Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Review

- **Hydrostatics**

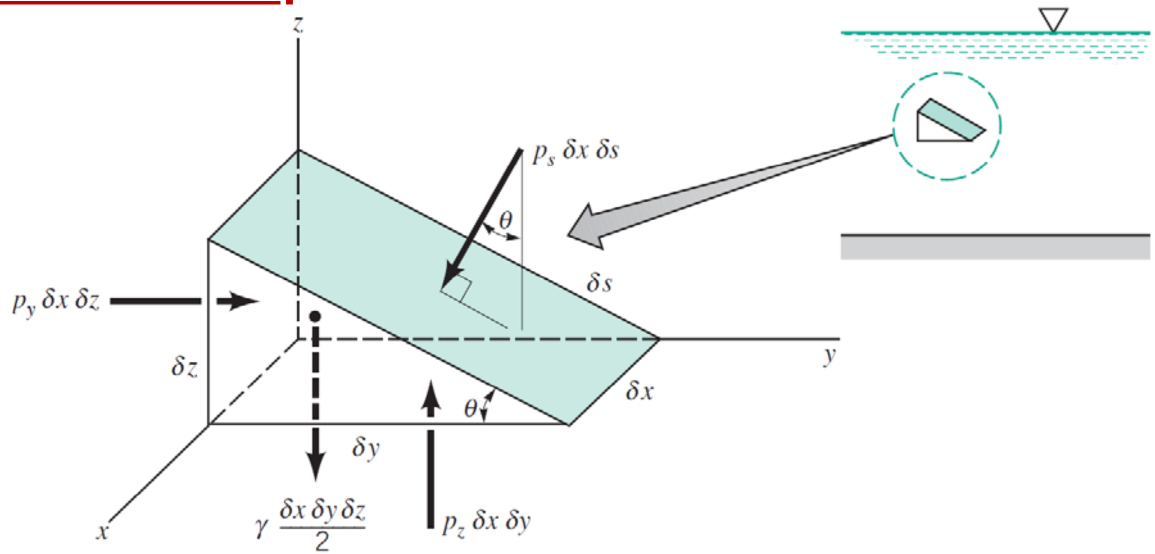
Pascal's law: Pressure at a point in a fluid is independent of direction as long as there are no shear stresses present, i.e., Pressure at a point has the same magnitude in all directions.

4.3 Pressure at a Point

By considering the equilibrium of a small triangular wedge of fluid extracted from a static fluid body, one can show that for *any* wedge angle θ, the pressures on the three faces of the wedge are equal in magnitude:

$$p_s = p_y = p_z \qquad \text{independent of } \theta$$

This result is known as Pascal's law, which states that the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shear stresses present.



4.3 Pressure at a Point

Shearing stresses=0: fluid element moves as a rigid body

According to Newton's second law: $\vec{F} = m\vec{a}$

$$\sum F_y = p_y \delta_x \delta_z - p_s \delta_x \delta_s \sin\theta = \rho \frac{\delta_x \delta_y \delta_z}{2} a_y$$

$$\sum F_z = p_z \delta_x \delta_y - p_s \delta_x \delta_s \cos\theta - \rho g \frac{\delta_x \delta_y \delta_z}{2} = \rho \frac{\delta_x \delta_y \delta_z}{2} a_z$$

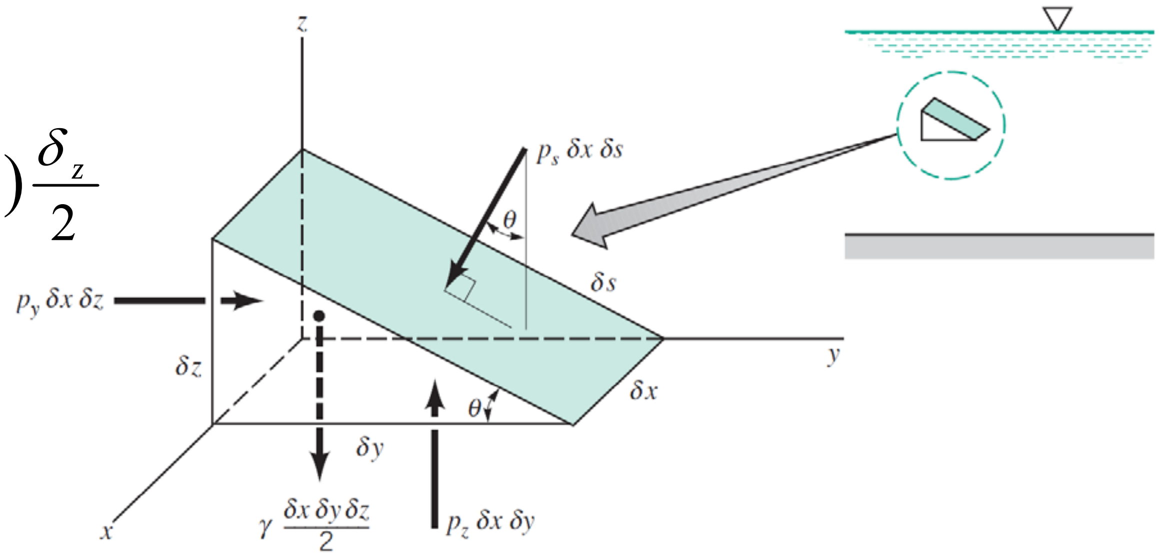
And, $\delta_y = \delta_s \cdot \cos\theta$, $\delta_z = \delta_s \cdot \sin\theta$

Then, $p_y - p_s = \rho a_y \dfrac{\delta_y}{2}$

$$p_z - p_s = (\rho a_z + \rho g)\frac{\delta_z}{2}$$

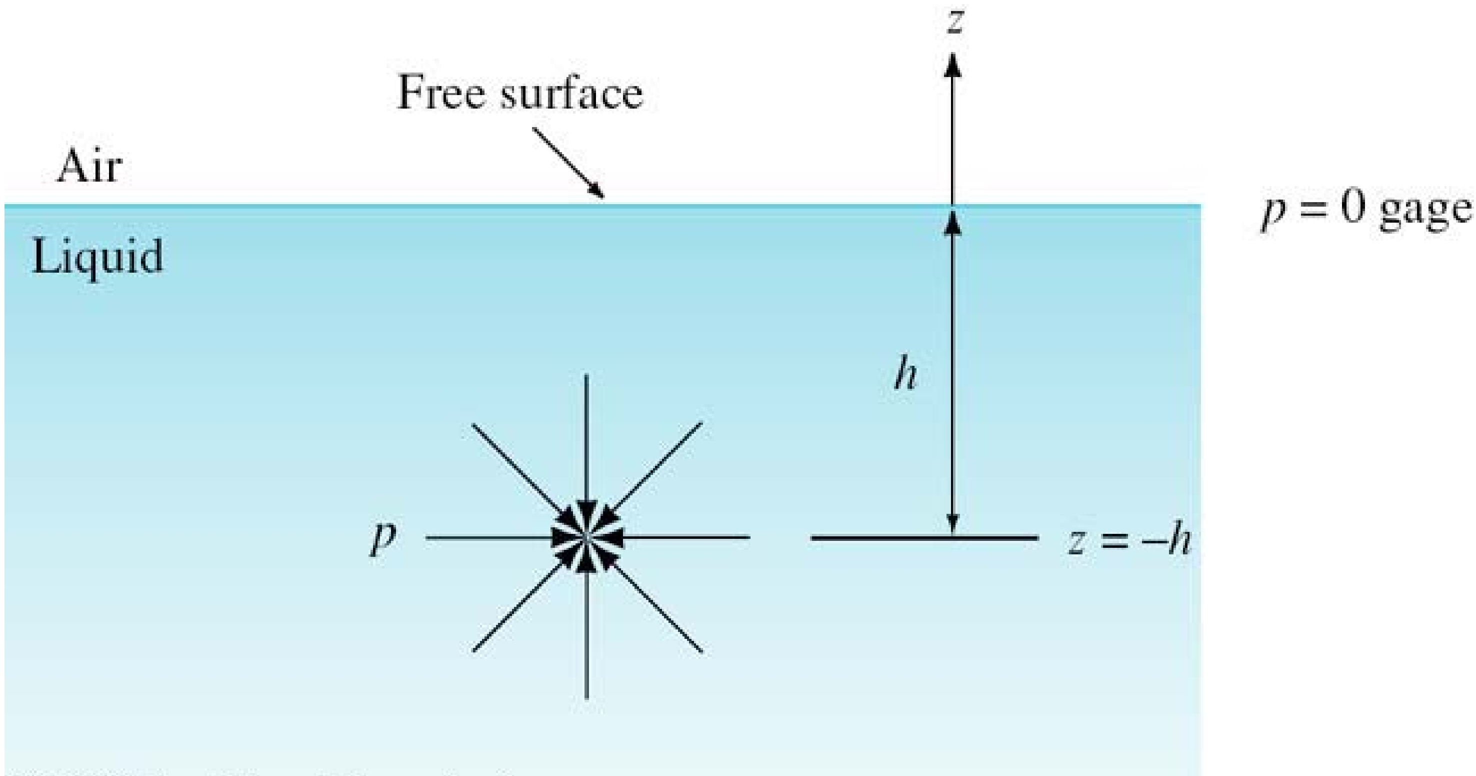
$\delta_y \to 0$, $\delta_z \to 0$:

$$p_s = p_y = p_z$$



4.3 Pressure at a Point

Pressure at a point has the <u>same</u> magnitude in all directions, this is called isotropic.



