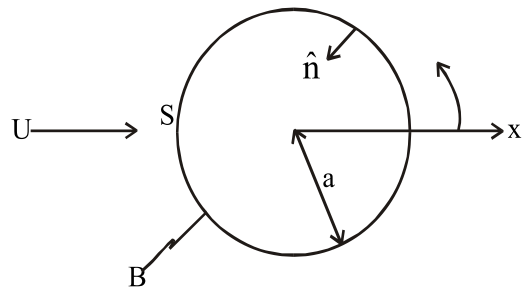
$$p_S = -\rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho V^2 + C_O(t) = -\rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho U^2 + C_O(t)$$

-U

$$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial t}(\phi' + Ux') = \frac{\partial\phi'}{\partial t} + U\frac{\partial x'}{\partial t} = -U^2$$

Therefore

$$p_S = -\rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho U^2 + C_O(t) = \frac{1}{2}\rho U^2 + C_O(t)$$

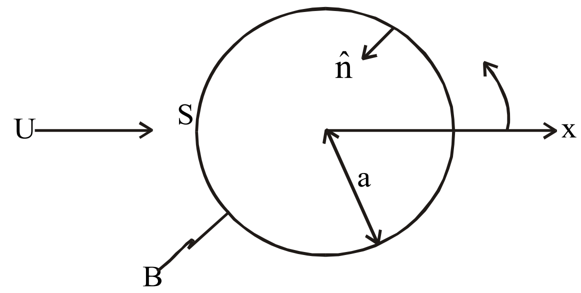


$$p_S - p_\infty = \frac{1}{2}\rho U^2$$

6.4 Potential Flow Generated by a Moving Body

In body-fixed system O', pressure at the *stagnation point* S can be obtained immediately

$$p_S = -\rho \frac{\partial \phi'}{\partial t} - \frac{1}{2}\rho V'^2 + C'_O = C'_O$$



that is,

$$p_S - p_\infty = \frac{1}{2}\rho U^2$$

Also pressures on the body surface can be readily obtained

$$p = -\rho \frac{\partial \phi'}{\partial t} - \frac{1}{2}\rho V'^2 + C'_O = -\frac{1}{2}\rho V'^2 + C'_O$$

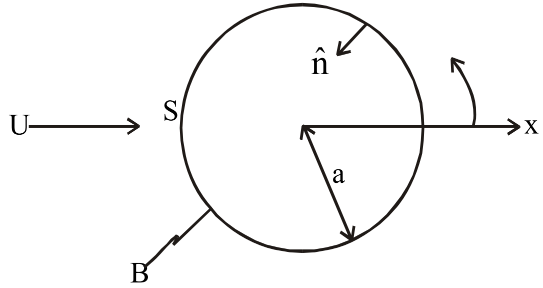
$$p - p_\infty = \frac{1}{2}\rho\left(U^2 - V'^2\right) = \frac{1}{2}\rho\left(U^2 - \left|\nabla \phi'\right|^2\right)$$

Eg. **Calculate resultant force on a circular cylinder moving at constant speed *U*.**

Solution: The resultant force is an integral of pressures on *B*

$$\mathbf{F}_S = \iint_B p\mathbf{n}dS = \frac{1}{2}\rho\iint_B \left(\frac{p_\infty}{\rho} + U^2 - \left|\nabla\phi'\right|^2\right)\mathbf{n}dS$$

$$= -\frac{1}{2}\rho\iint_B \left|\nabla\phi'\right|^2 \mathbf{n}dS = -\frac{1}{2}\rho\int_0^{2\pi} \left|\nabla\phi'\right|^2_{r=a} \mathbf{n}ad\theta$$



$$F_x = \mathbf{F}_S \cdot \mathbf{i} = -\frac{\rho a}{2}\int_0^{2\pi} \left|\nabla\phi'\right|^2_{r=a} \mathbf{n}\cdot\mathbf{i}\,d\theta = \frac{\rho a}{2}\int_0^{2\pi} \left|\nabla\phi'\right|^2_{r=a} \cos\theta\, d\theta$$

$$F_y = \mathbf{F}_S \cdot \mathbf{j} = -\frac{\rho a}{2}\int_0^{2\pi} \left|\nabla\phi'\right|^2_{r=a} \mathbf{n}\cdot\mathbf{j}\,d\theta = \frac{\rho a}{2}\int_0^{2\pi} \left|\nabla\phi'\right|^2_{r=a} \sin\theta\, d\theta$$

