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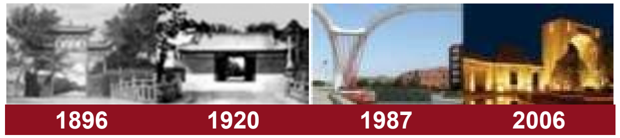


# Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

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# Fifth Assignment

1896 1920 1987 2006

- The fifth assignment can be downloaded from following website:

  Website: ftp://public.sjtu.edu.cn

  Username: dcwan

  Password: 2015mhydro

  Directory: IntroMHydro2015-Assignments

- Seven problems

- Submit the assignment on April 27th (in English, written on paper)

# Chapter 5
# Vorticity Dynamics

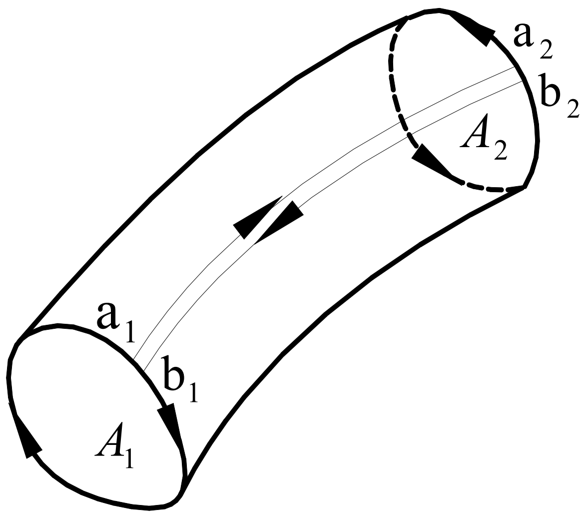
# Helmholtz' Vortex Theorems

# 5.8 Helmholtz' Vortex Theorems

Helmholtz proposed three theorems about the vortex motion, which explain basic properties of vortex. They are the fundamental theorems for studying rotational flow of ideal fluids.

**1. Helmholtz first theorem (Vortex strength keeps constant in space):** *in rotational flow field of ideal fluids, which is either incompressible or barotropic, under the action of body forces with potential, vortex strength along a vortex tube on any cross section will be a constant.*

Take two arbitrary cross sections $A_1$, $A_2$ from the same vortex tube, and take two infinitely close line segments $a_1a_2$, $b_1b_2$ on the surface of the vortex tube between $A_1$ and $A_2$.



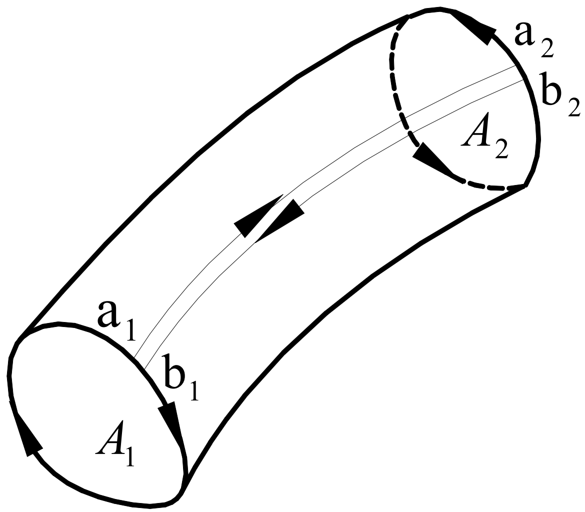
# 5.8 Helmholtz' Vortex Theorems

Because there is no vortex line through the surface of the vortex tube bounded by the closed curve $a_1a_2b_2b_1a_1$, the vortex strength is 0. Based on Stokes' theorem, the velocity circulation about a closed curve equals 0, i.e.,

$$\Gamma_{a_1a_2b_2b_1a_1} = \Gamma_{a_1a_2} + \Gamma_{a_2b_2} + \Gamma_{b_2b_1} + \Gamma_{b_1a_1} = 0$$

Because $\Gamma_{a_1a_2} + \Gamma_{b_2b_1} = 0$, and $\Gamma_{a_2b_2} = -\Gamma_{b_2a_2}$, then: $\Gamma_{b_1a_1} = \Gamma_{b_2a_2}$

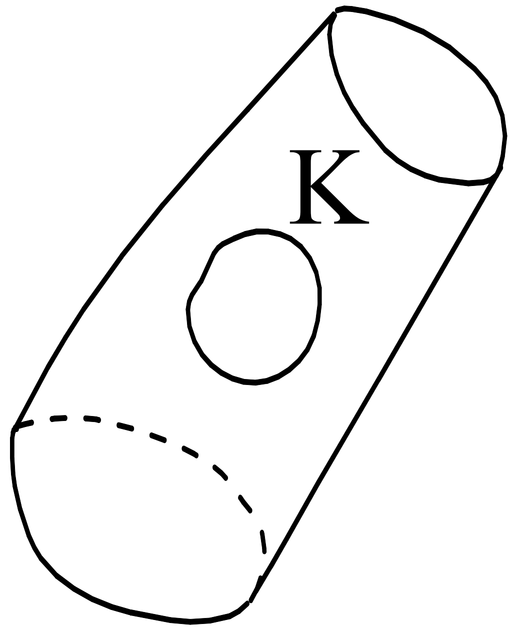
Similarly, this theorem indicates that in an ideal, incompressible or barotropic fluid with potential body forces, the vortex tube cannot begin or end in the interior of the fluid, but can form a vortex ring or loop, or start or end at boundaries.



# 5.8 Helmholtz' Vortex Theorems

**2. Helmholtz second theorem (Vortex tube keeps during motion):** *For an ideal fluid, either incompressible or barotropic, under the action of body forces with potential, fluid particles which form a vortex tube at an instant will always form a vortex tube during the fluid motion.*

K is a closed curve on the surface of a vortex tube, the vortex flux passing through the area bounded by K is 0. From Stokes' theorem, the velocity circulation around K is 0, and from Kelvin's theorem, the velocity circulation around K will keep 0, i.e., the patch of fluid particles bounded by curve K will be always on the wall of vortex tube. In other words, fluid particles that are a patch of a vortex tube will be always a patch of a vortex tube, and the vortex tube always consists of the same fluid particles, but its position and shape may change with time.

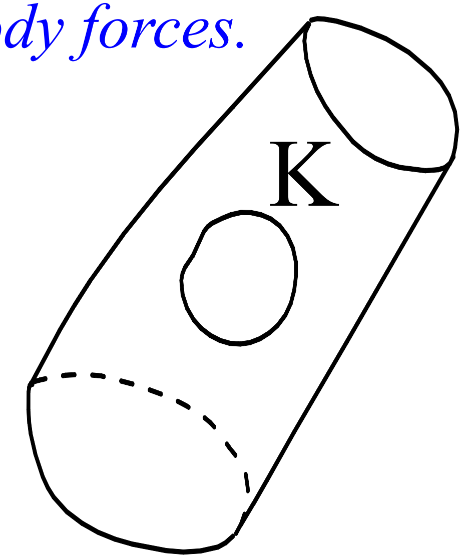


# 5.8 Helmholtz' Vortex Theorems

## Deduction:

**1) Theorem (Vortex surface keeps during motion):** *Fluid particles that form a vortex surface at an initial time will always form a vortex surface in both previous and subsequent time, i.e., a vortex surface keeps a vortex tube during the fluid motion, but its position and shape may change with time, provided the fluid is ideal, barotropic and under the action of potential body forces.*

**2) Theorem (Vortex line keeps during motion):** *Fluid particles that form a vortex line at an initial time will always form a vortex line in both previous and subsequent time, i.e., a vortex line keeps during motion, but its position and shape may change.*



# 5.8 Helmholtz' Vortex Theorems

**3. Helmholtz third theorem (Vortex strength keeps constant during motion):** *For an ideal fluid, either incompressible or barotropic, under the action of body forces with potential, not only any vortex tube keeps a vortex tube, but its vortex strength pertains during the course of motion.*

Take an arbitrary cross section A of a vortex tube. Curve K be the boundary of A. From Kelvin's theorem, velocity circulation along K will keep a constant during motion. According to Stokes' theorem, vortex strength through A takes the same constant. Thus strength of a vortex tube does not change its value during motion, though the position and shape may possibly change.