4.4 Pressure Variation with Depth

Consider a small vertical cylinder of fluid in equilibrium, where *positive z is pointing vertically upward*. Suppose the origin z=0 is set at the free surface of the fluid. Then the pressure variation at a depth $z = -h$ below the free surface is governed by:

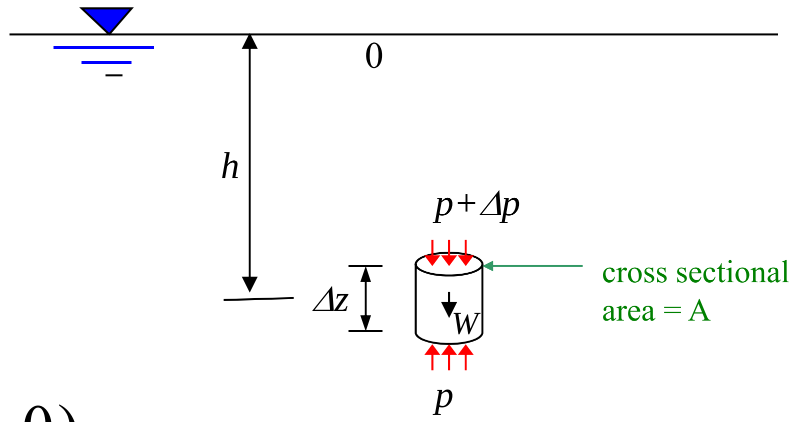
$$(p + \Delta p)A + W = pA$$

$$\Rightarrow \quad \Delta p A + \rho g A \Delta z = 0$$

$$\Rightarrow \quad \Delta p = -\rho g \Delta z$$

$$\Rightarrow \quad \frac{dp}{dz} = -\rho g$$

or $$\frac{dp}{dz} = -\gamma \qquad (\text{as } \Delta z \to 0)$$



Therefore, the hydrostatic pressure increases linearly with depth at the rate of the specific weight $\gamma \equiv \rho g$ of the fluid.

4.4 Pressure Variation with Depth

For a fluid with constant density,

$$p_{\text{below}} = p_{\text{above}} + \rho g \left|\Delta z\right|$$

As a diver goes down, the pressure on his ears increases. So, the pressure "below" is greater than the pressure "above".

4.4 Pressure Variation with Depth

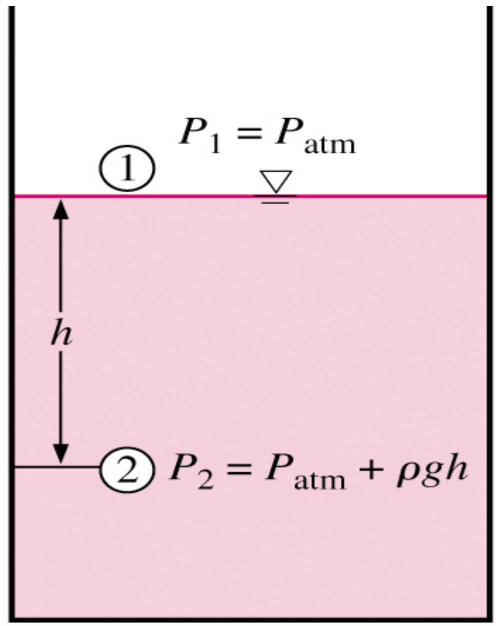
Homogeneous fluid: ρ is constant.

By simply integrating the equation above :

$$\int dp = -\int \rho g dz \qquad \Rightarrow \qquad p = -\rho g z + C$$

where C is an integration constant. When $z = 0$ (on the free surface), $p = C = p_0$ (the atmospheric pressure). Hence,

$$p = -\rho g z + p_0$$

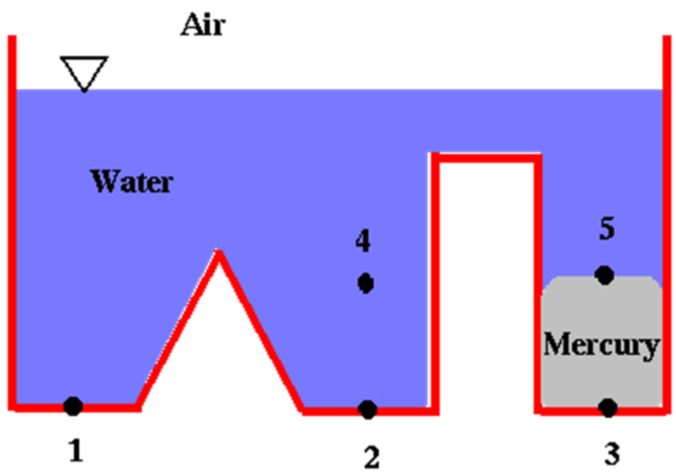


The equation derived above shows that when the density is constant, the pressure in a liquid at rest increases linearly with depth from the free surface.

4.5 Hydrostatic Pressure Difference between Two Points

There are several "rules" or comments which directly result from the equation above:

1) If you can draw a continuous line through the same fluid from point 1 to point 2, then $p_1 = p_2$ if $z_1 = z_2$.

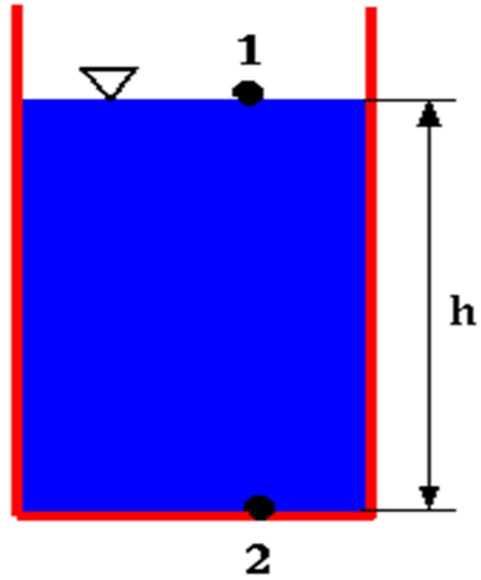


For example, consider the oddly shaped container. By this rule, $p_1 = p_2$ and $p_4 = p_5$ since these points are at the same elevation in the same fluid. However, p_2 does not equal p_3 even though they are at the same elevation, because one cannot draw a line connecting these points through the same fluid. In fact, p_2 is less than p_3 since mercury is denser than water.

4.5 Hydrostatic Pressure Difference between Two Points

2) Any free surface open to the atmosphere has atmospheric pressure, p_0.

(This rule holds not only for hydrostatics, but for any free surface exposed to the atmosphere, no matter the surface is moving, stationary, flat, or mildly curved.)

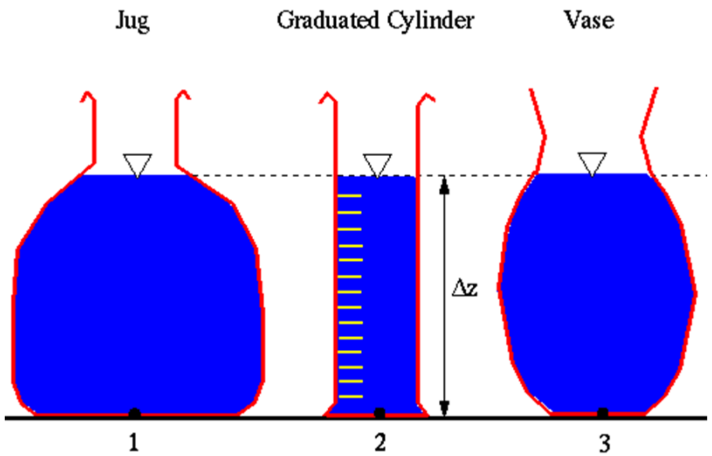


Consider the hydrostatics example of a container of water: The little upside-down triangle indicates a free surface, and means that the pressure there is atmospheric pressure, p_0. In other words, in this example, $p_1 = p_0$. To find the pressure at point 2, our hydrostatics equation is used: $p_2 = p_0+\rho gh$ (absolute pressure) or $p_2 = \rho gh$ (gage pressure).