6.4 Potential Flow Generated by a Moving Body

Where Φ' is the velocity potential of a uniform flow around a fixed body, *i.e.*

$$\phi' = U\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$\nabla\phi' = \left(\frac{\partial\phi'}{\partial r}, \frac{\partial\phi'}{r\partial\theta}\right)_{r=a} = (0, -2U\sin\theta)$$

therefore

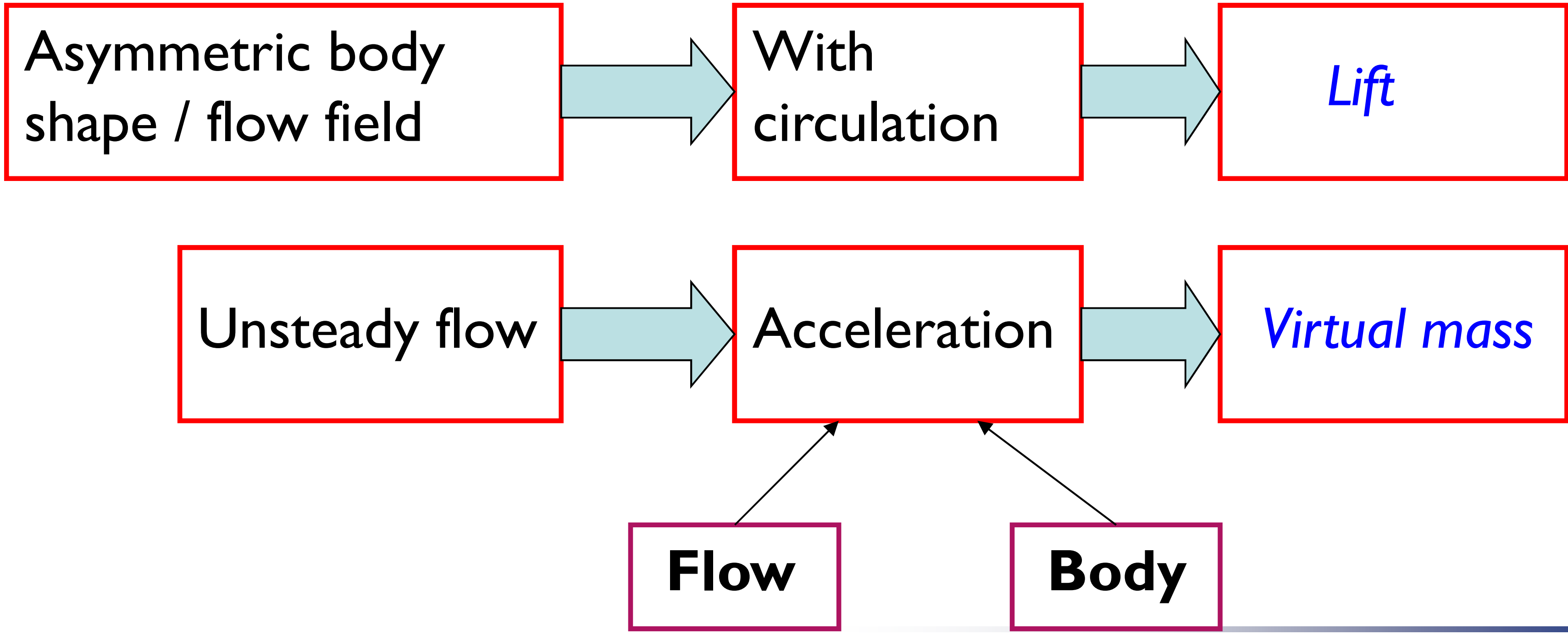
$$F_x = \frac{\rho a}{2}\int_0^{2\pi}\left|\nabla\phi'\right|_{r=a}^2\cos\theta d\theta = \frac{\rho a}{2}\int_0^{2\pi}\left(4U^2\sin^2\theta\cos\theta\right)d\theta = 0$$

$$F_y = \frac{\rho a}{2}\int_0^{2\pi}\left|\nabla\phi'\right|_{r=a}^2\sin\theta d\theta = \frac{\rho a}{2}\int_0^{2\pi}\left(4U^2\sin^2\theta\sin\theta\right)d\theta = 0$$

Conclusion: When a body moves in a calm water at a constant velocity, the resultant force on the body vanishes too, just like the one of a uniform flow flows passing a fixed body (***d'Alembert's paradox***).

6.5 Potential Flow with Circulation

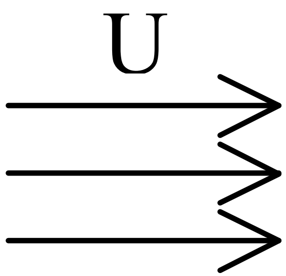
Now we know that whether from the point of view of a uniform flow flows past a fixed body, or from the point of view of a body moves in a calm water at constant velocity, the resultant forces are the same, all vanish (***d'Alembert's Paradox***). Then, **in what kind of potential flows, it will result a nonzero resultant force on the body?**

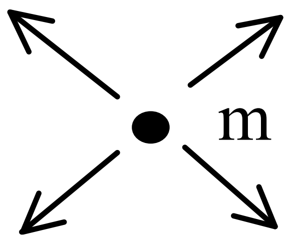


6.5 Potential Flow with Circulation

Let's look at flows with circulation. It results asymmetric flow fields.

