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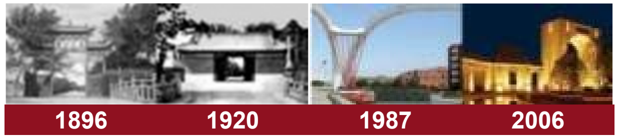


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Chapter 6
Potential Flow Theory

- **Fluid mechanics**
  - **Ideal fluid mechanics**
    - Rotational flows
    - Potential flows
    - Wave motion
  - **Viscous fluid mechanics**
    - Simple flows
    - Boundary layer flows
    - Laminar and turbulent flows

6.1 Governing Equations

We consider an ***incompressible*** ***inviscid*** ***potential flow***. The flow is governed by ***Euler's equation*** (*inviscid*) and ***continuum equation*** (*incompressible*).

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\frac{1}{\rho}\nabla p + \mathbf{f}, \qquad \nabla\cdot\mathbf{V} = 0$$

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{|\mathbf{V}|^2}{2}\right) - \mathbf{V}\times\cancel{\mathbf{\Omega}} = -\frac{1}{\rho}\nabla p + \mathbf{f}, \qquad \nabla\cdot\mathbf{V} = 0$$

For *irrotational* flow, there exists a velocity potential, ϕ ,

$$\mathbf{V} = \nabla\phi$$

Substituting it in continuum equation, ***Lapalce's equation*** results

$$\nabla\cdot\nabla\phi = 0 \quad \Rightarrow \quad \nabla^2\phi = 0$$

6.1 Governing Equations

In addition, if the body force is a potential force, then ***Euler's equation*** is rewritten

$$\nabla\left(\frac{\partial\phi}{\partial t}\right)+\nabla\left(\frac{|\nabla\phi|^2}{2}\right)=-\nabla\left(\frac{p}{\rho}\right)-\nabla\Pi$$

$$\nabla\left(\frac{\partial\phi}{\partial t}+\frac{|\nabla\phi|^2}{2}+\frac{p}{\rho}+\Pi\right)=0$$

$$\frac{\partial\phi}{\partial t}+\frac{|\nabla\phi|^2}{2}+\frac{p}{\rho}+\Pi=C(t)$$

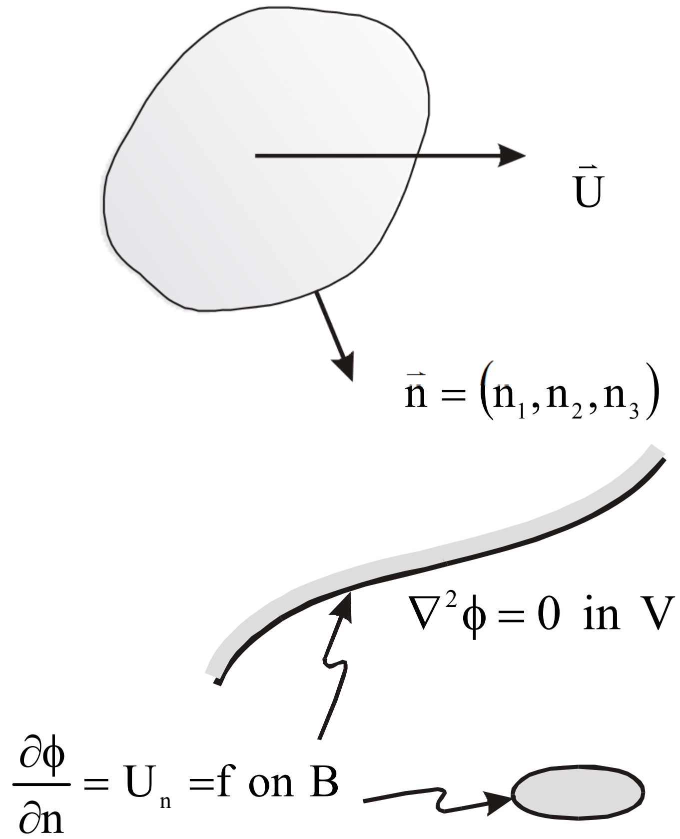
That is the general ***Bernoulli equation***, usually used as a ***dynamic boundary condition*** in potential flow solution, from which pressure distribution can be determined when velocity potential is obtained.

6.1 Governing Equations

Furthermore, generally fluid is assumed not able to enter into or flow out from the wall of a body, but has to move with it, i.e. body surface is considered as ***impermeable***. This is a boundary condition governed on body surface,

$$\mathbf{V}\cdot\mathbf{n}=\mathbf{U}\cdot\mathbf{n} \quad \Rightarrow \quad \nabla\phi\cdot\mathbf{n}=\mathbf{U}_n$$

$$\frac{\partial\phi}{\partial\mathbf{n}}=\mathbf{U}_n$$



Named ***kinematic boundary condition***.

6.1 Governing Equations

Besides, at very far field, both velocity and pressure have to be given. Also the whole flow field has to be given at an instant, generally when flow starts and named as an initial instant.

The ***far field condition*** at infinity

$$\nabla \phi = \mathbf{U}_{\infty}, \qquad p = p_{\infty}$$

The general ***initial condition***

$$\nabla \phi \Big|_{t=0} = \mathbf{U}_0(\mathbf{x}), \qquad p \Big|_{t=0} = p_0(\mathbf{x})$$

6.1 Governing Equations

In summary, equations and conditions which govern potential flows to make it well-posed are as follows.

A quadratic term encountered at the evaluation of pressure p.

Laplace's eq.

$$\nabla^2 \phi = 0$$

Dynamic cond. (Bernoulli eq.)

$$\frac{\partial \phi}{\partial t} + \frac{|\nabla \phi|^2}{2} + \frac{p}{\rho} + \Pi = C(t)$$

Kinematic cond.

$$\frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U}_n \quad \text{(on body surface)}$$

Far field cond.

$$\nabla \phi = \mathbf{U}_\infty, \quad p = p_\infty$$

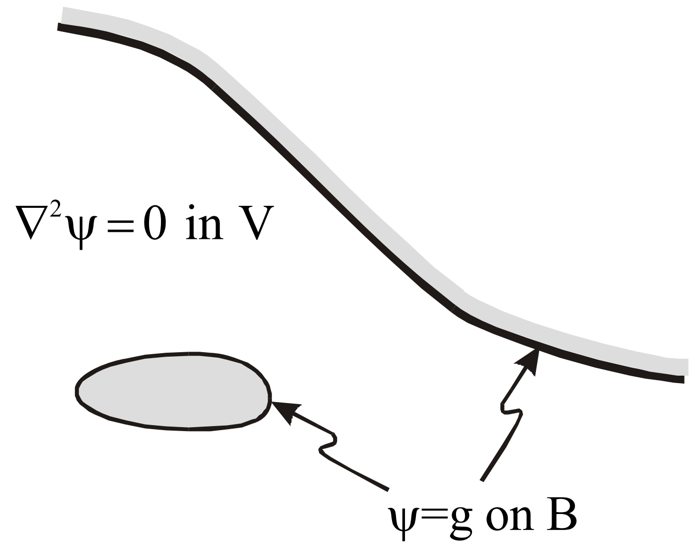
Initial condition

$$\nabla \phi \big|_{t=0} = \mathbf{U}_0(\mathbf{x}), \quad p \big|_{t=0} = p_0(\mathbf{x})$$