4.5 Hydrostatic Pressure Difference between Two Points

3) The shape of a container does not matter in hydrostatics.

(Except for very small diameter tubes, where surface tension becomes important)



Consider the three containers in the figure:

At first glance, it may seem that the pressure at point 3 would be greater than that at point 1 or 2, since the weight of the water is more "concentrated" on the small area at the bottom, but in reality, all three pressures are identical. The application of hydrostatics equation confirms this conclusion, i.e.:

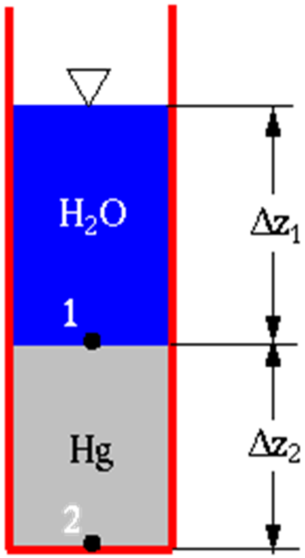
$$p_{\text{below}} = p_{\text{above}} + \rho g\left|\Delta z\right| \Rightarrow p_1 = p_2 = p_3 = p_0 + \rho g \Delta z$$

4.5 Hydrostatic Pressure Difference between Two Points

4) Pressure in layered fluid.

For example, consider the container in the figure below, which is partially filled with mercury, and partially with water:

In this case, our hydrostatics equation must be used twice, once in each of the liquids



$$p_{below} = p_{above} + \rho g |\Delta z|$$

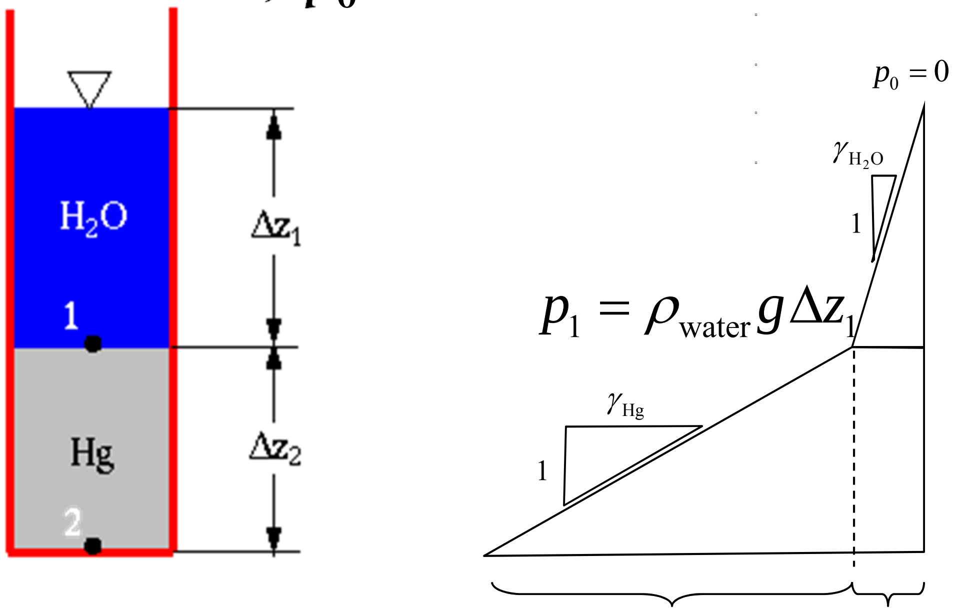
$$\Rightarrow p_1 = p_0 + \rho_{water} g \Delta z_1 \quad \text{and} \quad p_2 = p_1 + \rho_{mercury} g \Delta z_2$$

Combining, $$p_2 = p_0 + \rho_{water} g \Delta z_1 + \rho_{mercury} g \Delta z_2$$

4.5 Hydrostatic Pressure Difference between Two Points

Shanghai Jiao Tong University

Shown on the right side of the figure is the distribution of pressure with depth across the two layers of fluids, where the atmospheric pressure is taken to be zero, p_0=0.



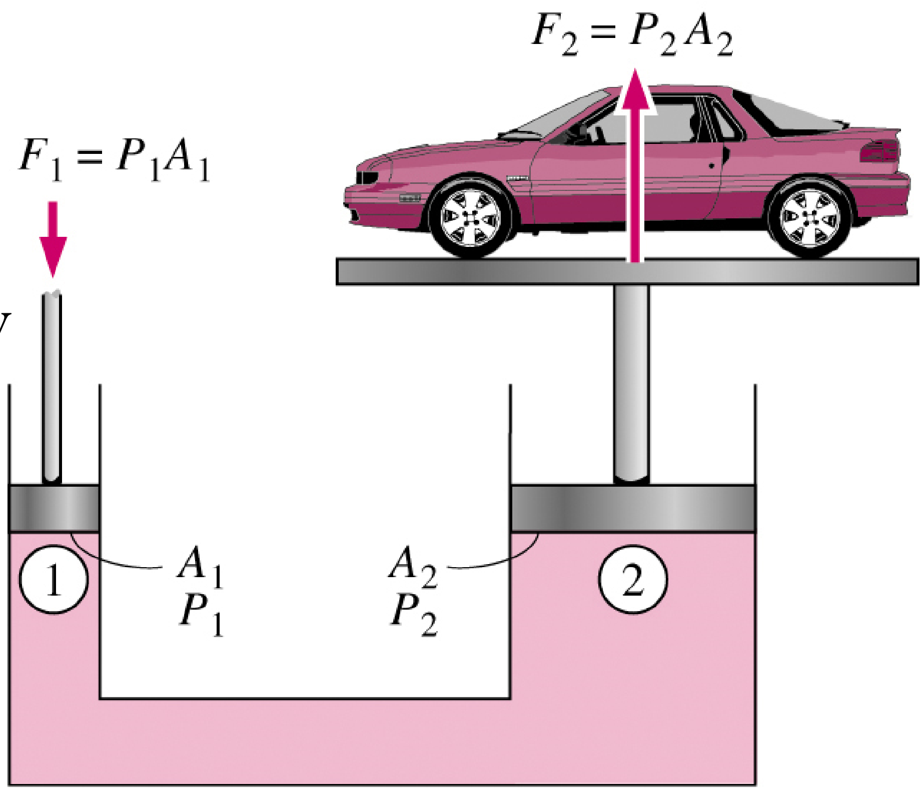
$$p_2 = \rho_{mercury} g \Delta z_2 + p_1$$

The pressure is continuous at the interface between water and mercury. Therefore, p_1, which is the pressure at the bottom of the water column, is the <u>starting pressure</u> at the top of the mercury column.

4.5 Hydrostatic Pressure Difference between Two Points

The fact that the pressure applied to a confined fluid increases the pressure throughout the fluid by the same amount (Pascal's law) has important applications, such as in the hydraulic lifting of heavy objects:

$$P_1 = P_2 \Rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2} \Rightarrow \frac{F_1}{F_2} = \frac{A_1}{A_2}$$