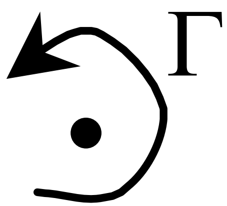
Without circulation: **A uniform flow + a point source**





Symmetric flow fields

With circulation: **A uniform flow + a point vortex**



Asymmetric flow fields

6.5 Potential Flow with Circulation

Consider a circular cylinder flow with circulation.

A uniform flow + a point dipole + a point vortex.

(a) A circular flow without circulation

$$\phi_1 = U\left(1+\frac{a^2}{r^2}\right)r\cos\theta, \quad \psi_1 = U\left(1-\frac{a^2}{r^2}\right)r\sin\theta$$

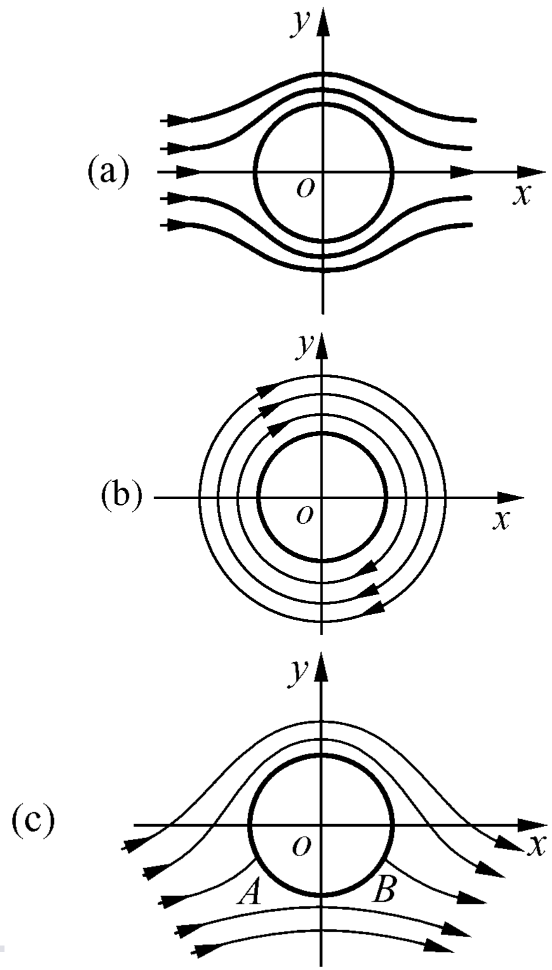
(b) A point vortex

$$\phi_2 = \frac{\Gamma}{2\pi}\theta, \qquad \psi_2 = -\frac{\Gamma}{2\pi}\ln r$$

(c) A circular flow with circulation

$$\phi = \phi_1 + \phi_2 = U\left(1+\frac{a^2}{r^2}\right)r\cos\theta + \frac{\Gamma}{2\pi}\theta$$

$$\psi = \psi_1 + \psi_2 = U\left(1-\frac{a^2}{r^2}\right)r\sin\theta - \frac{\Gamma}{2\pi}\ln r$$



6.5 Potential Flow with Circulation

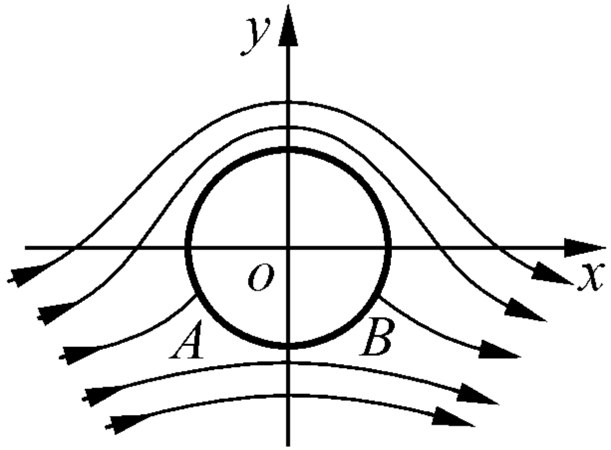
We can see that the superposed flow is asymmetric. On the upper side, the speed is getting higher, while on the lower side, getting slower, because the speed due to the point vortex coincides with or opposes to the ones of circular flow fields. The upper side pressure is reduced and the lower side one is increased. It results a resultant upward force, namely ***lift force***.

Velocities of the superposed flow are written as

$$V_r = \frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta,$$

$$V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

(c)



6.5 Potential Flow with Circulation

Now let's confirm the body surface condition and the far field condition.