6.1 Governing Equations

For 2d flows, a ***stream function***, ψ, is commonly introduced:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$



Thus, the ***irrotational condition*** can be written:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and, on ***impermeable*** body surface, stream function should be a constant, that is, fluid can only flow along it, but can not penetrate it:

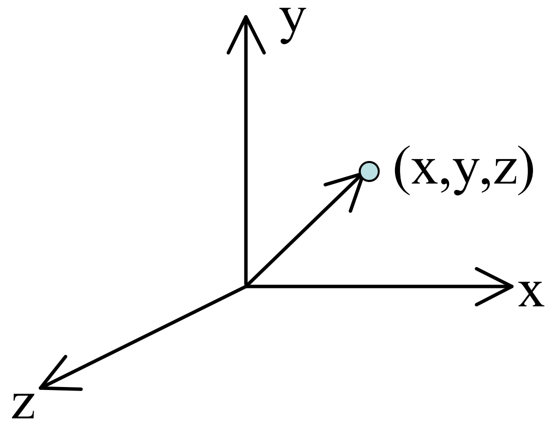
$$\psi = g$$

where, g is a constant, i.e. body surface is a ***streamline***.

6.1 Governing Equations

Now, expressions of the gradient and Laplacian of velocity potential are given below in ***Cartesian***, ***Cylindrical***, ***Spherical coordinate systems***.

$$V = \nabla \phi, \qquad \nabla^2 \phi = 0$$



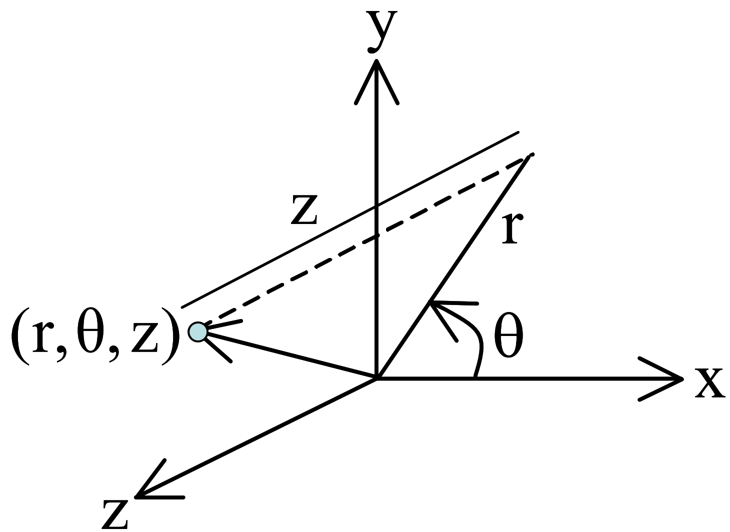
In ***Cartesian coordinate system*** (x, y, z), we have

$$\mathbf{V} = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

6.1 Governing Equations

In ***Cylindrical system*** **(*r*,*θ*, *z*)**, we have



$$r^2 = x^2 + y^2,$$

$$\theta = \tan^{-1}(y/x)$$

$$\mathbf{V} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

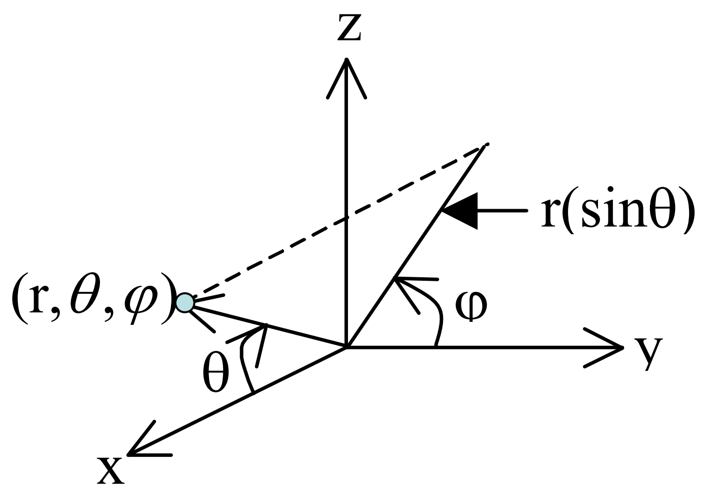
6.1 Governing Equations

In ***Spherical system*** **(*r*, *θ*, *φ*)**, we have

$$r^2 = x^2 + y^2 + z^2,$$

$$\theta = \cos^{-1}(x/r)$$

$$\varphi = \tan^{-1}(z/y)$$



$$\mathbf{V} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial\phi}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

6.2 Superposition Principle

In solution of velocity potential (and stream function in 2d problems), ***Laplace's equation*** and the ***kinematic boundary condition*** are all linear. Thus, ***velocity potentials*** (and ***stream functions***) can be superposed to form a new potential flow. It is called ***linear superposition principle***. ***Dynamic integral*** is only applied after completion of solution in order to evaluate pressures, and its nonlinearity does not affect the solution.

Therefore, superposition of elementary potential flows will result a new potential flow, of which velocity potential, stream function and velocity components are just equal to the sum of the ones of the elementary flows.

In this way, elementary potential flows can be superposed to generate a new flow. On the other hand, in principle a complicated flow can be resolved into a superposition of several elementary flows.

6.2 Superposition Principle

Let φ_1 and φ_2 be ***velocity potentials*** of two different planar flows, that is

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} = 0$$

$$\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial y^2} = 0$$

Then, their sum gives a new potential flow. The velocity potential is

$$\varphi = \varphi_1 + \varphi_2$$

6.2 Superposition Principle

Since $$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial^2(\varphi_1+\varphi_2)}{\partial x^2}+\frac{\partial^2(\varphi_1+\varphi_2)}{\partial y^2}$$

$$=\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}\right)+\left(\frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial y^2}\right)=0$$

so, it is also a ***potential flow*** with velocity components

$$u_x=\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial x}=u_{x1}+u_{x2}$$

$$u_y=\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_2}{\partial y}=u_{y1}+u_{y2}$$