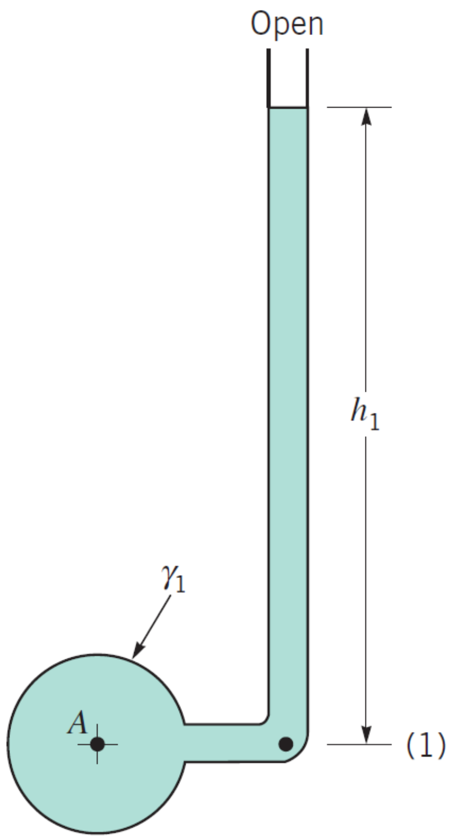


4.6 Pressure Measurement and Manometers

1) Piezometer tube

The simplest manometer is a tube, open at the top, which is attached to a vessel or a pipe containing liquid at a pressure (higher than atmospheric) to be measured. This simple device is known as a piezometer tube. As the tube is open to the atmosphere the pressure measured is relative to atmospheric so is gage pressure:

$$p_A = \gamma_1 h_1$$

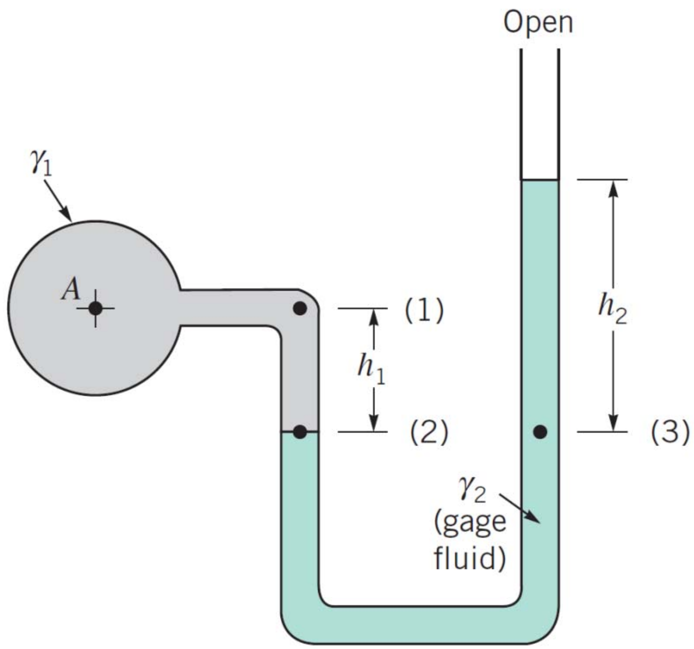


This method can only be used for liquids (i.e. not for gases) and only when the liquid height is convenient to measure. It must not be too small or too large and pressure changes must be detectable.

2) U-tube manometer

This device consists of a glass tube bent into the shape of a "U", and is used to measure some unknown pressure.

For example, consider a U-tube manometer that is used to measure pressure p_A in some kind of tank or machine.



Consider the left side and the right side of the manometer separately:

$$p_2 = p_1 + \gamma_1 h_1 = p_A + \gamma_1 h_1$$

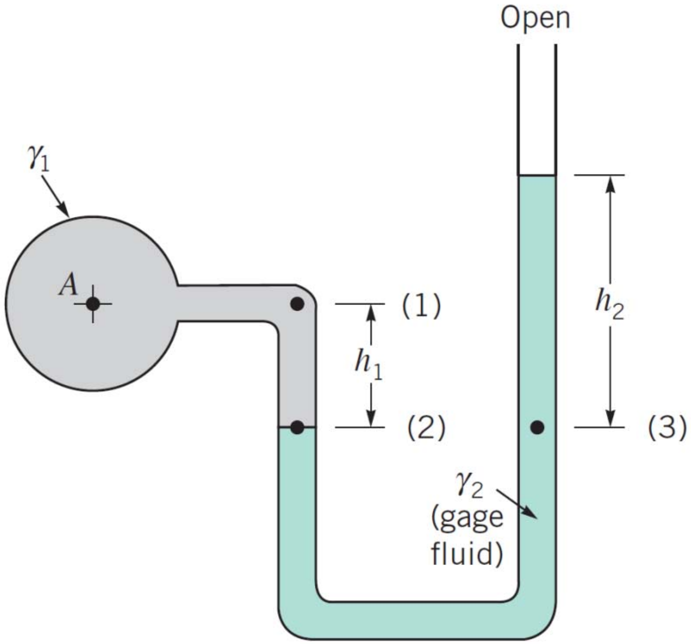
$$p_3 = \gamma_2 h_2$$

4.6 Pressure Measurement and Manometers

Since points labeled (2) and (3) in the figure are at the same elevation in the same fluid, they are at equivalent pressures, and the two equations above can be equated to give:

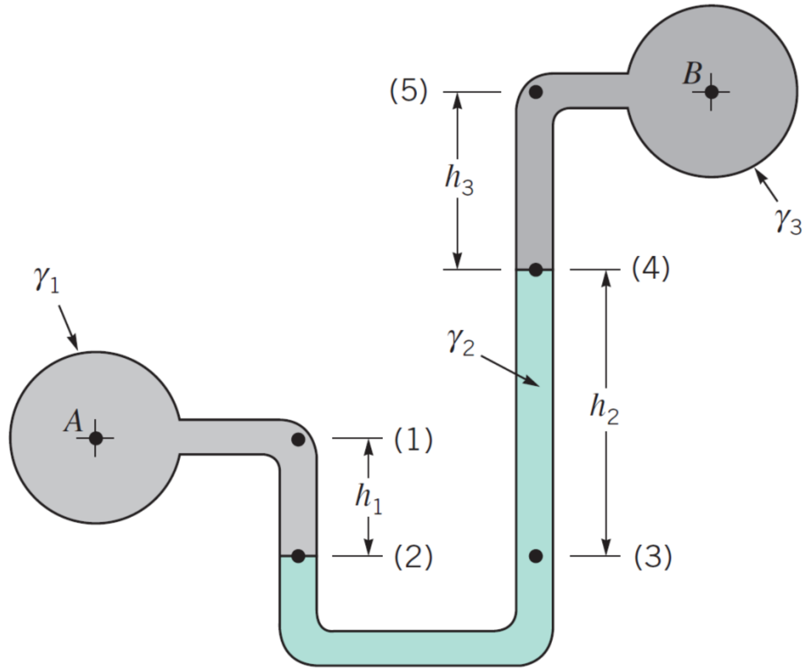
$$p_A = \gamma_2 h_2 - \gamma_1 h_1$$

Finally, note that in many cases (such as with air pressure being measured by a mercury manometer), the density of manometer fluid 2 is much greater than that of fluid 1. In such cases, the last term on the right is sometimes neglected.



3) Differential manometer

A differential manometer can be used to measure the difference in pressure between two containers or two points in the same system. Again, on equating the pressures at points labeled (2) and (3), we may get an expression for the pressure difference between *A* and *B*:



$$p_A - p_B = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

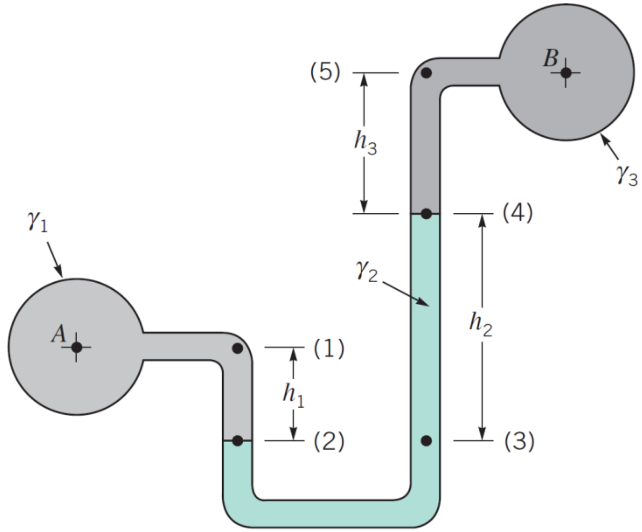
4.6 Pressure Measurement and Manometers

In the common case when *A* and *B* are at the same elevation

$(h_1 = h_2 + h_3)$ and the fluids in the two containers are the same $(\gamma_1 = \gamma_3)$

one may show that the pressure difference registered by a differential manometer is given by