1. The circle $r = a$ is a streamline, that is, $\psi = C$.

$$\psi = -\frac{\Gamma}{2\pi}\ln a = C$$

Or, on $r = a$, $V_r = 0$. Fulfill kinematic body surface condition.

2. At far field, $r = \infty$, $V = U$.

Fulfill the undisturbed condition.

Therefore, the body surface condition and far field condition are all fulfilled.

6.5 Potential Flow with Circulation

Velocity distribution on the circle, *r* = *a*.

$$\begin{cases} V_r = 0 \\ V_\theta = -2U\sin\theta + \dfrac{\Gamma}{2\pi a} \end{cases}$$

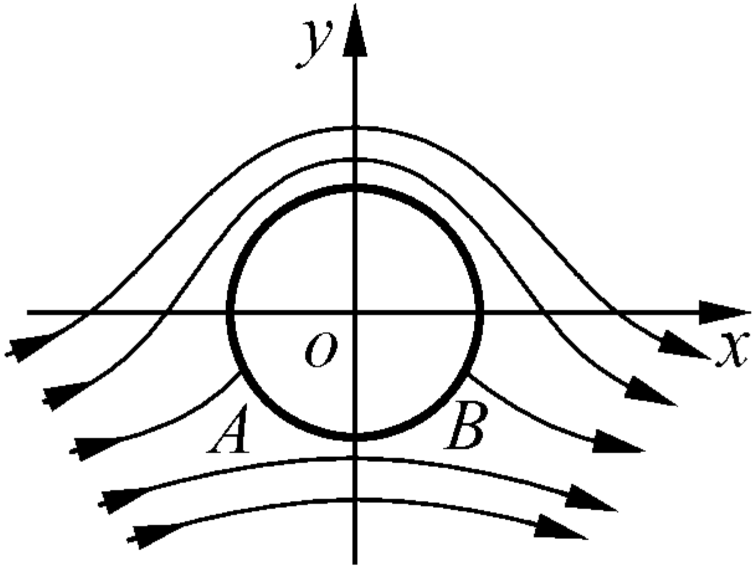
That is, the radial component vanishes, i.e. without separation from the surface, and the tangential component varies with a sine function of angle θ, which is the angle from the direction of the uniform flow to the radial line.

Location of the *stagnation points*.

$$V_\theta = 0 \quad \Rightarrow \quad \sin\theta = \frac{\Gamma}{4\pi a U}$$

6.5 Potential Flow with Circulation

How many ***stagnation points***?



$$V_\theta = 0 \quad \Rightarrow \quad \sin\theta = \frac{\Gamma}{4\pi a U}$$

(a) If $|\sin\theta| < 1$ and $|\Gamma| < 4\pi a U$, there are 2 stagnation points.

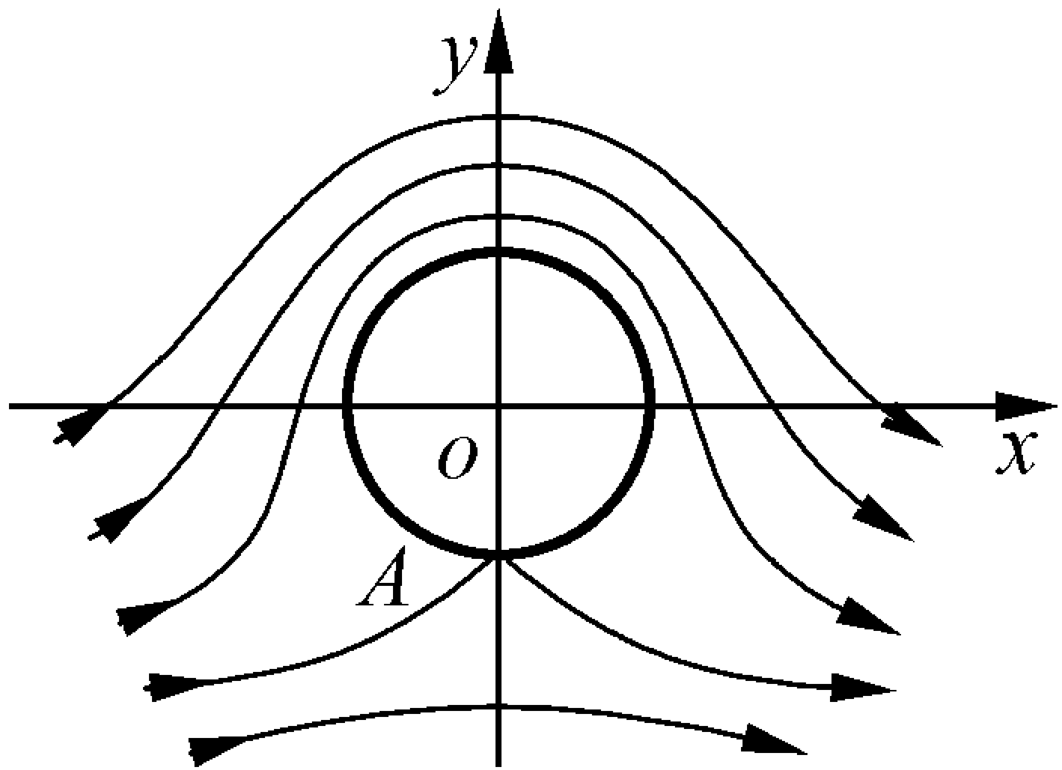
Since $\sin(-\theta) = \sin[-(\pi-\theta)]$, then $-\theta$ and $-(\pi-\theta)$ are a pair of stagnation points. The larger the circulation, Γ, the larger the angle, θ, that is, stagnation points are getting nearer to the bottom of the circle.