According to Helmholtz' three vortex theorems, the shear stresses of viscous fluids would be a factor, which will gradually reduce the strength of the vortex tube, since they consume some energy.

# Summary

**Properties of vorticity field in space**

**Vortex flux through an arbitrary closed curved surface is 0**

- **Helmholtz first theorem:** *Vortex strength (vortex flux) passing through any cross section of a vortex tube is an equal constant.*
- **A vortex tube in the flow field can neither begin nor end in the interior of the fluid.**

**Properties of vorticity field in time**

**Velocity circulation along any closed material curve moving with the fluid remains constant with time (Kelvin's theorem)**

- **Lagrange Theorem:** *Vortex can neither be created nor be destroyed.*
- **Helmholtz second theorem:** *Vortex tube, vortex surface and vortex line, as a material one, will be kept during motion.*
- **Helmholtz third theorem:** *Vortex strength of a material vortex tube will keep a constant during motion.*

# 5.9 Mechanism of Vortex

The conditions for vortex keeping constant in time and space are: fluid is ideal, either incompressible or barotropic, body forces are potential. Otherwise, vortex may possibly be created or destroyed. Three aspects following may generate vortex :

1) due to fluid viscosity

2) due to the non-barotropic, i.e. baroclinic of the fluid

3) due to non-potential body forces

# 5.9 Mechanism of Vortex

**1.Consider the situation that fluid is non-baratropic but still ideal, and body forces are also potential.**

The Euler equation is: $\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} = -\frac{\nabla p}{\rho} - \nabla \Pi$

Take a closed fluid line *L*. Material derivative of the velocity circulation is:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_L \frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t}\cdot d\mathbf{l} + \oint_L \mathbf{V}_r \cdot \frac{\mathrm{D}d\mathbf{l}}{\mathrm{D}t} = \oint_L \frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t}\cdot d\mathbf{l} + \oint_L \mathbf{V}_r \cdot d\mathbf{V}_r = \oint_L \frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t}\cdot d\mathbf{l} + \oint_L d\left(\frac{\mathbf{V}_r^2}{2}\right)$$

$$= -\oint_L \frac{\nabla p}{\rho}\cdot d\mathbf{l} - \oint_L \nabla \Pi \cdot d\mathbf{l} = -\oint_L \frac{\nabla p}{\rho}\cdot d\mathbf{l} - \oint_L d\Pi$$

$$= -\oint_L \frac{\nabla p}{\rho}\cdot d\mathbf{l} = \iint_S \nabla \times \left(\frac{\nabla p}{\rho}\right)\cdot d\mathbf{S} = \iint_S \frac{1}{\rho^2}\left(\nabla \rho \times \nabla p\right)\cdot d\mathbf{S}$$

*S* is a curved surface tented on *L*, whose normal is determined from the positive of *L* by the right-handed law.

# 5.9 Mechanism of Vortex

If the flow field is barotropic: $\rho = \rho(p) \Rightarrow \nabla\rho \times \nabla p = 0$

Namely, for barotropic fluids, gradient vector of density is parallel to the gradient vector of pressure. Accordingly, isopressure surface is identical with the isodensity surface.

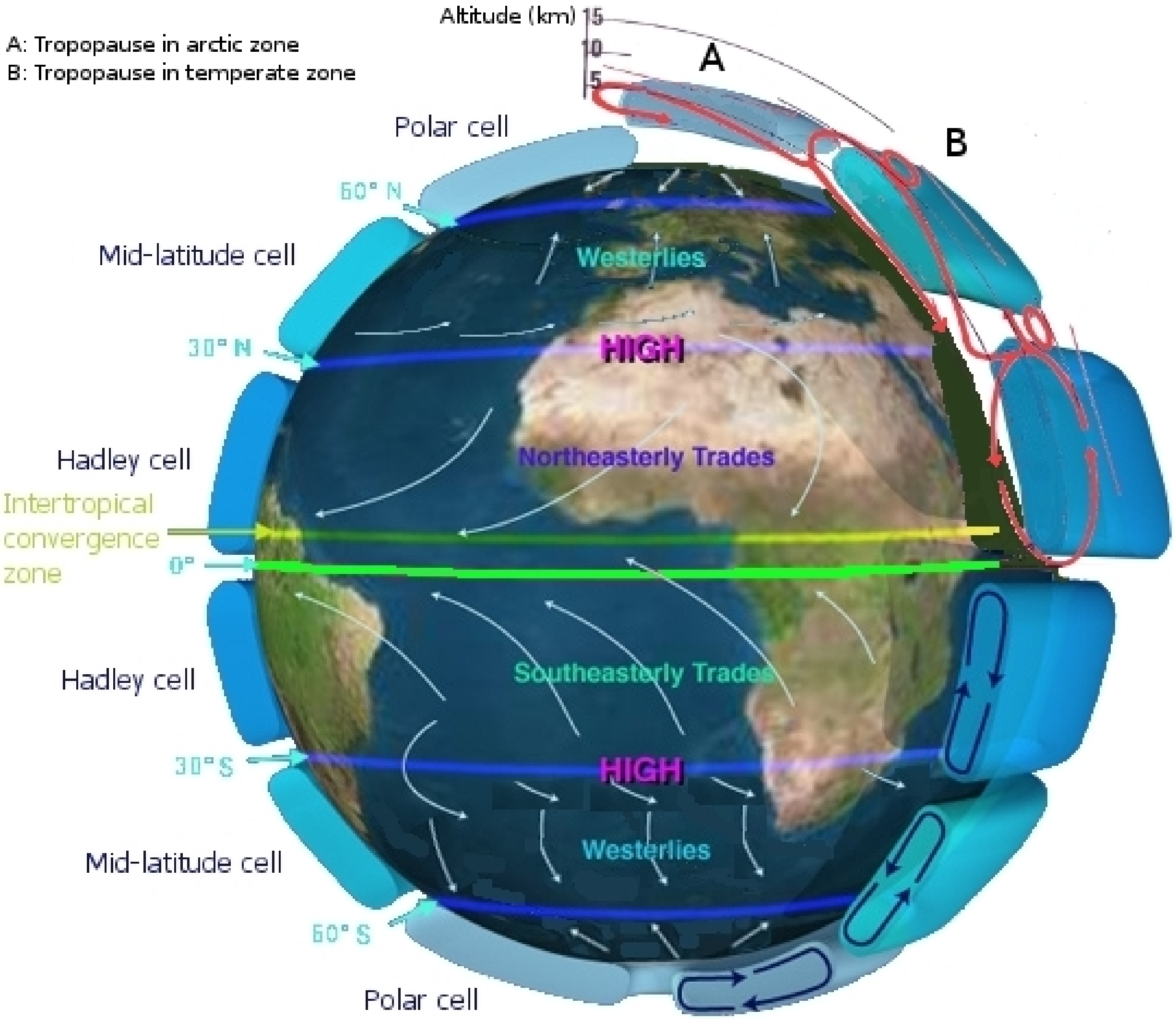
But here, $\nabla\rho \times \nabla p \neq 0$, thus, $\boxed{\dfrac{D\Gamma}{Dt} \neq 0}$