In the same way, its *stream function* is the sum of the other two's

$$\psi=\psi_1+\psi_2$$

6.2 Superposition Principle

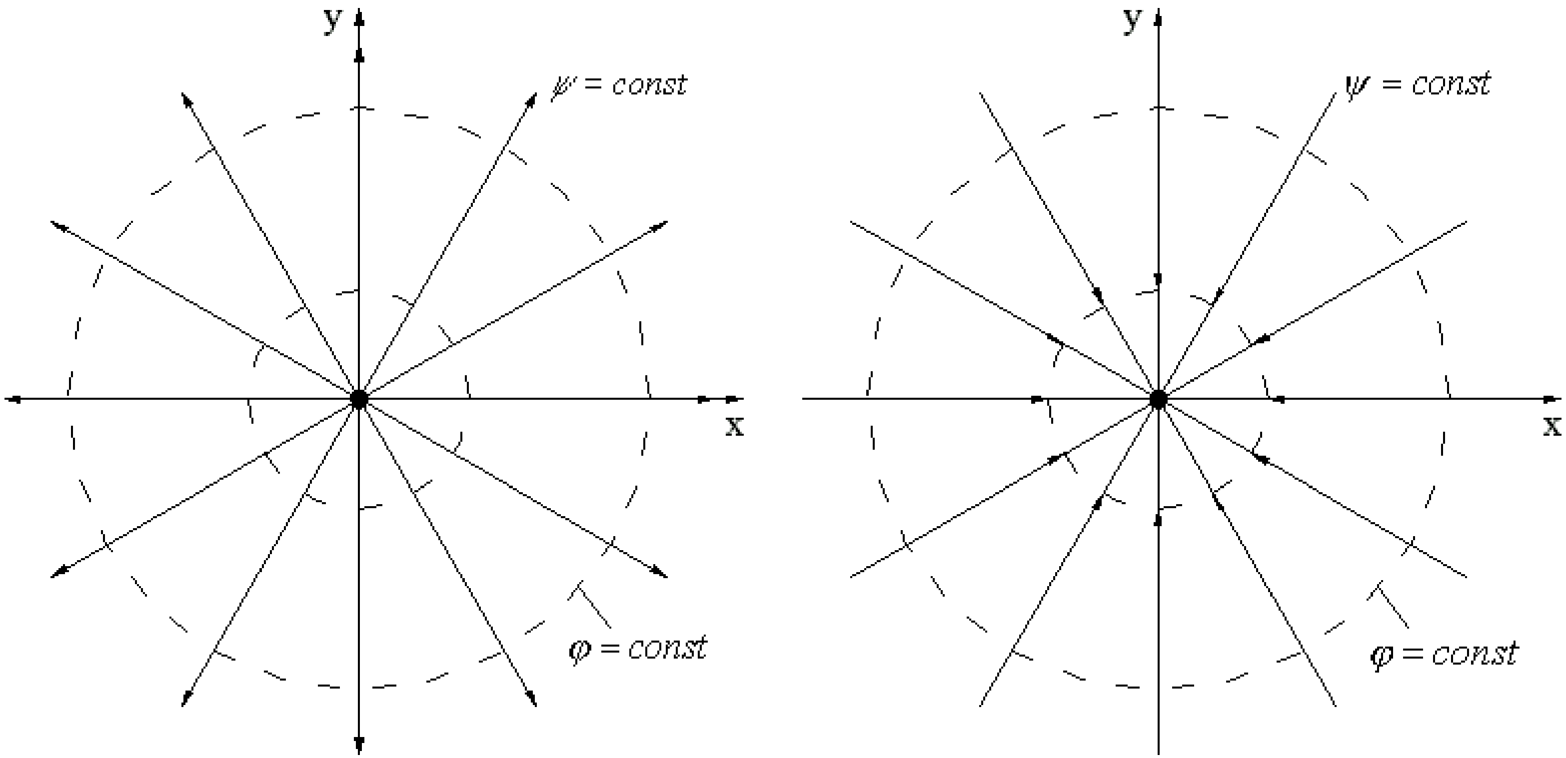
Source and Sink

2d Flow:

$$\phi = \frac{m}{2\pi}\ln r, \qquad \psi = \frac{m}{2\pi}\theta$$

3d Flow:

$$\phi = \frac{m}{4\pi r}$$

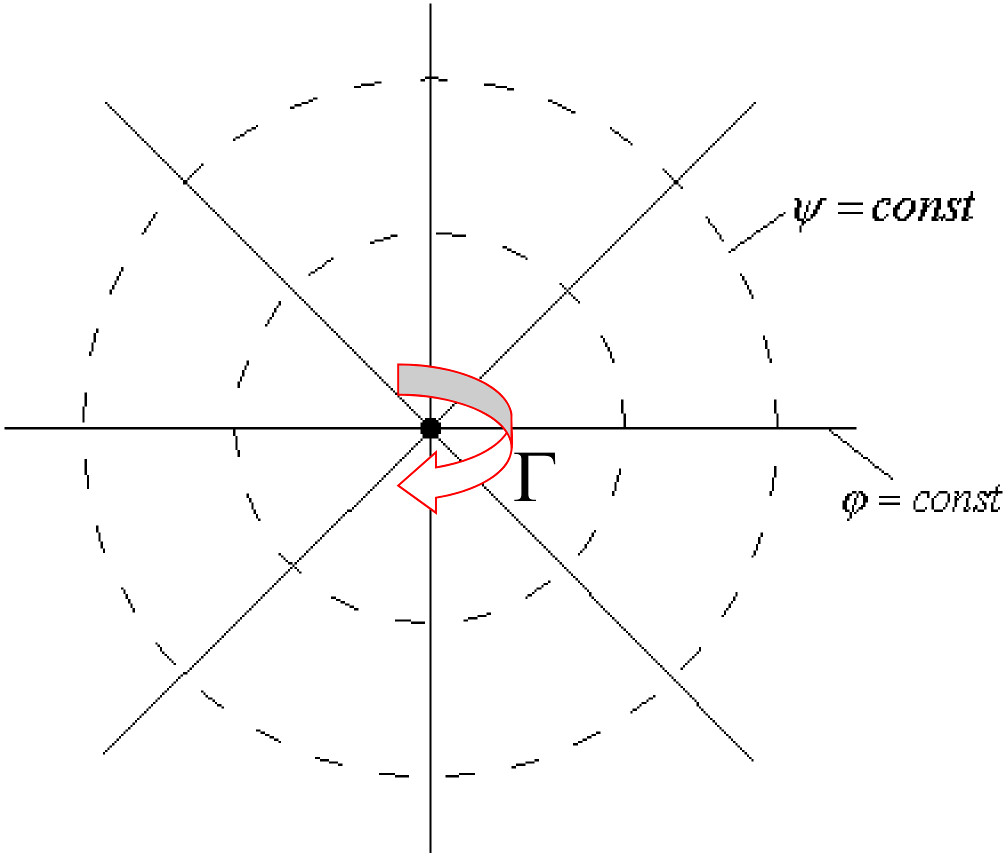


6.2 Superposition Principle

Point Vortex:

$$\phi = \frac{\Gamma}{2\pi}\theta$$

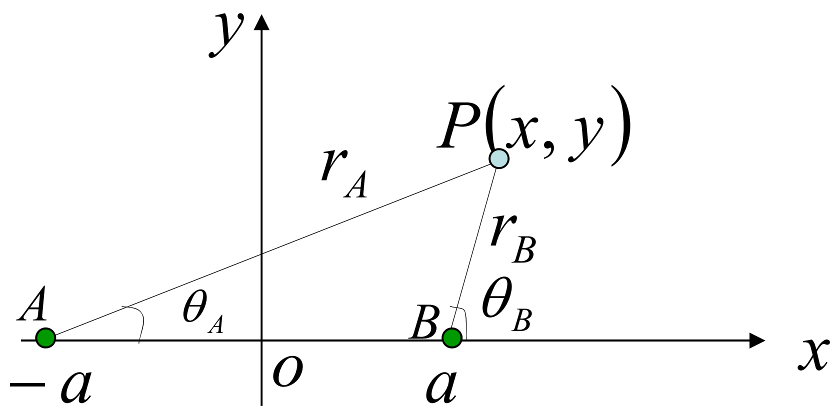
$$\psi = -\frac{\Gamma}{2\pi}\ln r$$



6.2 Superposition Principle

Eg.1: Dipole — Superposition of a point source and a point sink

Given a point sink at $A(-a, 0)$ with intensity Q, and a point source at $B(a, 0)$ with the same intensity. Let Φ_1 and Φ_2, Ψ_1 and Ψ_2 be their velocity potentials and stream functions. Please write down their superposed velocity potential at point $P(x, y)$.