6.5 Potential Flow with Circulation

(b) If $|\Gamma| = 4\pi aU$, we have $|\sin\theta| = 1 \Rightarrow \theta = -\frac{\pi}{2}$.

The 2 stagnation points are overlapped.

They become a single stagnation point.

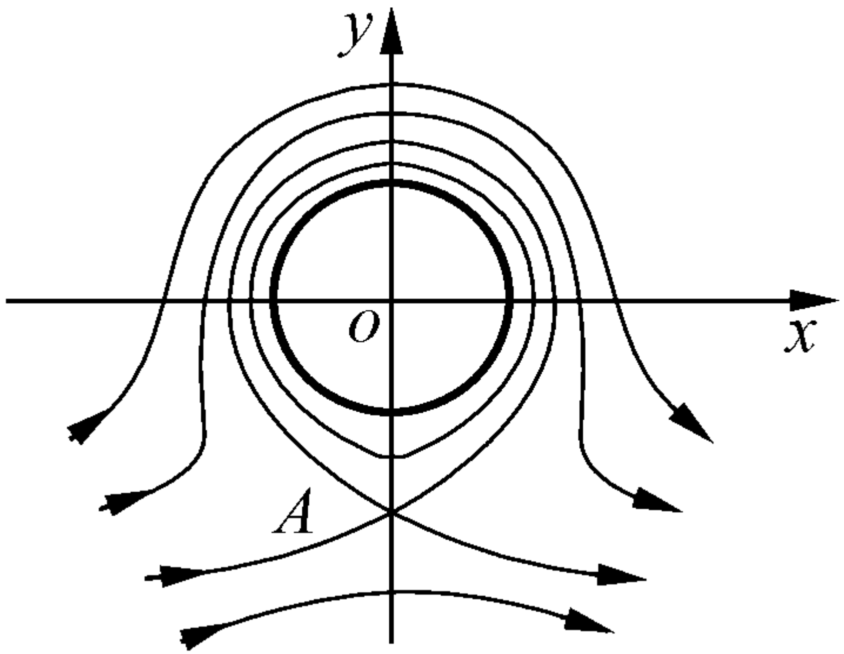


6.5 Potential Flow with Circulation

(c) If $|\Gamma| > 4\pi a U$, we have $|\sin\theta| > 1$, there will be no stagnation point on the circle. Solving the following equation, two stagnation points are obtained. One is located inside the circle, and the other is outside of it.

$$V_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta = 0,$$

$$V_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} = 0$$



6.5 Potential Flow with Circulation

Pressure distribution on the circle, *r* = *a*

According to ***Bernoulli's equation***, we have

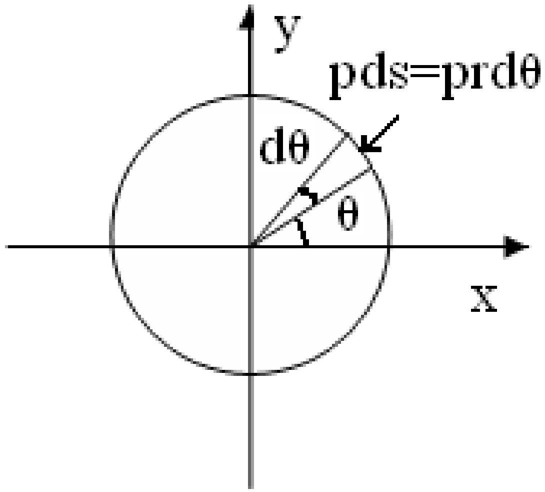
$$p + \frac{1}{2}\rho V_\theta^{\ 2} = p_\infty + \frac{1}{2}\rho U^{\ 2}$$

It follows

$$p = p_\infty + \frac{1}{2}\rho\left[U^2 - \left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2\right]$$