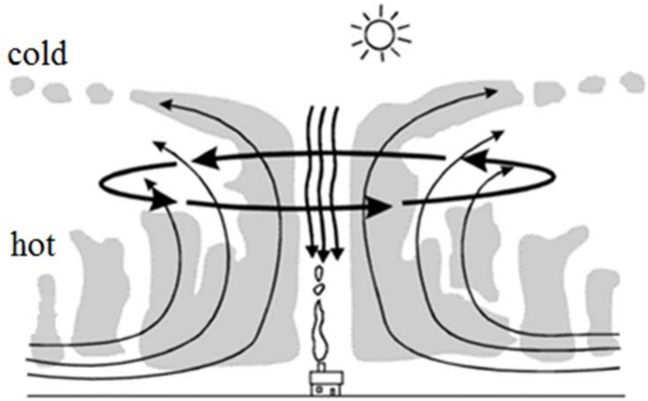
For baroclinic fluids, because the isopressure and isodensity surfaces do <u>not</u> coincide, velocity circulation will change with time, and vortex will generate or disappear during the course of fluid flowing.

When, $\dfrac{D\Gamma}{Dt} > 0$, vortex strength increases; conversely, decreases.

# 5.9 Mechanism of Vortex

## Trade wind



# 5.9 Mechanism of Vortex

**2. Consider the situation that body forces are non-potential, but fluid is ideal and barotropic.**

Take the motion of earth's atmosphere as an example. Consider the earth's rotation, Euler equation for the motion of the atmosphere relative to the earth is:

$$\frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t} = \mathbf{F} - \frac{\nabla p}{\rho} - \mathbf{a}_e - 2\left(\boldsymbol{\omega} \times \mathbf{V}_r\right)$$

Where $\mathbf{V}_r$ is the velocity of atmosphere relative to the earth, **F** is body force of gravity, $\mathbf{a}_e$ the convective acceleration, $\boldsymbol{\omega}$ the rotational angular velocity of the earth. The last term on the right side is Coriolis acceleration. Suppose the angular velocity of the earth $\boldsymbol{\omega}$ is constant, if the distance from a fluid particle to earth's axis is denoted by $\boldsymbol{R}$, then:

$$\mathbf{a}_e = -\nabla\left(\frac{\omega^2 R^2}{2}\right)$$

# 5.9 Mechanism of Vortex

Besides, the gravity is potential and the fluid is barotropic, so the material derivative of the velocity circulation is:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t}=\oint_L \frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t}\cdot d\mathbf{l}+\oint_L \mathbf{V}_r\cdot\frac{\mathrm{D}d\mathbf{l}}{\mathrm{D}t}=\oint_L \frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t}\cdot d\mathbf{l}+\oint_L \mathbf{V}_r\cdot d\mathbf{V}_r=\oint_L \frac{\mathrm{D}\mathbf{V}_r}{\mathrm{D}t}\cdot d\mathbf{l}+\oint_L d\left(\frac{\mathbf{V}_r^2}{2}\right)$$

$$=-\oint_L \nabla\cdot\left(\Pi+\frac{\omega^2R^2}{2}+\frac{p}{\rho}\right)\cdot d\mathbf{l}-\oint_L 2\left(\boldsymbol{\omega}\times\mathbf{V}_r\right)\cdot d\mathbf{l}$$

$$=-\oint_L d\left(\Pi+\frac{\omega^2R^2}{2}+\frac{p}{\rho}\right)-\oint_L 2\left(\boldsymbol{\omega}\times\mathbf{V}_r\right)\cdot d\mathbf{l}=-\oint_L 2\left(\boldsymbol{\omega}\times\mathbf{V}_r\right)\cdot d\mathbf{l}$$

It indicates the impacts of body force Coriolis on the velocity circulation, which is a factor of forming the vortex.

# 5.9 Mechanism of Vortex

## Typhoon

# 5.10 Vorticity Dynamic Equation

**A vortex motion must satisfy the basic equations of fluids, i.e., the Navier-Stokes equations (for real fluids), or Euler equation (for ideal fluids)**

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\dfrac{1}{\rho} \nabla p + \mathbf{f} + \nu \nabla^2 \mathbf{V}, \quad \nabla \cdot \mathbf{V} = 0 \\ \dfrac{\partial \mathbf{V}}{\partial t} + \nabla \left( \dfrac{|\mathbf{V}|^2}{2} \right) - \mathbf{V} \times \mathbf{\Omega} = -\dfrac{1}{\rho} \nabla p + \mathbf{f} + \nu \nabla^2 \mathbf{V}, \quad \nabla \cdot \mathbf{V} = 0 \end{cases}$$

If the flow is incompressible and the body forces are potential, then:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla \left( \frac{|\mathbf{V}|^2}{2} \right) - \mathbf{V} \times \mathbf{\Omega} = -\nabla \left( \frac{p}{\rho} \right) + \nu \nabla^2 \mathbf{V} + \nabla (\Pi), \quad \nabla \cdot \mathbf{V} = 0$$

# 5.10 Vorticity Dynamic Equation

Take the cross product (curl) for both sides of the equation, then:

$$\nabla\times\left(\frac{\partial\mathbf{V}}{\partial t}\right)+\nabla\times\nabla\left(\frac{|V|^2}{2}\right)-\nabla\times(\mathbf{V}\times\mathbf{\Omega})$$

$$=-\nabla\times\nabla\left(\frac{p}{\rho}\right)+\nabla\times\left(\nu\nabla^2\mathbf{V}\right)+\nabla\times\nabla(\Pi),$$

Tensor formula: $\nabla\times\nabla\phi=0$ , the curl of a gradient is 0:

$$\nabla\times(\mathbf{V}\times\mathbf{\Omega})=(\mathbf{\Omega}\cdot\nabla)\mathbf{V}-(\mathbf{V}\cdot\nabla)\mathbf{\Omega}+\mathbf{V}(\nabla\cdot\mathbf{\Omega})-\mathbf{\Omega}(\nabla\cdot\mathbf{V})$$

$$\nabla^2\mathbf{V}=\nabla(\nabla\cdot\mathbf{V})-\nabla\times(\nabla\times\mathbf{V})=-\nabla\times\mathbf{\Omega}$$

$$\nabla^2\mathbf{\Omega}=\nabla(\nabla\cdot\mathbf{\Omega})-\nabla\times(\nabla\times\mathbf{\Omega})$$

# 5.10 Vorticity Dynamic Equation

$$\nabla\times\left(\frac{\partial \mathbf{V}}{\partial t}\right)=\frac{\partial\left(\nabla\times\mathbf{V}\right)}{\partial t}=\frac{\partial \mathbf{\Omega}}{\partial t}$$

Reorganize:

$$\frac{\partial \mathbf{\Omega}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{\Omega}-\left(\mathbf{\Omega}\cdot\nabla\right)\mathbf{V}=\nu\nabla^2\mathbf{\Omega}$$

$$\frac{\mathrm{D}\mathbf{\Omega}}{\mathrm{D}t}=\left(\mathbf{\Omega}\cdot\nabla\right)\mathbf{V}+\nu\nabla^2\mathbf{\Omega}$$

This is the **vorticity dynamic equation** for incompressible fluids with potential body forces, which is also called **vorticity transport equation**. The greatest advantage of this equation is that there is no pressure, density and body force, it only includes velocity and vorticity, that means the pressure has no direct relation with the transportation of the vorticity.