$$\Delta p = \left(\frac{\rho_m}{\rho} - 1 \right) \rho g h_2$$



where

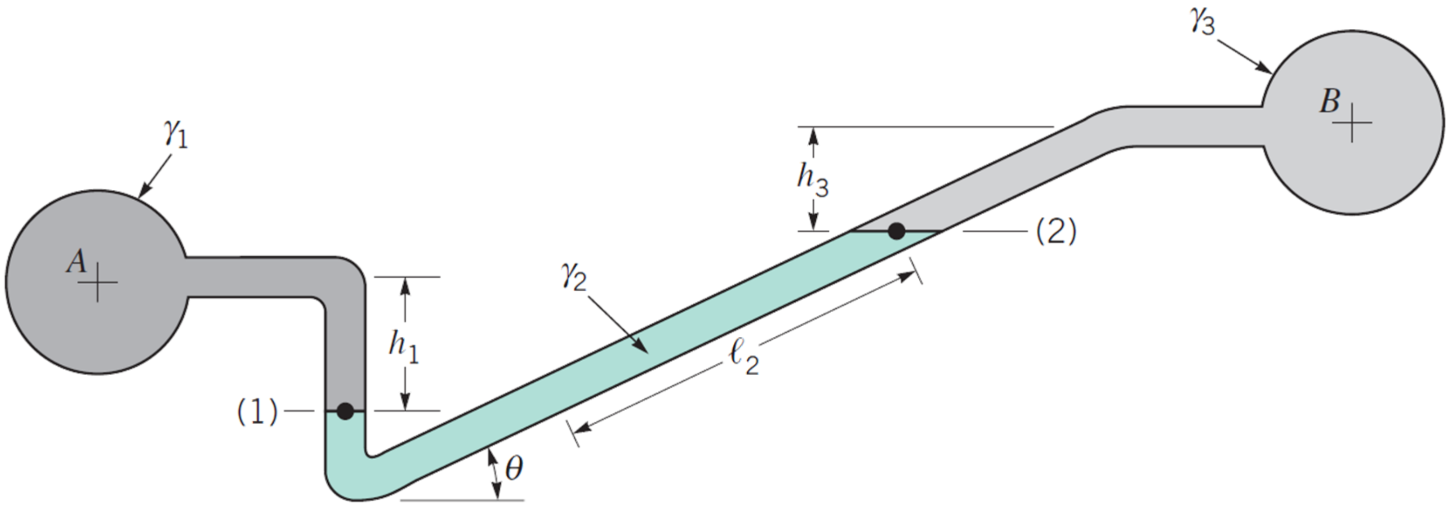
ρ_m is the density of the manometer fluid,

ρ is the density of the fluid in the system

h_2 is the manometer differential reading

4.6 Pressure Measurement and Manometers

4) Inclined-tube manometer



As shown above, the differential reading is proportional to the pressure difference. If the pressure difference is very small, the reading may be too small to be measured with good accuracy. To increase the sensitivity of the differential reading, one leg of the manometer can be inclined at an angle θ .

And the differential reading is measured along the inclined tube.

4.6 Pressure Measurement and Manometers

As shown above, $h_2 = \ell_2 \sin\theta$ **and hence**

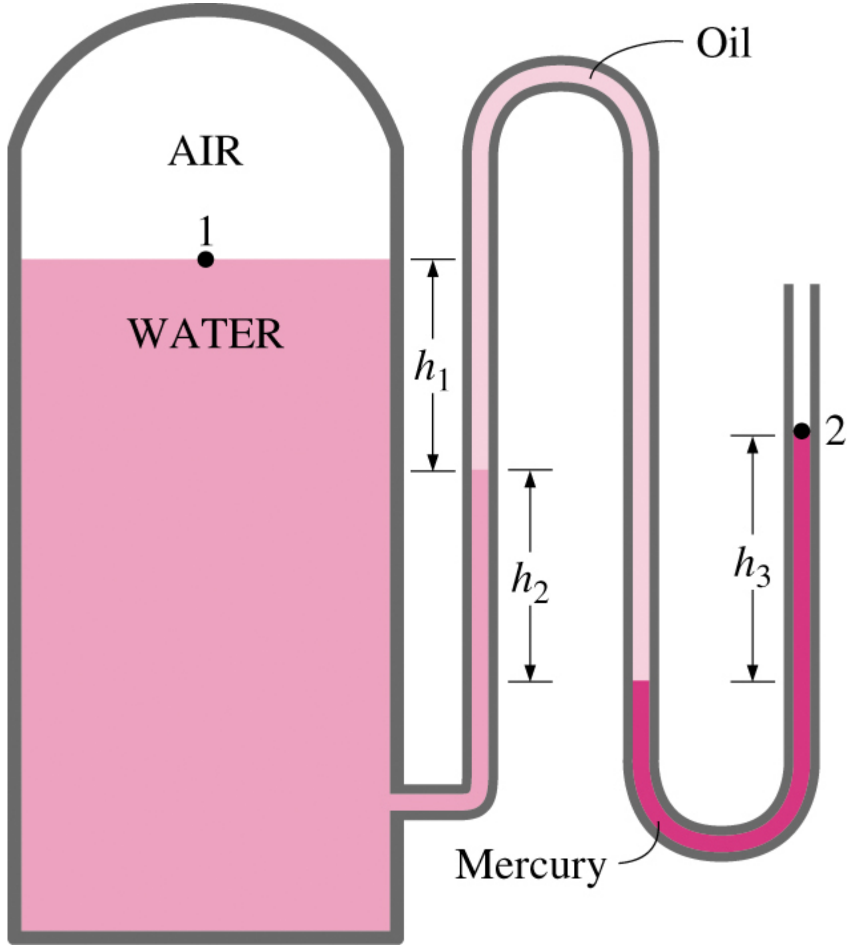
$$p_A - p_B = \gamma_2 \ell_2 \sin\theta + \gamma_3 h_3 - \gamma_1 h_1$$

Obviously, the smaller the angle θ **, the more the reading** ℓ_2 **is magnified.**

5) Multifluid manometer

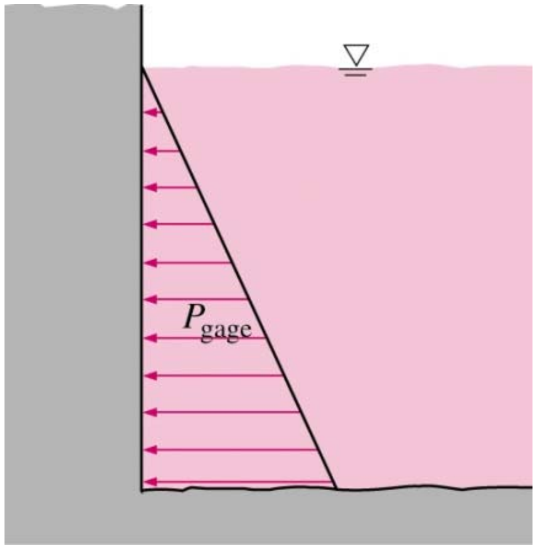
The pressure in a pressurized tank is measured by a multifluid manometer, as is shown in the figure. The air pressure in the tank is given by:

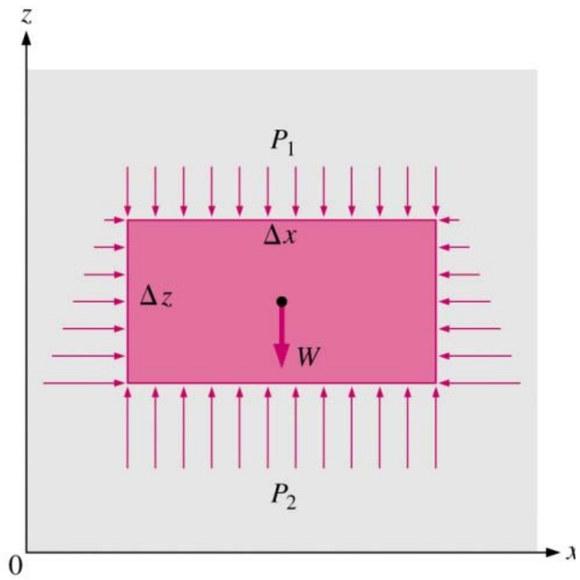
$$P_{\text{air}} = P_{\text{atm}} + g\left(\rho_{\text{mercury}} h_3 - \rho_{\text{oil}} h_2 - \rho_{\text{water}} h_1\right)$$