Solution: The superposed velocity potential is the sum of the ones of a point sink at A and a point source at B:

$$\phi = \phi_1 + \phi_2 = \frac{Q}{2\pi}\ln r_B - \frac{Q}{2\pi}\ln r_A$$

$$= \frac{Q}{2\pi}\ln\frac{r_B}{r_A} = \frac{Q}{4\pi}\ln\frac{y^2+(x-a)^2}{y^2+(x+a)^2}$$

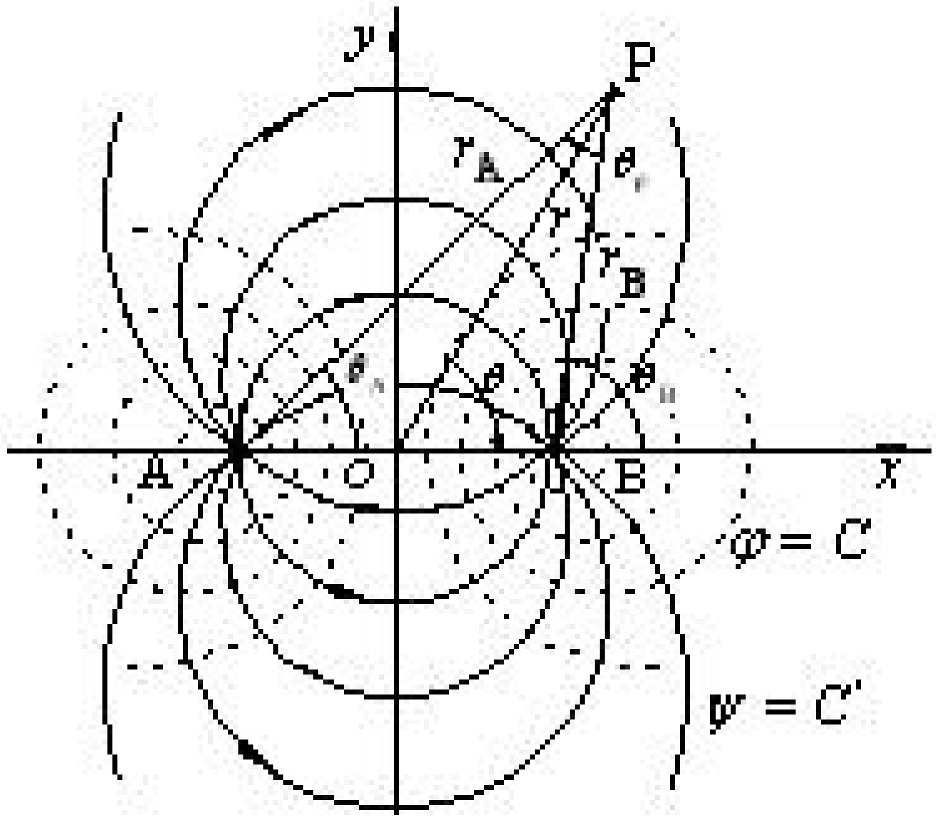
6.2 Superposition Principle

The ***stream function*** is

$$\psi = \frac{Q}{2\pi}(\theta_B - \theta_A)$$

$$= \frac{Q}{2\pi}\theta_p$$

$$= \frac{Q}{2\pi}\arctan\frac{2ay}{x^2 + y^2 - a^2}$$



Denote θ_p the angle between *AP* and *BP*. A *streamline*, on which Ψ= const., thus θ_p = const., is a circle passing through the *source point* and *sink point*.

6.2 Superposition Principle

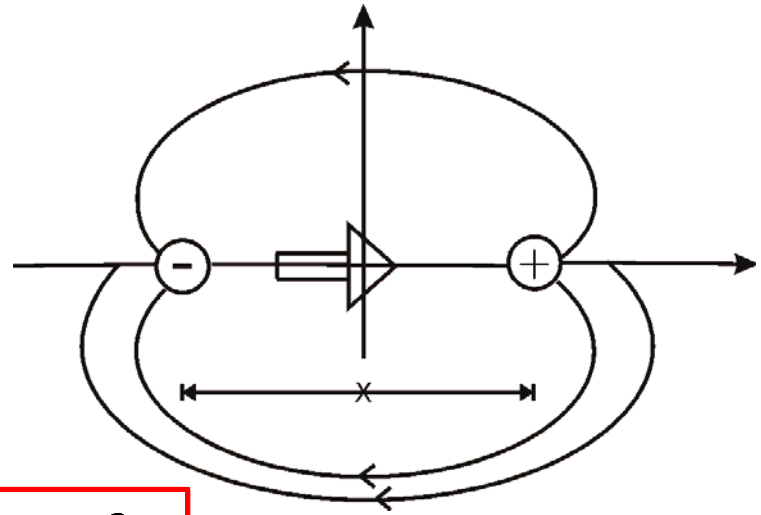
As $a \to 0$, the distance between the *source* and the *sink* tends to zero, the intensity needs to be infinitely large to guarantee $\lim_{a\to 0,\ Q\to\infty} Q\cdot 2a = M$ of finite value. That sort of flow is called a ***doublet flow***, or a ***dipole***. M is known as ***moment***, pointing from the sink to the source. For small value of ε, we have

$$\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + \varepsilon^3/3 - \ldots \approx \varepsilon$$

So, velocity potential of the *doublet flow* can be immediately derived.

$$\phi = \lim_{\substack{a\to 0\\ Q\to\infty}} \{\frac{Q}{4\pi}\ln[1+\frac{-4xa}{(x+a)^2+y^2}]\}$$

$$= \lim_{\substack{a\to 0\\ Q\to\infty}} [-\frac{Q}{4\pi}\frac{4xa}{(x+a)^2+y^2}]$$



i.e.

$$\phi = -\frac{M}{2\pi}\frac{x}{(x^2+y^2)} = -\frac{M}{2\pi}\frac{\cos\theta}{r}$$

6.2 Superposition Principle

Accordingly, velocity distribution is obtained.

$$V_r = \frac{\partial \phi}{\partial r} = \frac{M}{2\pi}\frac{\cos\theta}{r^2} \qquad V_\theta = \frac{\partial \phi}{r\partial \theta} = \frac{M}{2\pi}\frac{\sin\theta}{r^2}$$

And the corresponding *stream function* is written as

$$\psi = \lim_{\substack{a\to 0\\ Q\to\infty}}\left(\frac{Q}{2\pi}\arctan\frac{2ay}{x^2+y^2-a^2}\right) = \lim_{\substack{a\to 0\\ Q\to\infty}}\left(\frac{Q}{2\pi}\frac{2ay}{x^2+y^2-a^2}\right)$$