# 5.10 Vorticity Dynamic Equation

**For ideal fluids, only if the fluid is barotropic and the body force is potential, then:**

$$\frac{\partial \mathbf{\Omega}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{\Omega} - (\mathbf{\Omega} \cdot \nabla)\mathbf{V} + \underline{\mathbf{\Omega}(\nabla \cdot \mathbf{V})} = 0$$

$$\frac{\mathrm{D}\mathbf{\Omega}}{\mathrm{D}t} = (\mathbf{\Omega} \cdot \nabla)\mathbf{V} - \underline{\mathbf{\Omega}(\nabla \cdot \mathbf{V})}$$

This is the **vorticity dynamic equation** for ideal and barotropic fluids with potential body force, which is also called **Helmholtz equation**. Similarly, the greatest advantage of this equation is that there is no pressure, density and body force, it only includes velocity and vorticity. This equation is also valid for compressible fluids.

# 5.11 Induced Velocity Field

For a fluid at rest, to make it in motion, there should be either source/sink in the flow field (divergence of velocity is not 0, $\nabla \cdot \mathbf{V} \neq 0$), or vortex in the flow field (vorticity is not 0, $\nabla \times \mathbf{V} \neq 0$ ). Therefore, source/sink and vortex are two factors to induce the motion of fluid.

In a mathematical sense: if there is divergence field or vorticity field in the flow field, then the velocity field **V** of a fluid can be uniquely determined, i.e.,

$$\left.\begin{aligned} &\nabla \cdot \mathbf{V} = H(x, y, z, t), && \text{divergence field} \\ &\nabla \times \mathbf{V} = \boldsymbol{\Omega}(x, y, z, t), && \text{vorticity field} \\ &\mathbf{V} \cdot \mathbf{n}\big|_{\Sigma} = \mathbf{U}_{\Sigma}(x, y, z, t), && \text{boundary condition on solid surfaces} \end{aligned}\right\} \Rightarrow$$

Determine the velocity field **V** uniquely

# 5.11 Induced Velocity Field

**Verification of the uniqueness**: Suppose there are two velocity fields $V_1$ and $V_2$, both satisfying equations of divergence field, vorticity field and boundary conditions. Verify that $V_1=V_2$.

**Verification:** Let $V = V_1 - V_2$ , then V satisfies:

$$\nabla \cdot \mathbf{V} = 0, \quad \nabla \times \mathbf{V} = 0, \quad \left(\mathbf{V} \cdot \mathbf{n}\right)\Big|_{\Sigma} = 0$$

Because the curl of V is 0, i.e., for irrotational flow, there must be a velocity potential ϕ, i.e., $\mathbf{V} = \nabla \phi$

And since the divergence of V is 0, then the velocity potential ϕ satisfies Laplace equation: $\nabla^2 \phi = 0, \quad \frac{\partial \phi}{\partial \mathbf{n}} = \left(\mathbf{V} \cdot \mathbf{n}\right)\Big|_{\Sigma} = 0$

Based on the theory of Laplace equation, solution for the equation above is: ϕ= Const, thus: $\mathbf{V} = \nabla \phi \equiv 0$

Which verifies that $V_1=V_2$, the solution is unique.

# 5.11 Induced Velocity Field

**Problem**: In an infinite region, if there is no solid boundary, and it satisfies the divergence equation and curl equation. Determine the velocity field, i.e., :

$$\left.\begin{aligned}\nabla \cdot \mathbf{V} &= H(x, y, z, t) \\ \nabla \times \mathbf{V} &= \mathbf{\Omega}(x, y, z, t)\end{aligned}\right\} \Rightarrow \boxed{\text{Determine velocity field V}}$$

**Solution:** According to the uniqueness of the solution, the velocity field exists and is unique. Because both equations of divergence field and vorticity field are linear, their solution, the velocity field can be decomposed into two parts: $V = V_e + V_v$, where:

1) $V_e$ satisfies: $\nabla \cdot \mathbf{V}_e = H(x, y, z, t), \quad \nabla \times \mathbf{V}_e = 0$

2) $V_v$ satisfies: $\nabla \cdot \mathbf{V}_v = 0, \quad \nabla \times \mathbf{V}_v = \mathbf{\Omega}(x, y, z, t)$

# 5.11 Induced Velocity Field

In this way, the problem is changed to a linear superposition of two problems:

$$\left.\begin{aligned}\nabla\cdot\mathbf{V}&=H(x,y,z,t)\\\nabla\times\mathbf{V}&=\boldsymbol{\Omega}(x,y,z,t)\end{aligned}\right\}\Leftrightarrow\left.\begin{aligned}\nabla\cdot\mathbf{V}_e&=H(x,y,z,t)\\\nabla\times\mathbf{V}_e&=0\end{aligned}\right\}+\left.\begin{aligned}\nabla\cdot\mathbf{V}_v&=0\\\nabla\times\mathbf{V}_v&=\boldsymbol{\Omega}(x,y,z,t)\end{aligned}\right\}$$

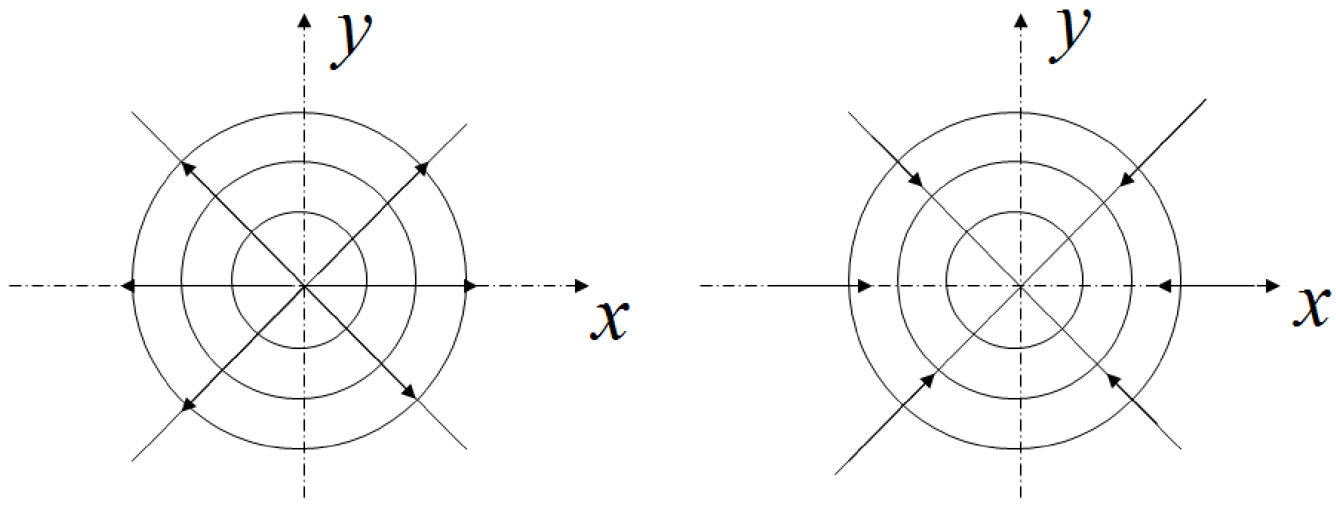
## 1) First to solve $\mathbf{V}_e$

$$\mathbf{V}_e=\frac{1}{4\pi}\iiint H(\xi,\eta,\zeta,t)\frac{\mathbf{r}}{r^3}d\xi\,d\eta\,d\zeta$$

# 5.11 Induced Velocity Field

## Source and sink

The fluid flows radially outward through a line from the origin, is called a **source** flow, the origin is a source point; if the fluid flows radially inward through a line toward the origin, is called a **sink** flow, the origin is a sink point. For both source and sink, there is only radial velocity, since the flow is a purely radial flow.