6.5 Potential Flow with Circulation

Drag and *lift* forces on the circular cylinder of unit length



$$F_D = F_x = -\int_0^{2\pi} pad\theta \cos\theta$$

$$= -\int_0^{2\pi} \left\{ p_\infty + \frac{1}{2}\rho \left[U^2 - \left(-2U\sin\theta + \frac{\Gamma}{2\pi a} \right)^2 \right] \right\} a\cos\theta d\theta$$

$$F_L = F_y = -\int_0^{2\pi} pad\theta \sin\theta$$

$$= -\int_0^{2\pi} \left\{ p_\infty + \frac{1}{2}\rho \left[U^2 - \left(-2U\sin\theta + \frac{\Gamma}{2\pi a} \right)^2 \right] \right\} a\sin\theta d\theta$$

6.5 Potential Flow with Circulation

$$F_D = -\int_0^{2\pi} \left\{ p_\infty + \frac{1}{2}\rho\left[U^2 - (-2U\sin\theta + \frac{\Gamma}{2\pi a})^2 \right]\right\} a\cos\theta d\theta$$

$$= -\int_0^{2\pi} \left\{ \left[p_\infty + \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\right] + \frac{1}{2}\rho\left[\frac{2\Gamma U}{\pi a}\sin\theta - \left(\frac{\Gamma}{2\pi a}\right)^2\right]\right\} a\cos\theta d\theta$$

$$= -\frac{\rho\Gamma U}{\pi}\int_0^{2\pi} \sin\theta\cos\theta d\theta = 0$$

As a result, there is no resultant drag force on the circle!

We shall see that the lift force does not vanish.

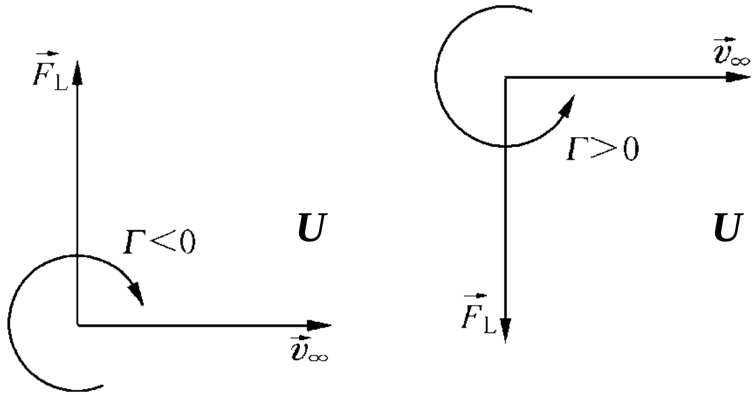
6.5 Potential Flow with Circulation

$$F_L = -\int_0^{2\pi}\left\{p_\infty + \frac{1}{2}\rho\left[U^2 - (-2U\sin\theta + \frac{\Gamma}{2\pi a})^2\right]\right\} a\sin\theta d\theta$$

$$= -\int_0^{2\pi}\left\{\left[p_\infty + \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)\right] + \frac{1}{2}\rho\left[\frac{2\Gamma U}{\pi a}\sin\theta - \left(\frac{\Gamma}{2\pi a}\right)^2\right]\right\} a\sin\theta d\theta$$

$$= -\frac{\rho\Gamma U}{\pi}\int_0^{2\pi}\sin^2\theta\, d\theta = -\rho\Gamma U$$

For the potential flow around a circle with circulation, a *lift force*, which is perpendicular to the uniform flow, is resulted.

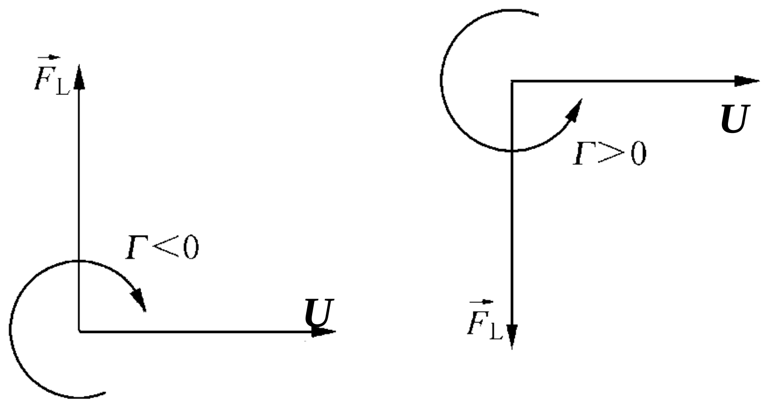


Its magnitude equals the product of **fluid density, speed of the uniform flow** and the **circulation**. Its direction is 90° turning from the direction of the uniform flow against the rounding direction of the circulation.