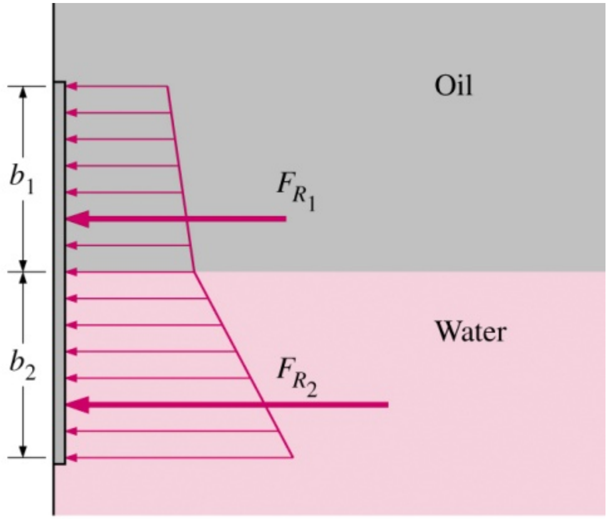


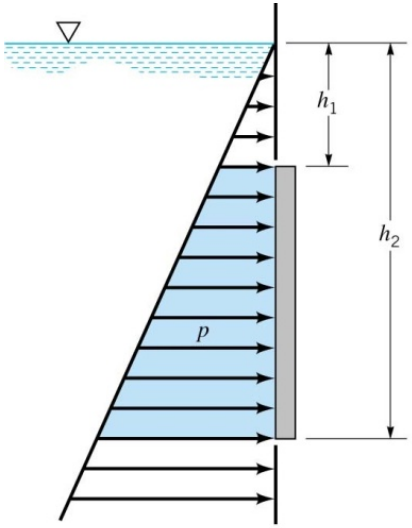
4.7 Pressure Distributions

1) Flat surfaces

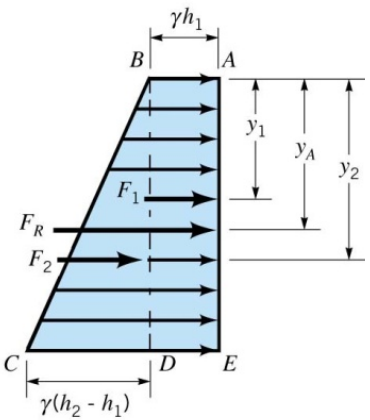




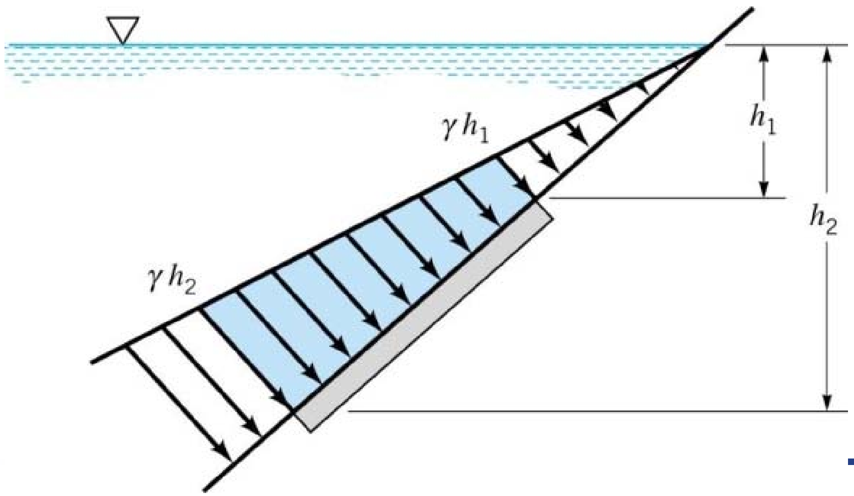




(a)



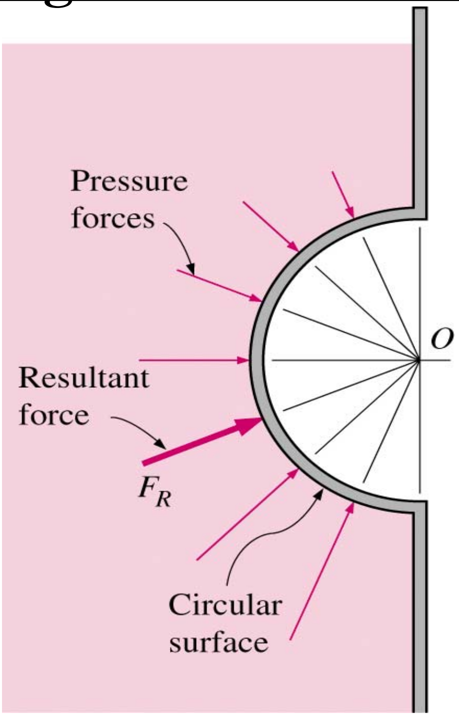
(b)

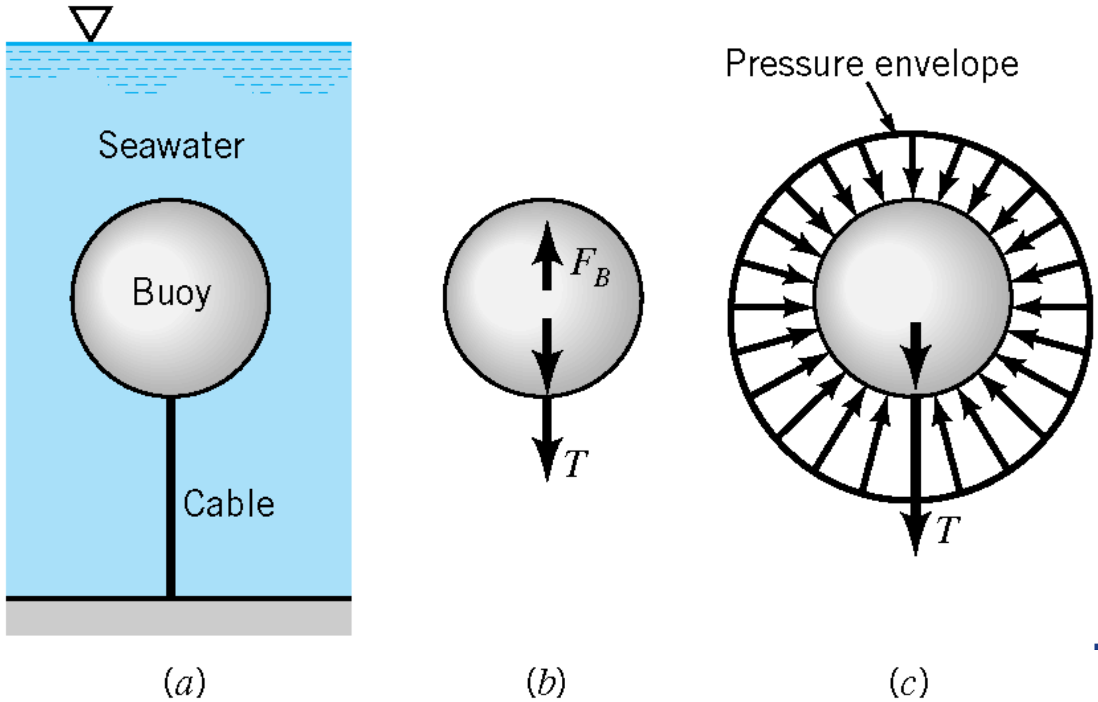


4.7 Pressure Distributions

2) Curved Surfaces

When the curved surface is a *circular arc* (full circle or any part of it), the resultant hydrostatic force acting on the surface always passes through the center of the circle. This is because the elemental pressure forces are normal to the surface, and by the well-known geometrical property that all lines normal to the surface of a circle must pass through the center of the circle.





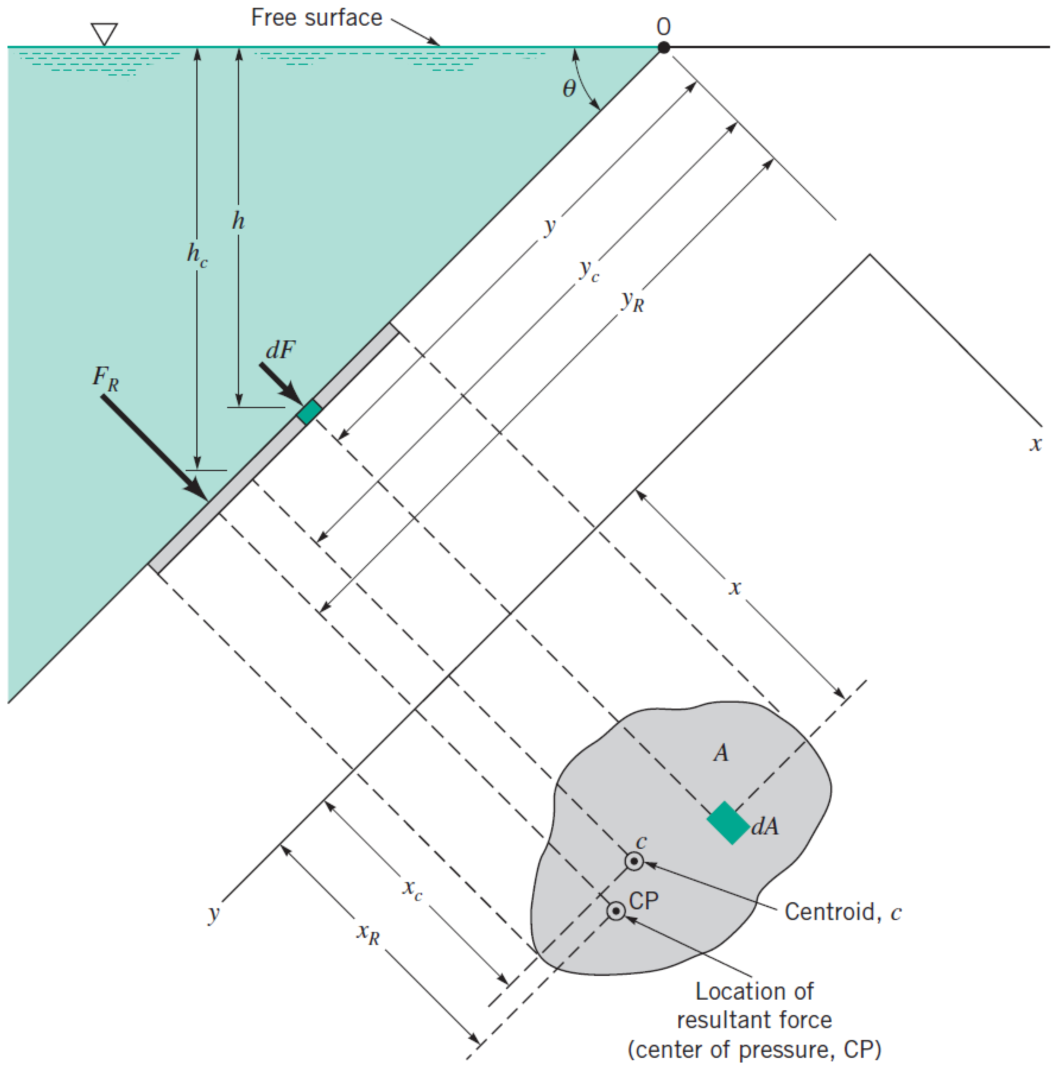
4.8 Hydrostatic Force on a Plane Surface