(a) (b)

# 5.11 Induced Velocity Field

From the continuity principle of the fluid: in polar (spherical) coordinates, the **volume flow rate *m*** (also called as the strength of the source/sink flow) per unit height of a fluid through any cylindrical surface (spherical surface) is constant.

For 2D: $$V_r \cdot 2\pi r = \pm m \quad \Rightarrow \quad V_r = \pm \frac{m}{2\pi r}, \quad V_\theta = 0$$

For 3D: $$V_r \cdot 4\pi r^2 = \pm m \quad \Rightarrow \quad V_r = \pm \frac{m}{4\pi r^2}, \quad V_\theta = 0, \quad V_z = 0$$

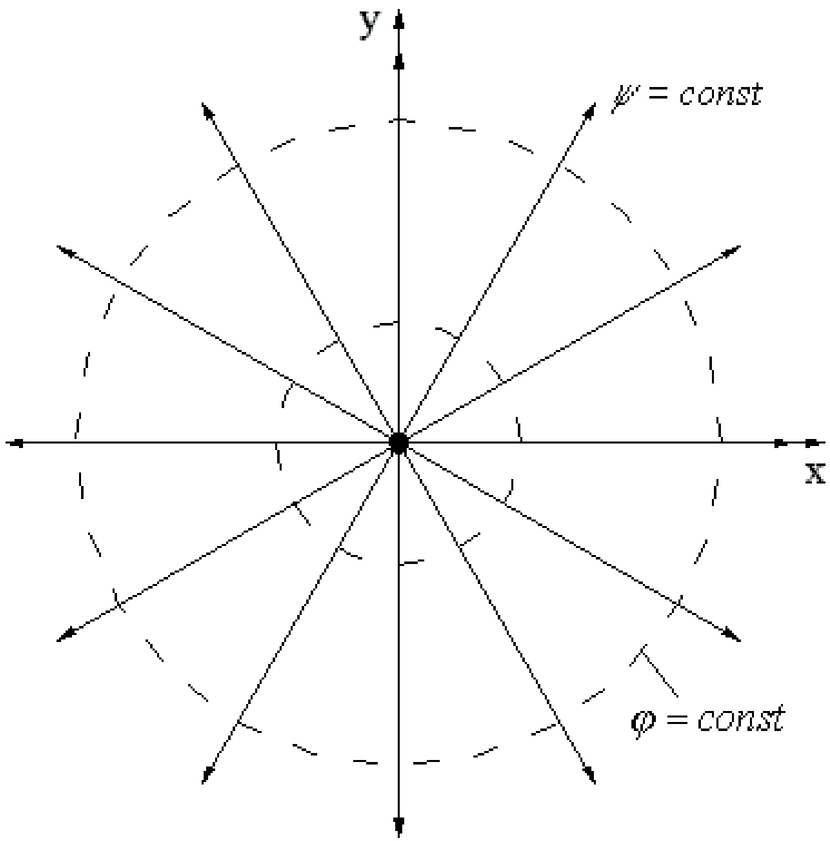
It can be verified that except for the source and sink, the above-mentioned flow field is irrotational and the divergence is 0, then:

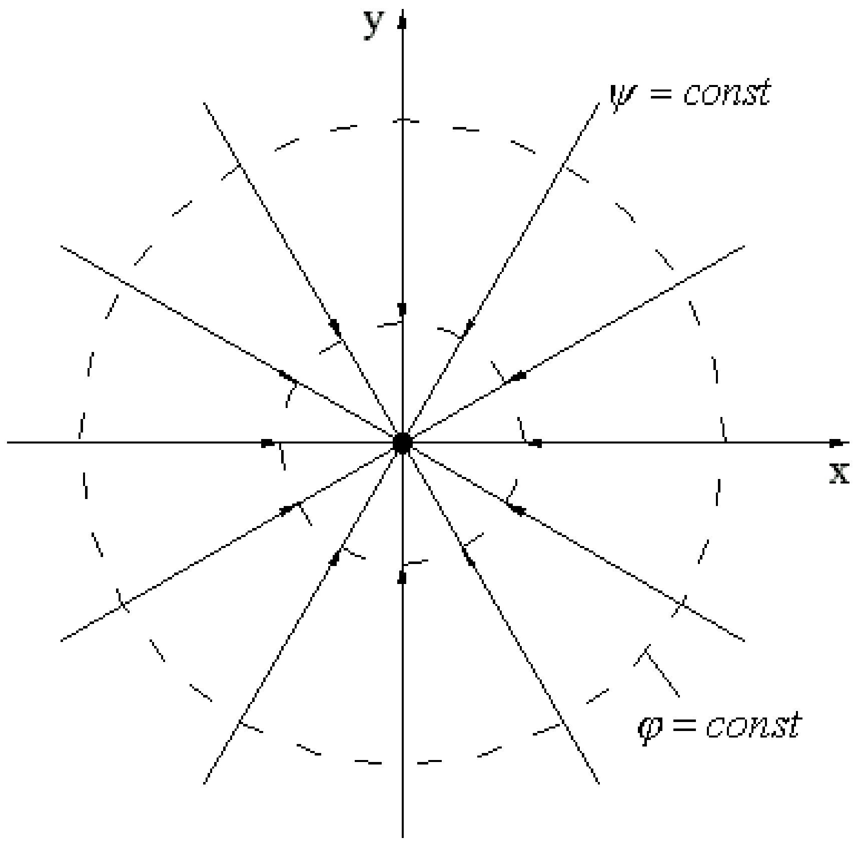
For 2D: $$\phi = \frac{m}{2\pi} \ln r, \quad \psi = \frac{m}{2\pi} \theta$$

For 3D: $$\phi = \frac{m}{4\pi r}$$

# 5.11 Induced Velocity Field

According to the derived stream function and velocity potential function, equipotential lines are a series of concentric circles with different radii, streamlines are a series of radial lines with different polar angles.



# 5.11 Induced Velocity Field

**2) Secondly, to solve $\mathbf{V}_v$,** which satisfies:

$$\nabla \cdot \mathbf{V}_v = 0, \quad \nabla \times \mathbf{V}_v = \mathbf{\Omega}(x, y, z, t)$$

The divergence of curl is 0, and $\nabla \cdot \mathbf{V}_v = 0$, so there must be a vector potential (function) **B**, satisfying:

$$\mathbf{V}_v = \nabla \times \mathbf{B}$$

Substituting it into the vorticity equation, and applying the following tensor formula:

$$\nabla^2 \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V})$$

We get: $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mathbf{\Omega}$

$$\begin{cases} \nabla^2 \mathbf{B} = -\mathbf{\Omega} \\ \nabla \cdot \mathbf{B} = 0 \end{cases} \Longrightarrow \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mathbf{\Omega}$$

# 5.11 Induced Velocity Field

The first equation is a Poisson equation, its solution is:

$$\mathbf{B} = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\Omega}(\xi,\eta,\zeta,t)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\xi d\eta d\zeta$$

Next to check if the solution satisfies the second equation:

$$\nabla\cdot\mathbf{B} = \frac{1}{4\pi}\iiint \nabla\cdot\left(\frac{\boldsymbol{\Omega}}{r}\right)d\xi d\eta d\zeta = \frac{1}{4\pi}\iint \frac{\boldsymbol{\Omega}\cdot\mathbf{n}}{r}\,dS$$

To make $\nabla\cdot\mathbf{B}=0$ then there must be: $\boldsymbol{\Omega}\cdot\mathbf{n}\big|_S = 0$, which requires the boundary at infinity or in local region should be a vortex surface. In this case, **B** is the original solution to a vorticity equation.