That is,

$$\psi = \frac{M}{2\pi}\frac{y}{x^2+y^2} = \frac{M}{2\pi}\frac{\sin\theta}{r}$$

6.2 Superposition Principle

Let Φ be a constant C_1, it results an *equi-potential line.*

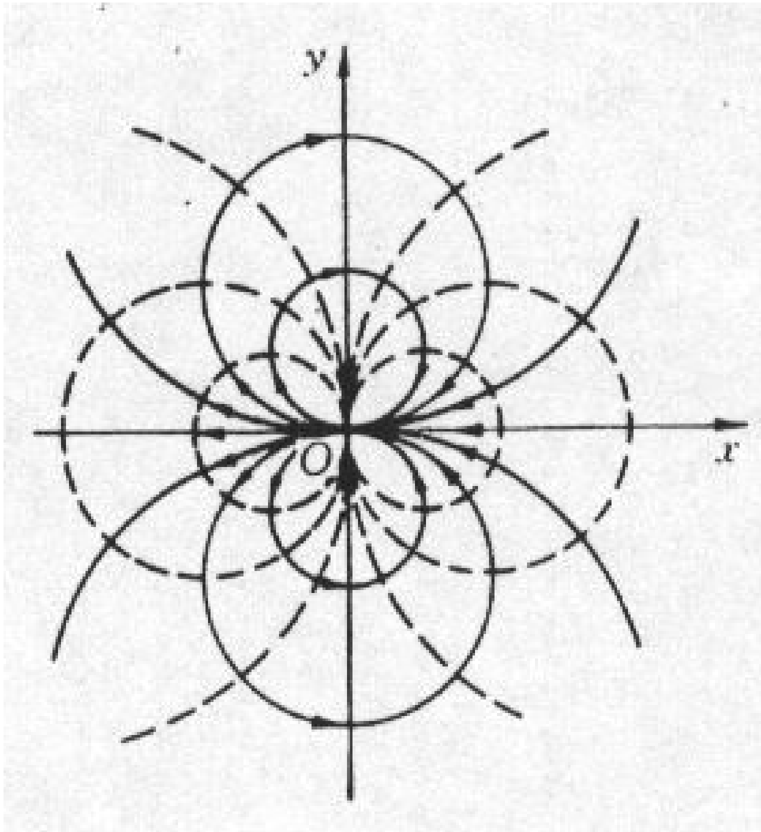
$$(x+\frac{M}{4\pi C_1})^2+y^2=(\frac{M}{4\pi C_1})^2$$

They are the circles with center at point $(-\frac{M}{4\pi C_1}, 0)$, and radius $\left|\frac{M}{4\pi C_1}\right|$, tangent to the y-axis at the origin, as the dash lines shown in the figure.

Let Ψ be constant C_2, we get a streamline

$$x^2+(y-\frac{M}{4\pi C_2})^2=(\frac{M}{4\pi C_2})^2$$

Streamlines are circles with centre at point $(0, \frac{M}{4\pi C_2})$, radius $\left|\frac{M}{4\pi C_2}\right|$, tangent to the x-axis at the origin, as the solid lines shown in the figure.



6.2 Superposition Principle

The *velocity potential* of the *dipole* can also be derived in another way.

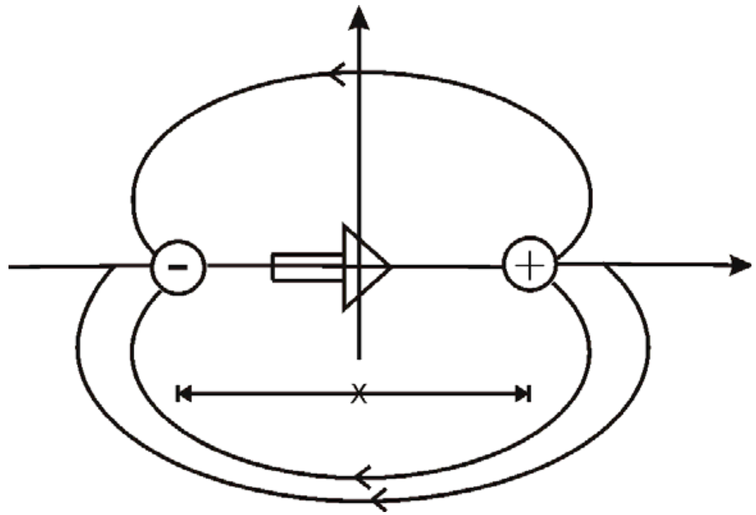
$$\phi = \lim_{\substack{a\to 0\\ Q\to\infty}} (\phi_1 + \phi_2) = \lim_{\substack{a\to 0\\ Q\to\infty}} \left[\frac{Q}{2\pi} \left(\ln\sqrt{(x-a)^2 + y^2} - \ln\sqrt{(x+a)^2 + y^2} \right) \right]$$

$$= -\frac{M}{2\pi}\frac{\partial}{\partial x}\ln\sqrt{x^2+y^2} = -\frac{M}{2\pi}\frac{x}{x^2+y^2} = -\frac{M}{2\pi}\frac{x}{r^2}$$

where M is the *moment*

$$\lim_{a\to 0,\ Q\to\infty} Q\cdot 2a = M$$

$$\phi = -\frac{M}{2\pi}\frac{x}{r^2}$$



The *stream function* can be derived in the same way.

6.2 Superposition Principle

For *3d doublet flow*, we can superpose the *3d point source* with the *3d point sink*, that is

$$\phi = \lim_{\substack{a\to 0\\ Q\to\infty}} (\phi_1 + \phi_2) = \lim_{\substack{a\to 0\\ Q\to\infty}} \left[-\frac{Q}{4\pi} \left(\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right) \right]$$

$$= -\frac{M}{4\pi} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{M}{4\pi} \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \frac{M}{4\pi} \frac{x}{r^3}$$

where M is the *moment.*

$$\lim_{a\to 0,\ Q\to\infty} Q \cdot 2a = M$$

Therefore, the *velocity potential* of the *3d doublet flow* is expressed as

$$\phi = \frac{M}{4\pi} \frac{x}{r^3}$$

6.2 Superposition Principle

Eg.2: Spiral Flow —superposition of a *point source* and a *point vortex*

Given a *point source* at the origin with *intensity* Q, and a *point vortex* at the origin with *intensity* Γ. Let Φ_1 and Φ_2 be their *velocity potentials*, Ψ_1 and Ψ_2 be their *stream functions*. Write down the *velocity potential* and *stream function* of the superposed ***spiral flow***.