Suppose a *submerged* plane surface is inclined at an angle θ to the free surface of a liquid.

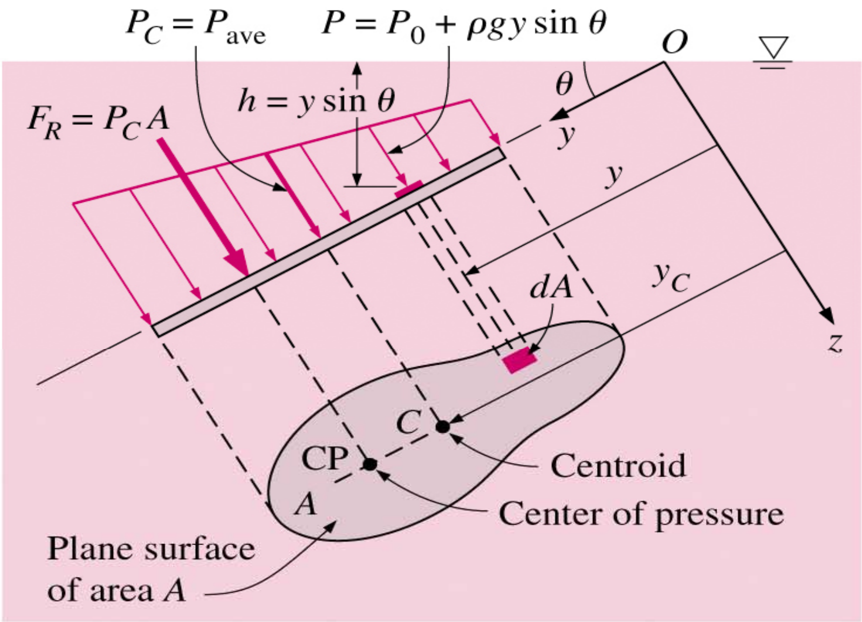
F_R:

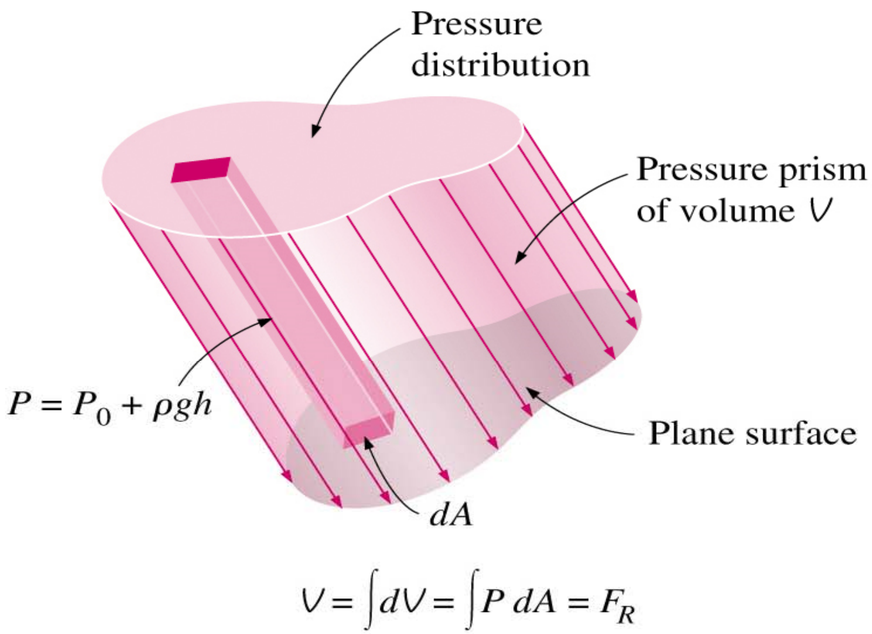
Magnitude

Location



4.8 Hydrostatic Force on a Plane Surface





A - area of the plane surface

O - the line where the plane in which the surface lies intersects the free surface,

C - centroid (or center of area) of the plane surface,

CP - center of pressure (point of application of the resultant force on the plane surface),

F_R - magnitude of the resultant force on the plane surface (acting normally),

h_R - vertical depth of the center of pressure CP,

h_c - vertical depth of the centroid C,

y_R - inclined distance from O to CP, y_c - inclined distance from O to C.

4.8 Hydrostatic Force on a Plane Surface

1) Find magnitude of resultant force

The resultant force is found by integrating the force due to hydrostatic pressure on an element *dA* at a depth *h* over the whole surface:

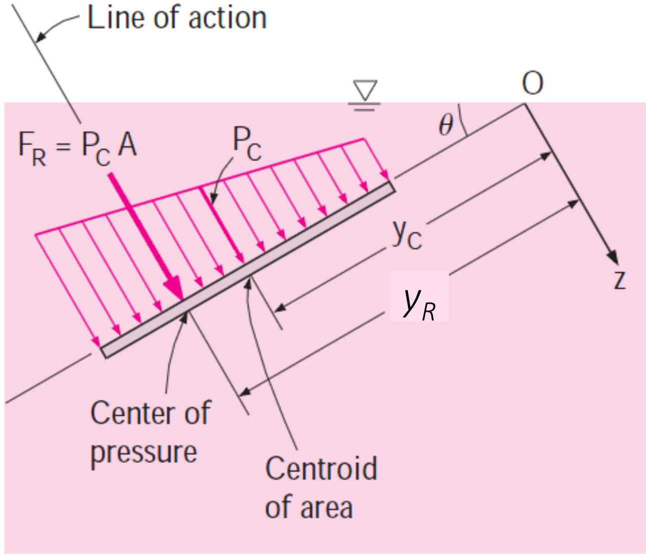
$$F_R = \int_A dF = \int_A \rho g h dA = \rho g \sin\theta \int_A y dA$$

where by the first moment of area $\int_A y dA = y_c A$, hence

$$F_R = \rho g \left(y_c \sin\theta \right) A = \rho g h_c A$$

The resultant force on one side of any plane submerged surface in a uniform fluid is therefore equal to the pressure at the centroid of the surface times the area of the surface, independent of the shape of the plane or the angle θ at which it is inclined.

4.8 Hydrostatic Force on a Plane Surface



2) Find location of center of pressure

Taking moment about *O*,

$$F_R y_R = \int_A y dF$$

$$\Rightarrow \left(\rho g y_c \sin\theta A\right) y_R = \int_A y\left(\rho g y \sin\theta dA\right)$$

$$\Rightarrow \left(y_c A\right) y_R = \int_A y^2 dA$$

$$\int_A y^2 dA = I_O = I_c + A y_c^2$$ by parallel axis theorem

where:

I_O = second moment of area (or moment of inertia) of the surface about *O*.

I_c = second moment of area (or moment of inertia) about an axis through the centroid *C* and parallel to the axis through *O*.