# 5.11 Induced Velocity Field

Under this condition, the induced velocity distribution of a vorticity field can be obtained:

$$\mathbf{V}_v = \nabla\times\mathbf{B} = \nabla\times\left(\frac{1}{4\pi}\iiint\frac{\mathbf{\Omega}(\xi,\eta,\zeta,t)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\varsigma)^2}}d\xi d\eta d\zeta\right)$$

$$= -\frac{1}{4\pi}\iiint\mathbf{\Omega}(\xi,\eta,\zeta,t)\times\nabla\left(\frac{1}{r}\right)d\xi d\eta d\zeta = \frac{1}{4\pi}\iiint\frac{\mathbf{\Omega}(\xi,\eta,\zeta,t)\times\mathbf{r}}{r^3}d\xi d\eta d\zeta$$

i.e.,

$$\mathbf{V}_v = \frac{1}{4\pi}\iiint\frac{\mathbf{\Omega}(\xi,\eta,\zeta,t)\times\mathbf{r}}{r^3}d\xi d\eta d\zeta$$

Therefore, if the vorticity field $\mathbf{\Omega}$ is known, the induced velocity distribution of the motion of surrounding spatial points by a vortex can be determined.

# 5.11 Induced Velocity Field

**Thus, the total induced velocity by divergence field (source/sink) and vorticity field (vortex) is:**

$$\mathbf{V} = \mathbf{V}_e + \mathbf{V}_v$$

$$= \frac{1}{4\pi}\iiint H(\xi,\eta,\zeta,t)\frac{\mathbf{r}}{r^3}d\xi d\eta d\zeta + \frac{1}{4\pi}\iiint \frac{\mathbf{\Omega}(\xi,\eta,\zeta,t)\times\mathbf{r}}{r^3}d\xi d\eta d\zeta$$

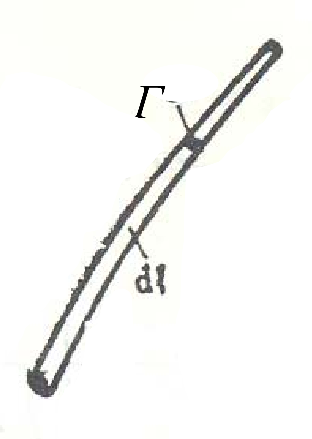
$$= \frac{1}{4\pi}\iiint \frac{1}{r^3}\left[H(\xi,\eta,\zeta,t)\mathbf{r} + \mathbf{\Omega}(\xi,\eta,\zeta,t)\times\mathbf{r}\right]d\xi d\eta d\zeta$$

where $(x, y, z)$ is a spatial point, $(\xi, \eta, \varsigma)$ is a integral variable

$$\mathbf{r} = (x-\xi, y-\eta, z-\zeta)^{\mathrm{T}}, \quad r = |\mathbf{r}| = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

# 5.11 Induced Velocity Field

Application 2: Determine the induced velocity distribution of a vortex filament (length is *L*, vortex strength is $\Gamma$)




**Solution:** A vortex filament is the vorticity concentrates on a vortex tube whose cross-section is of extremely small dimension, so the vortex filament can be approximated as a line. Take a small segment *dl* from this thin tube, the cross-sectional area is *S*, the volume is *Sdl*. Let the vorticity be Ω, then:

$$\lim_{\substack{\Delta S\to 0\\ |\mathbf{\Omega}|\to\infty}} S\left|\mathbf{\Omega}\right| = \Gamma, \qquad \mathbf{\Omega} d\xi d\eta d\varsigma = \mathbf{\Omega} d\tau = \left|\mathbf{\Omega}\right| S d\mathbf{l} = \Gamma d\mathbf{l}$$

Substitute into the vortex-induced velocity equation, thus:

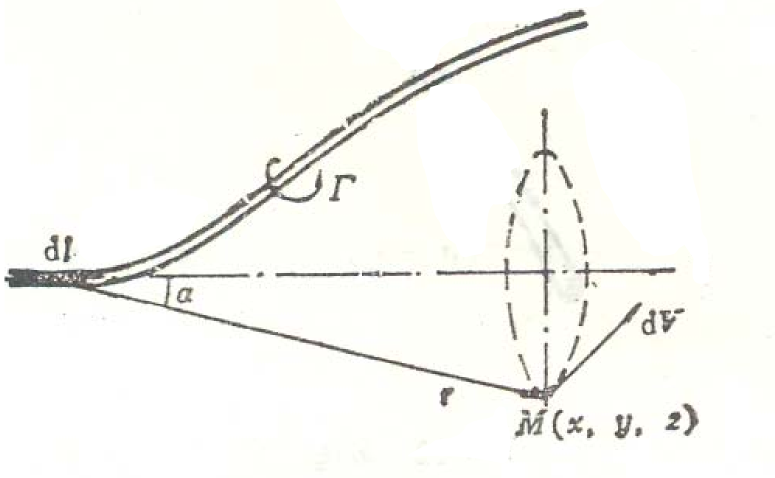
$$\mathbf{V}_v = \frac{1}{4\pi}\iiint \frac{\mathbf{\Omega}(\xi,\eta,\varsigma,t)\times\mathbf{r}}{r^3} d\xi d\eta d\varsigma = \frac{\Gamma}{4\pi}\int_L \frac{d\mathbf{l}\times\mathbf{r}}{r^3}$$

# 5.11 Induced Velocity Field

**This equation is identical to the Biot-Savart equation in electromagnetics, which can be used for liquid-electricity analogy. In electromagnetics, the Biot-Savart equation is used to determine the induced magnetic strength field around a live wire; while in fluid mechanics, this equation is used to determine the induced velocity field around a vortex. Let the angle between r and dl be $\alpha$, then Biot-Savart equation turns to be:**

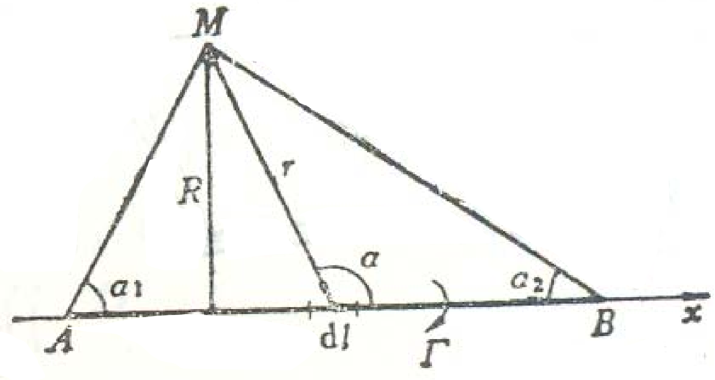
$$\left|\mathbf{V}_v\right| = \frac{\Gamma}{4\pi}\int_L \frac{\sin\alpha}{r^2}\,dl$$

Direction of induced velocity is the direction of the cross product of *dl* and r.



# 5.11 Induced Velocity Field

Application 3: A section of straight vortex filament AB is shown in the figure. Its vortex strength is $\Gamma$, the direction is the same as the positive direction of *x* axis. The distance between a spatial point M and the vortex filament is *R*. Determine the induced velocity at point M by the vortex filament.