Solution: The velocity potential and stream function of the superposed *spiral flow* can be given immediately

$$\phi = \phi_1 + \phi_2 = -\frac{1}{2\pi}(Q\ln r - \Gamma\theta)$$

$$\psi = \psi_1 + \psi_2 = -\frac{1}{2\pi}(Q\theta + \Gamma\ln r)$$

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6.2 Superposition Principle

Let the velocity potential and stream function be constants, we get the *equi-potential lines* and *streamlines*

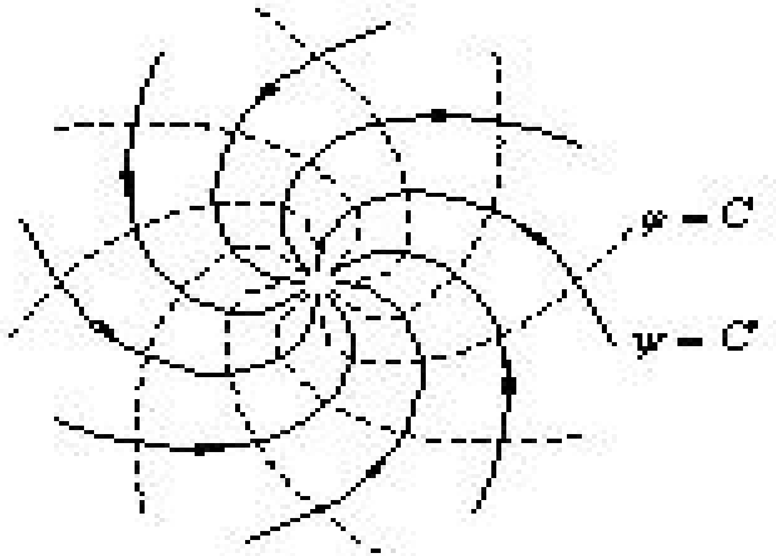
$$r = C_1 e^{\frac{\Gamma}{Q}\theta} \qquad\qquad r = C_2 e^{-\frac{Q}{\Gamma}\theta}$$

They are *spiral curves* and perpendicular with each other at any point of intersection. As streamlines are spirals, this sort of flow is call a ***spiral flow***.

The velocity field is as follows

$$V_r = \frac{\partial\phi}{\partial r} = -\frac{Q}{2\pi r}$$

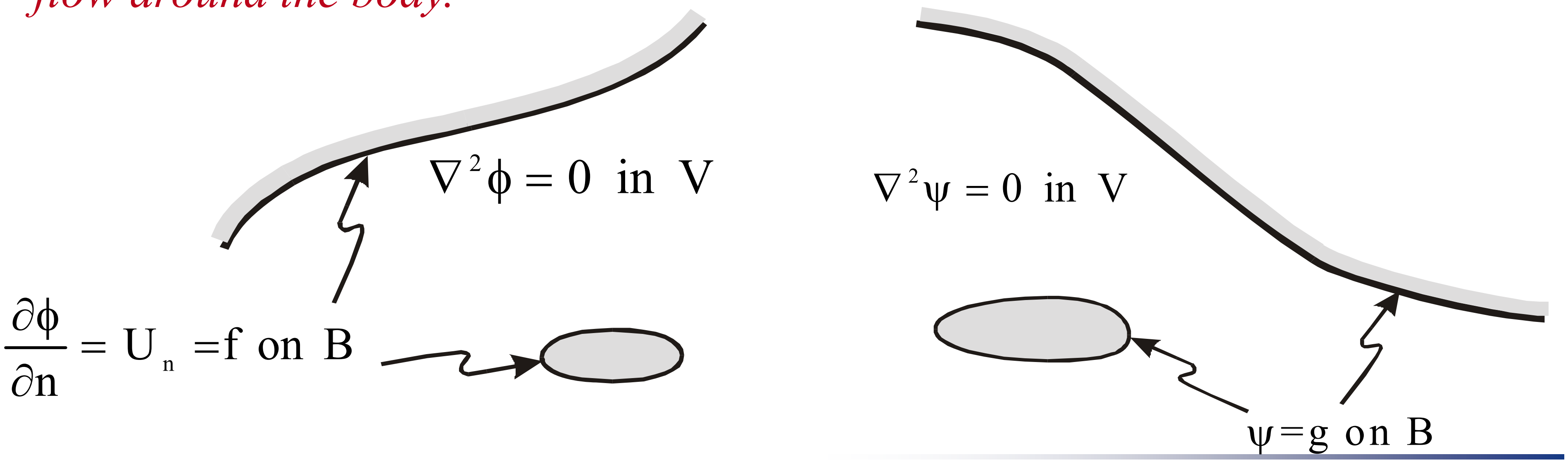
$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$$



6.3 Potential Flow around a Body

Now we further consider the flow field with a body in it. On the body surface a *kinematic boundary condition*, i.e. *impermeable* condition should be fulfilled. In the interior of the flow field, *Laplace's equation* is still to be satisfied.

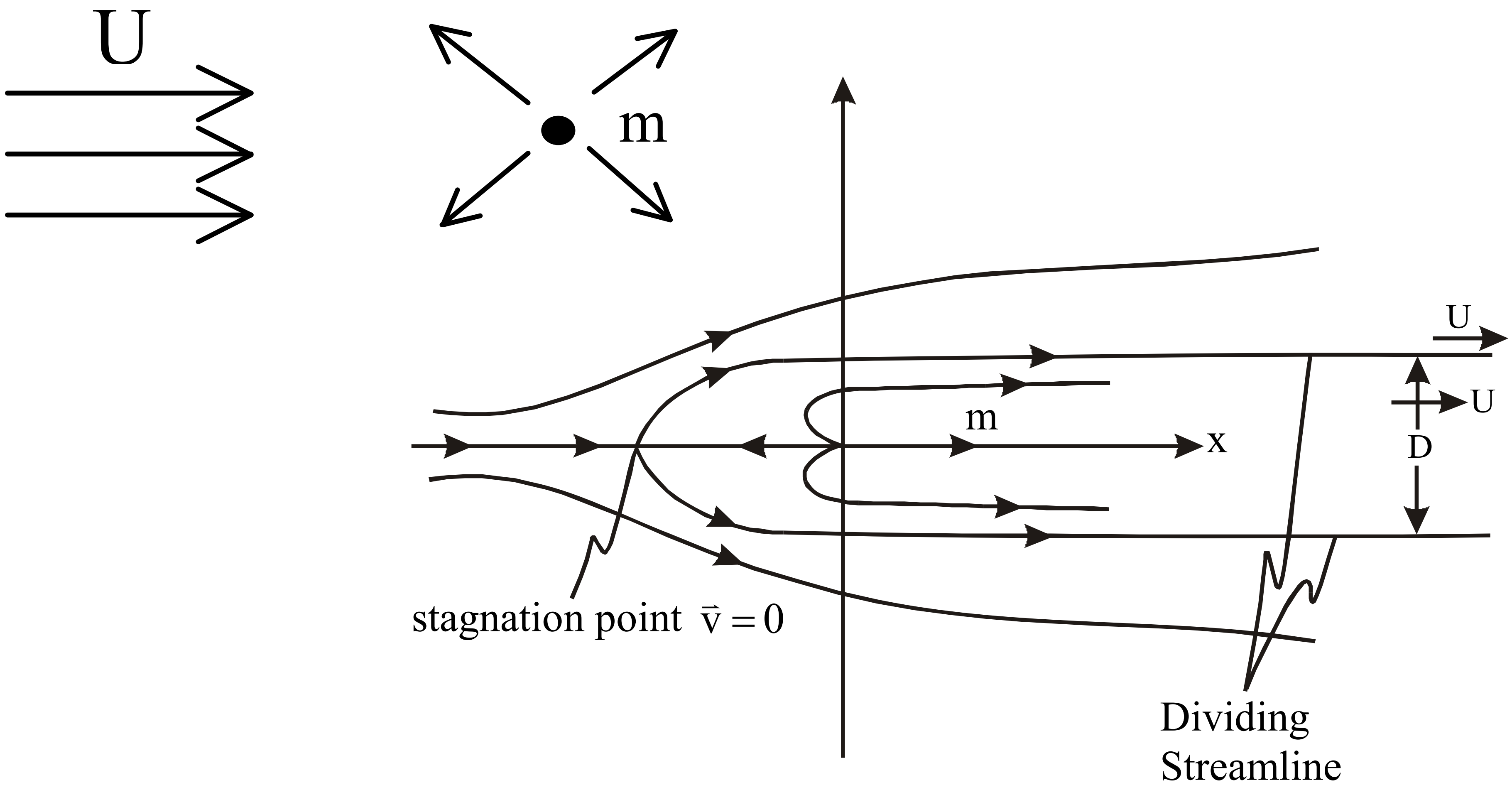
If a superposition of elementary flows fulfill the boundary condition, the sum of the velocity potentials and the sum of the stream functions of those elementary flows give out the velocity potential and the stream function of the flow around the body.