6.5 Potential Flow with Circulation

It is concluded that a uniform flow, *U*, flows passing a body with circulation Γ, a ***lift*** force is generated. It is perpendicular to the uniform flow and its magnitude equals the product of fluid density, uniform flow speed and the circulation, which is named as ***Kutta-Joukowski formula***.

$$\mathbf{F}_L = \rho \mathbf{U} \times \vec{\Gamma}$$



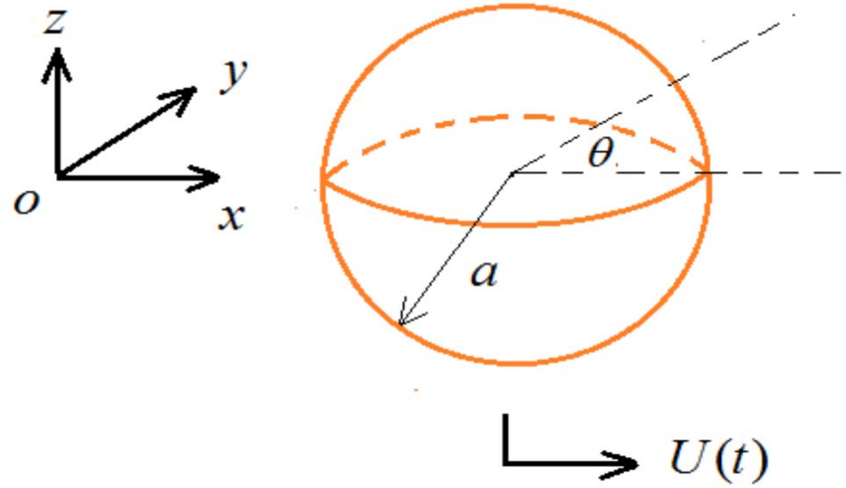
Generally, If a potential flow accompanies with *n* (>1) vortices, the circulation will be replaced by the sum of their circulations. Thus, we get the ***general Kutta-Joukowski formula*** of ***lift*** force.

$$\mathbf{F}_L = \rho \mathbf{U} \times \left(\sum_{i=1}^{n} \vec{\Gamma}_i \right)$$

6.6 Potential Flows due to A Body Moving at Varying Speeds

Now we consider a sphere moving in calm water at speed $U(t)$ varying with time along positive *x*-direction, and calculate resultant hydrodynamic force on it.

The reference frame $O(x, y, z)$ is fixed on the earth. Then, the flow is governed by

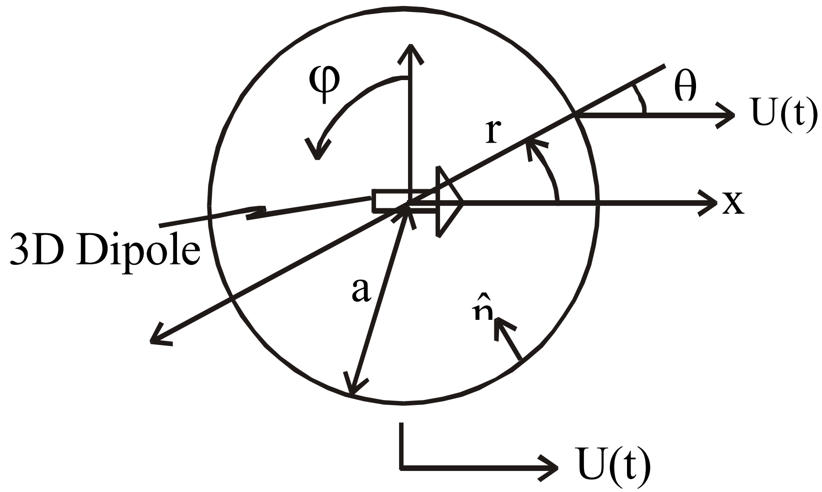


$$\begin{cases} \nabla^2\phi = 0 \\ \left.\dfrac{\partial\phi}{\partial r}\right|_{r=a} = \mathbf{U}\cdot\mathbf{n} = U(t)\cos\theta, & \text{on sphere} \\ \nabla\phi \to 0, \quad \text{if } |\vec{x}| \to \infty, & \text{at far field} \\ \nabla\phi = \mathbf{U}_0, \quad \text{if } t = 0, & \text{initial cond.} \end{cases}$$

6.6 Potential Flows due to A Body Moving at Varying Speeds

This flow is equivalent to a *3d* dipole moving along *x*-axis with a velocity potential

$$\phi = \frac{M}{4\pi}\frac{x}{r^3} = \frac{M}{4\pi}\frac{\cos\theta}{r^2}$$



From ***impermeable*** condition on the sphere, moment M of the dipole is determined.