**Solution:** Take a 1-D vortex filament dl from AB, from the geometrical relationship:

$$x = R\,\mathrm{tg}\left(\alpha - \frac{\pi}{2}\right) = -R\cot\alpha,\ dl = dx = R\,\mathrm{cse}^2\alpha\, d\alpha,\ r = R\sec\left(\alpha - \frac{\pi}{2}\right) = -R\,\mathrm{cse}\,\alpha$$

Substitute it into the Biot-Savart equation of vortex-induced velocity:

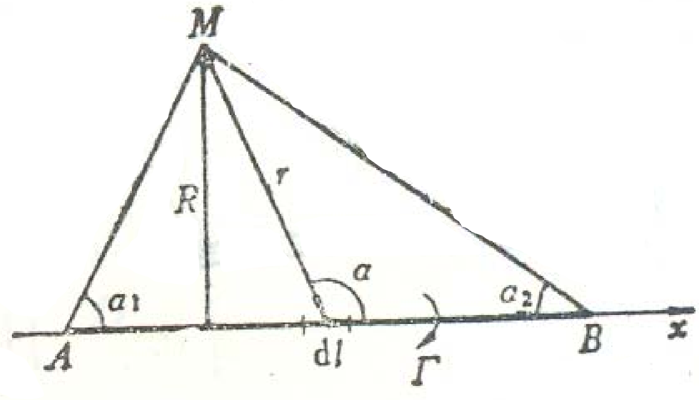
$$V = \frac{\Gamma}{4\pi}\int_L \frac{\sin\alpha}{r^2} dl = \frac{\Gamma}{4\pi R}\int_{\alpha_1}^{\pi-\alpha_2} \sin\alpha\, d\alpha = \frac{\Gamma}{4\pi R}(\cos\alpha_1 + \cos\alpha_2)$$

# 5.11 Induced Velocity Field

For semi-infinite vortex filament: $\alpha_1 = \frac{\pi}{2}, \ \alpha_2 = 0$

Substitute into the equation:

$$V = \frac{\Gamma}{4\pi R}$$




For infinite vortex filament: $\alpha_1 = 0, \ \alpha_2 = 0$

Substitute into the equation:

$$V = \frac{\Gamma}{2\pi R}$$

For an infinite vortex filament, in any plane perpendicular to the vortex filament, the induced velocity are the same, therefore, it can be treated as the 2D vortex induces a 2D flow, the velocity distribution is:

$$V_r = 0, \ V_\theta = \frac{\Gamma}{2\pi r}$$

where *r* is the distance between a spatial point and the vortex.

# 5.11 Induced Velocity Field

## Vortex:

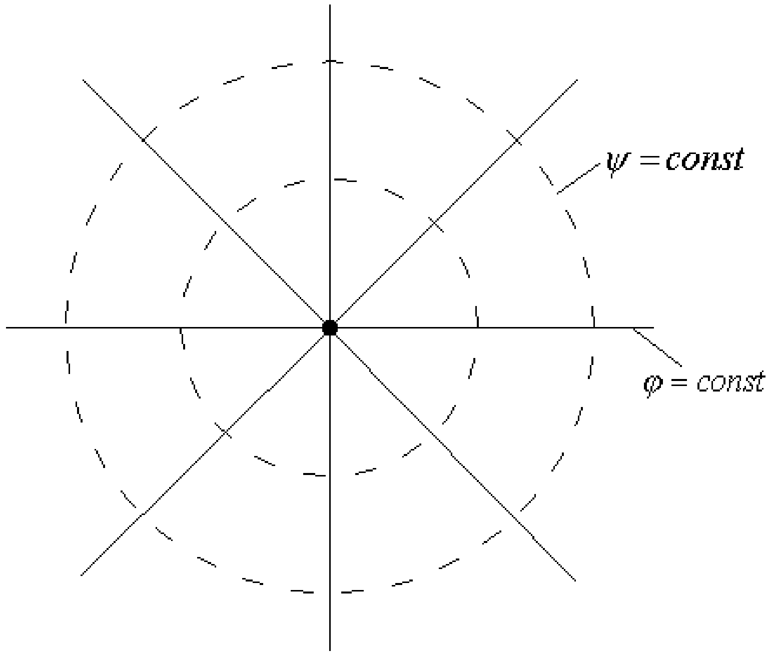
If the radius of a straight vortex filament $r_b \to 0$, then the flow in the plane perpendicular to this vortex filament is called vortex or free vortex flow, the center of the vortex flow is called the vortex point. Except for the vortex point, the flow field is irrotational and its divergence is 0, then:

$$V_r = \frac{\partial \phi}{\partial r} = 0, \quad V_\theta = \frac{\partial \phi}{r \partial \theta} = \frac{\Gamma}{2\pi r} \Rightarrow \phi = \frac{\Gamma}{2\pi}\theta$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0, \quad V_\theta = -\frac{\partial \psi}{\partial r} \Rightarrow \psi = -\frac{\Gamma}{2\pi}\ln r$$

# 5.11 Induced Velocity Field

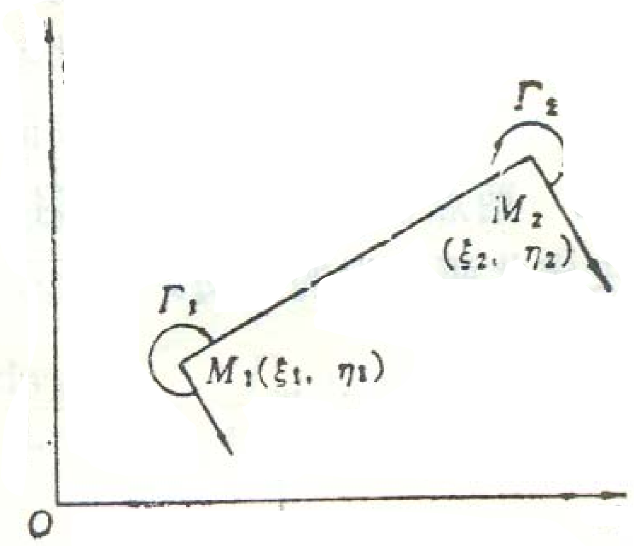
The equipotential lines of a vortex flow field are radial lines with different polar angles, i.e., ϕ=constant; The streamlines are concentric circles with different radii, i.e., ψ=constant. **Opposite** to the source/sink. The strength of a vortex Γ is the circulation around the axis of the vortex.




When Γ>0, the circulation is counterclockwise; when Γ<0, the circulation is clockwise. From Stokes' theorem, the strength Γ of a vortex depends on the vortex strength (vortex flux).

# 5.11 Induced Velocity Field

Application 4: The original position of two vortices is shown in the figure. Their vortex strengths are $\Gamma_1$ and $\Gamma_2$. Analyze the motion of these two vortices.




**Solution:** If regard the two vortices as a whole.

When $\Gamma_1 + \Gamma_2 = 0$, their vortex strengths are the same but with different directions, the two vortices only have the translational motion as a whole, there is no rotation or relative motion. Conversely, there will be rotation and relative motion.

The motion of point $M_1$, is induced by the vortex at point $M_2$, from the vortex-induced Biot-Savart equation, then:

$$u_{M_1} = \frac{d\xi_1}{dt} = -\frac{\Gamma_2}{2\pi}\frac{\eta_1 - \eta_2}{r^2}, \quad v_{M_1} = \frac{d\eta_1}{dt} = \frac{\Gamma_2}{2\pi}\frac{\xi_1 - \xi_2}{r^2}$$

# 5.11 Induced Velocity Field

The motion of point $M_2$, is induced by the vortex at point $M_1$, from the vortex-induced Biot-Savart equation, then:

$$u_{M_2} = \frac{d\xi_2}{dt} = -\frac{\Gamma_1}{2\pi}\frac{\eta_2 - \eta_1}{r^2}, \quad v_{M_2} = \frac{d\eta_2}{dt} = \frac{\Gamma_1}{2\pi}\frac{\xi_2 - \xi_1}{r^2}$$

Where $r = \sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2}$ is the distance between the two vortices.