6.3 Potential Flow around a Body

Eg.1： *Rankine Half-body Flow* — Flow around 2d semi-infinite body

(A superposition of a 2d uniform flow and a 2d point source)

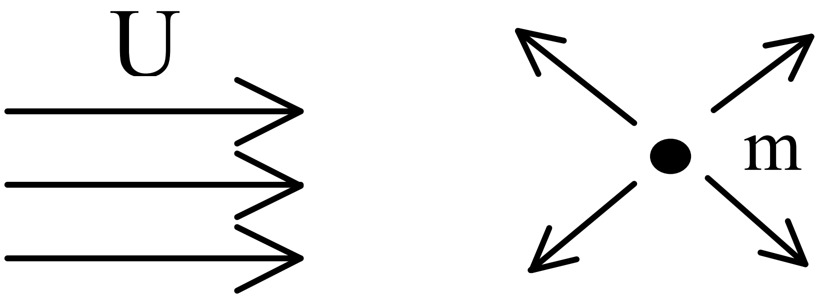


6.3 Potential Flow around a Body

Solution: Velocity potentials and stream functions of a uniform flow ($u = U$, $v = 0$) along the direction of x-axis and a point source at the origin are known as

Uniform flow:

$$\begin{cases} \phi_1 = Ur\cos\theta \\ \psi_1 = Ur\sin\theta \end{cases}$$



Point Source:

$$\begin{cases} \phi_2 = \dfrac{m}{2\pi}\ln r \\ \psi_2 = \dfrac{m}{2\pi}\theta \end{cases}$$

6.3 Potential Flow around a Body

The superposed flow has the *velocity potential* and *stream function*.

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r \qquad \psi = Ur\sin\theta + \frac{m}{2\pi}\theta$$

Velocity field:

$$V_r = \frac{\partial\phi}{\partial r} = U\cos\theta + \frac{m}{2\pi r}, \quad V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta$$

Streamlines:

$$\psi = Ur\sin\theta + \frac{m}{2\pi}\theta = C$$

For different streamlines, constant C will take different values.

As the body surface is *impermeable*, it should be a part of a streamline.

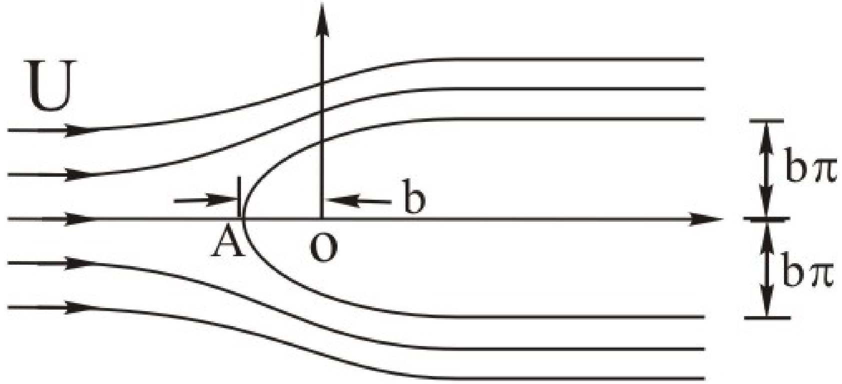
6.3 Potential Flow around a Body

Consider the negative *x*-axis ($\theta = \pi$, $x < 0$). There exists a point, say *A*(-*b*, 0), where velocity is zero. That point is called a ***stagnation point***.

$$V_{r,\theta=\pi} = \left(U\cos\theta + \frac{m}{2\pi r} \right)_{\theta=\pi} = -U + \frac{m}{2\pi b} = 0$$

It results

$$b = \frac{m}{2\pi U}$$



The streamline passing through stagnation point *A*(-*b*, 0) takes a specified value for streamline function as follows.

$$\psi_{A,\theta=\pi} = \left(Ur\sin\theta + \frac{m}{2\pi}\theta \right)_{\theta=\pi} = \frac{m}{2}$$

6.3 Potential Flow around a Body

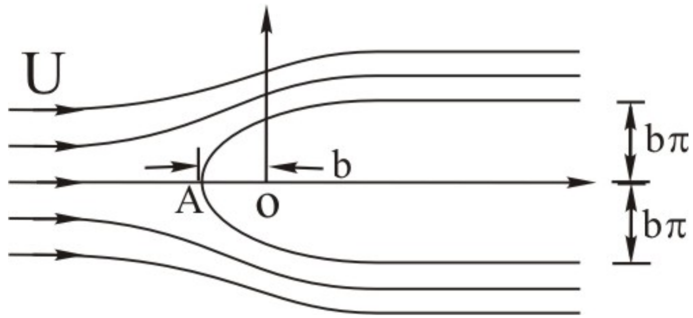
Therefore, that streamline is expressed as

$$Ur\sin\theta + \frac{m}{2\pi}\theta = \frac{m}{2}$$

or

$$r = \frac{m}{2\pi U}\frac{\pi-\theta}{\sin\theta} = \frac{b(\pi-\theta)}{\sin\theta}$$

The right part of it can be looked as a body, as the diagram shows, known as ***Rankine half-body***. At infinity **(θ→0** and **2π),** the upper and lower branches tend to be parallel to the *x*-axis. The distances to the *x*-axis can be evaluated respectively.



$$y_0 = (r\sin\theta)\Big|_{\theta=0,\,2\pi} = \left[b(\pi-\theta)\right]\Big|_{\theta=0,\,2\pi} = \pm b\pi = \pm\frac{m}{2U}$$