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \frac{\partial}{\partial r}\left(\frac{M}{4\pi}\frac{\cos\theta}{r^2}\right)\Bigg|_{r=a} = -\frac{M}{2\pi}\frac{\cos\theta}{a^3} = U(t)\cos\theta$$

$\Longrightarrow$ $$M = -2\pi a^3 U(t)$$

Thus, the velocity potential is explicitly expressed as

$$\phi = \frac{M}{4\pi}\frac{\cos\theta}{r^2} = -\frac{U(t)a^3}{2}\frac{\cos\theta}{r^2}$$

Substituting it in ***Bernoulli's equation*** (omitting body force), the dynamic pressure is obtained

$$p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2}\right) + C(t)$$

On *x* direction, the resultant horizontal hydrodynamic force is expressed as

$$F_x = \iint_B p\big|_{r=a} n_x dS = -\rho\iint_B \left(\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2}\right)_{r=a} n_x\, dS$$

6.6 Potential Flows due to A Body Moving at Varying Speeds

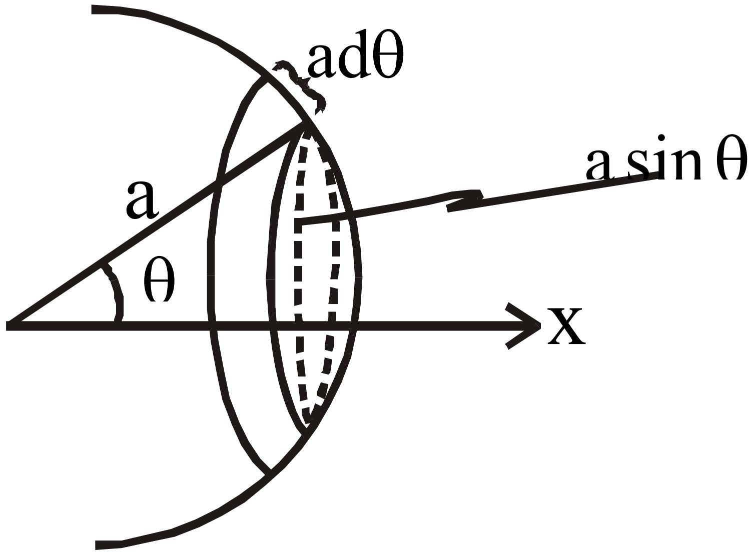
From the derived expression of the velocity potential function, terms in the integral can be immediately given on the sphere.

$$\left.\frac{\partial\phi}{\partial t}\right|_{r=a} = -\frac{\mathrm{d}U}{\mathrm{d}t}\left.\frac{a^3\cos\theta}{2r^2}\right|_{r=a} = -\frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}t}a\cos\theta$$

$$\left.\nabla\phi\right|_{r=a} = \left(\frac{\partial\phi}{\partial r},\frac{1}{r}\frac{\partial\phi}{\partial\theta},\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) = \left(U\cos\theta,\frac{1}{2}U\sin\theta,0\right)$$

$$\left.\left|\nabla\phi\right|^2\right|_{r=a} = U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta$$

$$\iint_B dS = \int_0^{\pi}(a\,d\theta)(2\pi a\sin\theta)$$



Filling above terms in the integral for horizontal force on the sphere, it gives

$$F_x = -\rho \iint_B \left[\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} \right]_{r=a} n_x \, dS$$

$$= -\rho \int_0^{\pi} \left[\underbrace{-\frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}t} a\cos\theta}_{\frac{\partial \phi}{\partial t}} + \frac{1}{2}\left(\underbrace{U^2 \cos^2\theta + \frac{1}{4}U^2 \sin^2\theta}_{|\nabla \phi|^2} \right) \right] \left(\underbrace{-\cos\theta}_{n_x} \right) \underbrace{\left(2\pi a^2 \sin\theta\right) d\theta}_{dS}$$

$$= -\pi\rho \frac{\mathrm{d}U}{\mathrm{d}t} a^3 \underbrace{\int_0^{\pi} \left(\sin\theta\cos^2\theta\right) d\theta}_{2/3} + \rho U^2 \pi a^2 \underbrace{\int_0^{\pi} \left(\sin\theta\cos\theta\right)\left(\cos^2\theta + \frac{1}{4}\sin^2\theta \right) d\theta}_{0}$$

$$= -\dot{U}(t)\left[\rho \frac{2}{3}\pi a^3 \right]$$

That is, the horizontal force is proportional to the sphere's acceleration

$$F_x = -\dot{U}(t)\left[\rho \frac{2}{3}\pi a^3\right]$$

Mass dimension

Case 1: If the speed is constant, *i.e.* $\mathrm{d}U(t)/\mathrm{d}t = 0$, again we get $F_x = 0$, the same result as the fore mentioned ***d'Alembert's paradox***.

Case 2: If the sphere moves with an acceleration, the resultant horizontal force will be

$$F_x = -\dot{U}(t)\left[\rho \frac{2}{3}\pi a^3\right] = -\dot{U}(t)\cdot m_A$$

Denote $m_A = \rho \frac{2}{3}\pi a^3 = \frac{1}{2}(\rho V)$, called ***added mass*** / ***virtual mass***.

Volume of the sphere