**If $\Gamma_1 + \Gamma_2 = 0$**, obviously, $u_{M_1} = u_{M_2}, \ v_{M_1} = v_{M_2}$ the two vortices have the uniform translational motion, there is no relative motion.

**If $\Gamma_1 + \Gamma_2 \neq 0$**, there are relative motion and rotational motion of the two vortices.

The first equation times $\Gamma_1$, the second equation times $\Gamma_2$, then sum up the two equations and by integrating:

$$\Gamma_1\xi_1 + \Gamma_2\xi_2 = \text{const}, \quad \Gamma_1\eta_1 + \Gamma_2\eta_2 = \text{const}$$

# 5.11 Induced Velocity Field

Because $\Gamma_1+\Gamma_2 \neq 0$, and $\Gamma_1+\Gamma_2=$ constant, then:

$$\xi_c = \frac{\Gamma_1\xi_1 + \Gamma_2\xi_2}{\Gamma_1 + \Gamma_2} = \text{const}, \qquad \eta_c = \frac{\Gamma_1\eta_1 + \Gamma_2\eta_2}{\Gamma_1 + \Gamma_2} = \text{const}$$

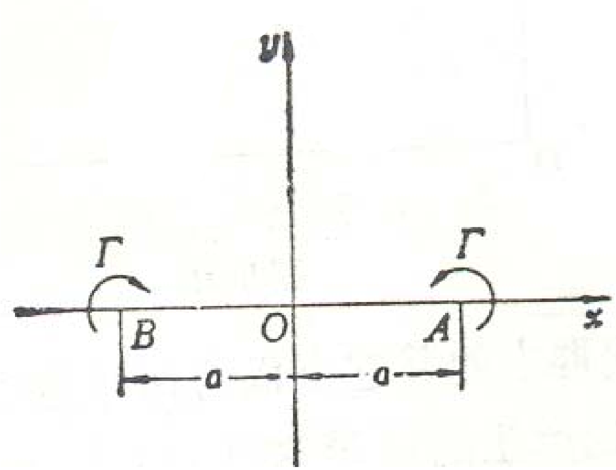
If there are several vortices, and $\sum \Gamma_i \neq 0$ , $\sum \Gamma_i = \text{const}$ , similarly, it can be derived that:

$$\xi_c = \frac{\sum \Gamma_i\xi_i}{\sum \Gamma_i} = \text{const}, \quad \eta_c = \frac{\sum \Gamma_i\eta_i}{\sum \Gamma_i} = \text{const}$$

Where point $(\xi_c, \eta_c)$ is called the center of gravity of the vortex group, which is similar to the equation of computing the centroid of several material particle. It should be noted that, although there are relative motions between vortices, the center of gravity of the vortex group does not change during the motion. The rotational motion and relative motion of vortices are around the center of gravity of the vortex group as the origin.

# 5.11 Induced Velocity Field

Application 5: The original position of two vortices is shown in the figure. Their vortex strengths have the same magnitude but opposite directions. Determine the motion of these two vortices.




**Solution:** The motion of a vortex must be induced by other vortex.

Because $\Gamma_A+\Gamma_B=0$, the two vortices have a translational motion as a whole. The motion of point A, is induced by the vortex at point B, from the vortex-induced Biot-Savart equation, then:

$$u_A = \frac{dx_A}{dt} = 0, \quad v_A = \frac{dy_A}{dt} = -\frac{\Gamma}{2\pi \cdot 2a} = -\frac{\Gamma}{4\pi a}$$

Integrating: $x_A = C_1, \quad y_A = -\frac{\Gamma}{4\pi a} t + C_2$

At $t=0$, $x_A=a$, $y_A=0$, then: $C_1 = a$, $C_2 = 0$

# 5.11 Induced Velocity Field

The motion of point B, is induced by the vortex at point A, from the vortex-induced Biot-Savart equation, then:

$$u_B = \frac{dx_B}{dt} = 0, \quad v_B = \frac{dy_B}{dt} = -\frac{\Gamma}{2\pi \cdot 2a} = -\frac{\Gamma}{4\pi a}$$

Integrating: $x_B = C_3, \quad y_B = -\frac{\Gamma}{4\pi a} t + C_4$

At $t$=0, $x_B$= -$a$, $y_B$=0, then: $C_3$ = -$a$, $C_4$ = 0

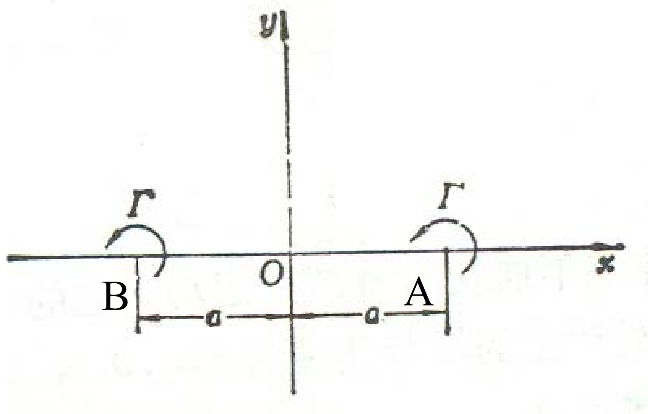
Therefore, the motion equation s of points A and B are:

$$x_A = a, \quad y_A = -\frac{\Gamma}{4\pi a} t$$

$$x_B = -a, \quad y_B = -\frac{\Gamma}{4\pi a} t$$

# 5.11 Induced Velocity Field

Application 6: The original position of two vortices is shown in the figure. Their vortex strengths have the same magnitude and direction. Determine the motion of these two vortices.




**Solution:** Because $\Gamma_A+\Gamma_B \neq 0$, the two vortices have relative motion and rotational motion. The center of gravity of the vortex group (fixed point) is:

$$\xi_c = \frac{\Gamma a - \Gamma a}{2\Gamma} = 0, \quad \eta_c = \frac{0-0}{2\Gamma} = 0$$

So the two vortices rotates around the origin 0.

# 5.11 Induced Velocity Field

The motion of point A, is induced by the vortex at point B, from the vortex-induced Biot-Savart equation, then:

$$u_A = \frac{dx_A}{dt} = 0, \quad v_A = \frac{dy_A}{dt} = \frac{\Gamma}{2\pi \cdot 2a} = \frac{\Gamma}{4\pi a}$$

Similarly, the motion of point B, is induced by the vortex at point A, from the vortex-induced Biot-Savart equation, then:

$$u_B = \frac{dx_B}{dt} = 0, \quad v_B = \frac{dy_B}{dt} = -\frac{\Gamma}{2\pi \cdot 2a} = -\frac{\Gamma}{4\pi a}$$

Both point A and B rotate around the origin 0 with the same angular velocity $\omega$:

$$\omega = \frac{\left|v_{A(B)}\right|}{a} = \frac{\Gamma}{4\pi a^2}$$

# 5.11 Induced Velocity Field

Expressed in polar coordinates, the equation of motion for point A and B are:

$$r_A = a, \quad \theta_A = \frac{\Gamma}{4\pi a^2} t$$

$$r_B = a, \quad \theta_B = \pi + \frac{\Gamma}{4\pi a^2} t$$