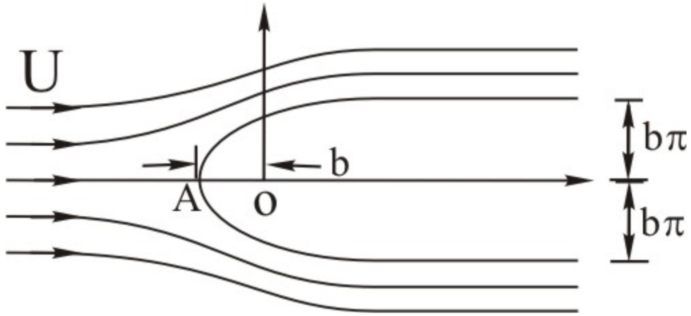
6.3 Potential Flow around a Body

In summary, a superposition of *uniform flow* and a *point source* has potential function

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r = Ux + \frac{m}{2\pi}\ln\sqrt{x^2+y^2}$$

velocity:

$$u = \frac{\partial\phi}{\partial x} = U + \frac{m}{2\pi}\frac{x}{x^2+y^2}$$



stagnation point A:

$$u\Big|_{x=x_A,\ y=0} = U + \frac{m}{2\pi x_A} = 0 \quad\Rightarrow\quad x_A = -\frac{m}{2\pi U}$$

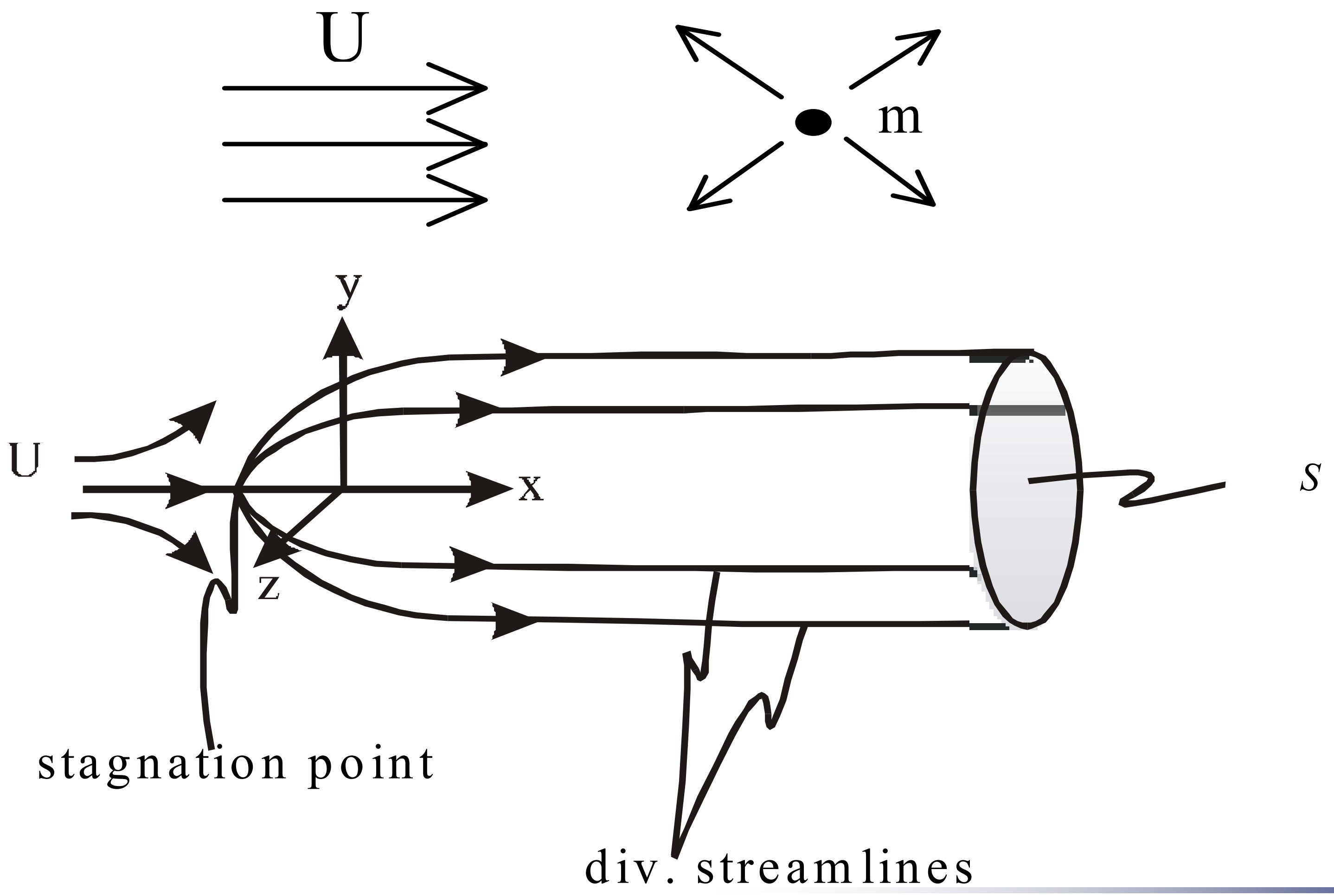
As $x\to\infty$, we have $u\to U$, the body surface become a *stream tube*. The *mass conservation law* tells us that the flow rate equals the point source intensity m, that is

$$2|y_0|U = m \quad\Rightarrow\quad y_0 = \pm\frac{m}{2U}$$

6.3 Potential Flow around a Body

Eg.2: *Rankine Half-body Flow*

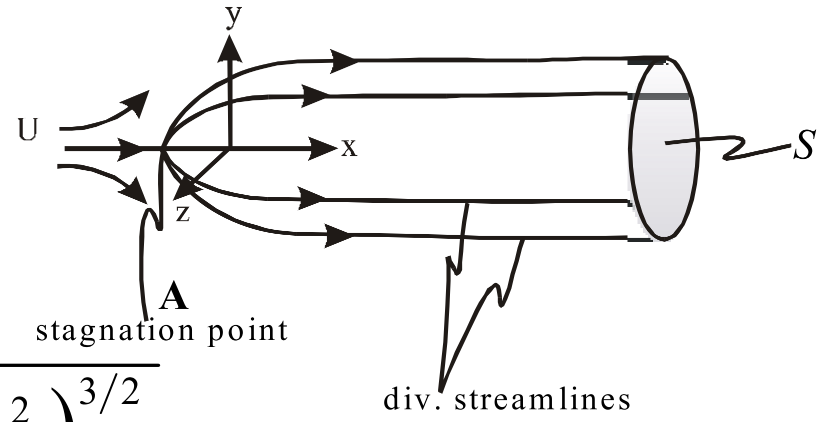
—— A superposition of *3d uniform flow* and a *point source* flow



6.3 Potential Flow around a Body

In Cartesian coordinates, the superposed velocity potential is

$$\phi = Ux - \frac{m}{4\pi\sqrt{x^2+y^2+z^2}}$$



velocity:

$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}$$

stagnation point A:

$$u\Big|_{x=x_A,\ y=z=0} = U + \frac{m}{4\pi}\frac{x_A}{\left|x_A\right|^3} = 0 \quad \Rightarrow \quad x_A = -\sqrt{\frac{m}{4\pi U}}$$

As $x \rightarrow \infty$, we have $u \rightarrow U$, the body surface become a *stream tube*. The *mass conservation law* tells us that the flow rate equals the point source intensity m. Denote S the cross section area, we have

$$S \cdot U = m \quad \Rightarrow \quad S = \frac{m}{U}$$