4.8 Hydrostatic Force on a Plane Surface

Therefore, on substituting,

$$(y_c A)\, y_R = A y_c^2 + I_c$$

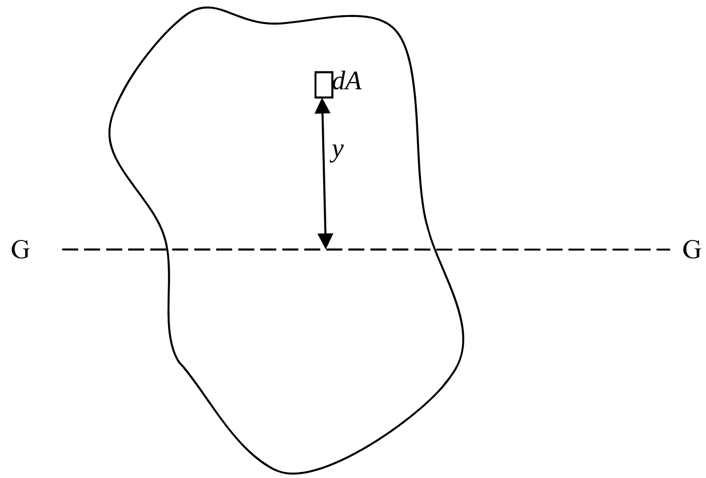
$$\Rightarrow \boxed{y_R = y_c + \frac{I_c}{y_c A}} \qquad \text{or} \qquad h_R = h_c + \frac{I_c \sin^2\theta}{h_c A}$$

Now, the depth of the center of pressure depends on the shape of the surface and the angle of inclination, and is always below the depth of the centroid of the plane surface.

Moment of Area

For a plane surface of arbitrary shape, we may define the n^{th} (n = 0, 1, 2, 3, …) moment of area about an axis GG by the integral:

$$\int_A y^n \, dA$$



Then,

- the zeroth moment of area = total area of the surface,
- the first moment of area = 0, if GG passes through the centroid of the surface,
- the second moment of area (moment of inertia) gives the variance of the distribution of area about the axis.

4.8 Hydrostatic Force on a Plane Surface

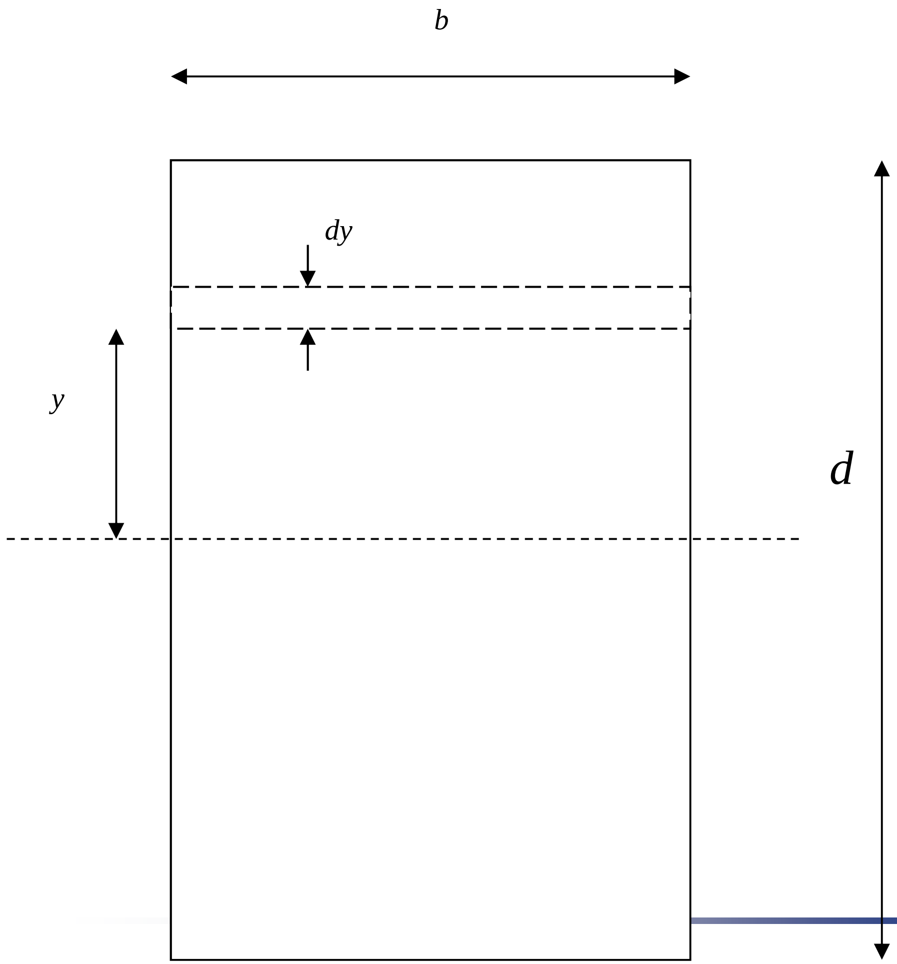
For example, for a rectangular surface, the second moment of area about the axis that passes through the centroid is:

$$I_c = \int_A y^2 dA$$

$$= \int_{-d/2}^{d/2} y^2 (bdy)$$

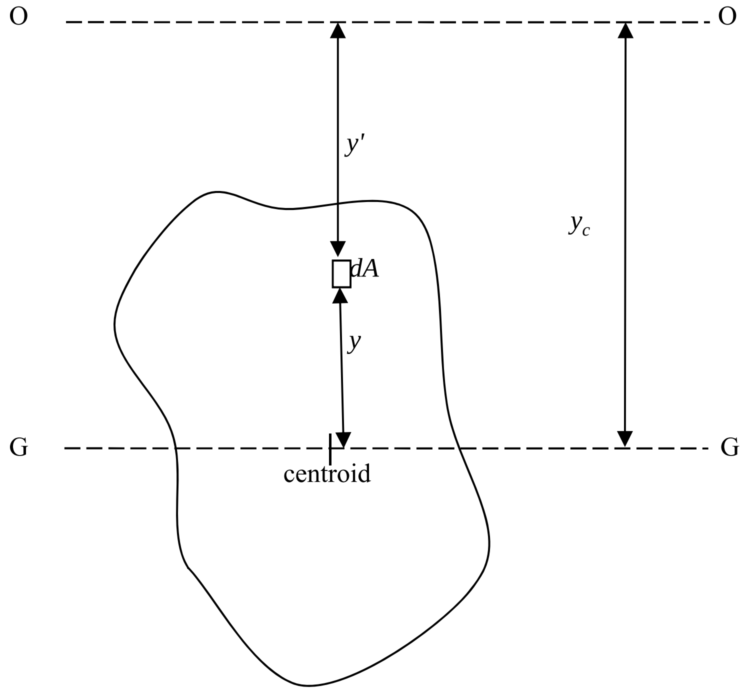
$$= \left[\frac{by^3}{3}\right]_{-d/2}^{d/2}$$

$$= \frac{bd^3}{12}$$



Parallel Axis Theorem

If OO is an axis that is parallel to the axis GG, which passes through the centroid of the surface, then the second moment of area about OO is equal to that about GG plus the square of the distance between the two axes times the total area:

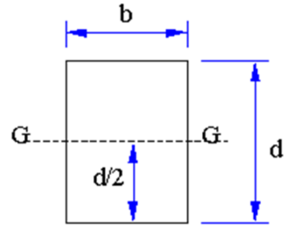
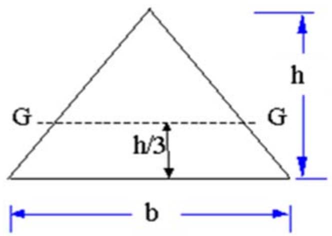
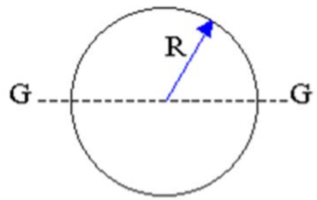
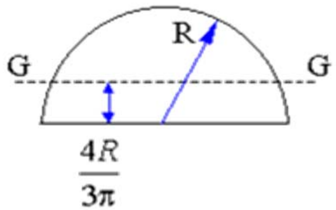


$$I_o = \int_A y'^2 dA = \int_A (y_c - y)^2 dA = \int_A \left(y_c^2 - 2y_c y + y^2\right) dA$$

$$= y_c^2 A - 2y_c \underbrace{\int_A y dA}_{0} + \underbrace{\int_A y^2 dA}_{I_c} = y_c^2 A + I_c$$

Properties for some common sectional areas

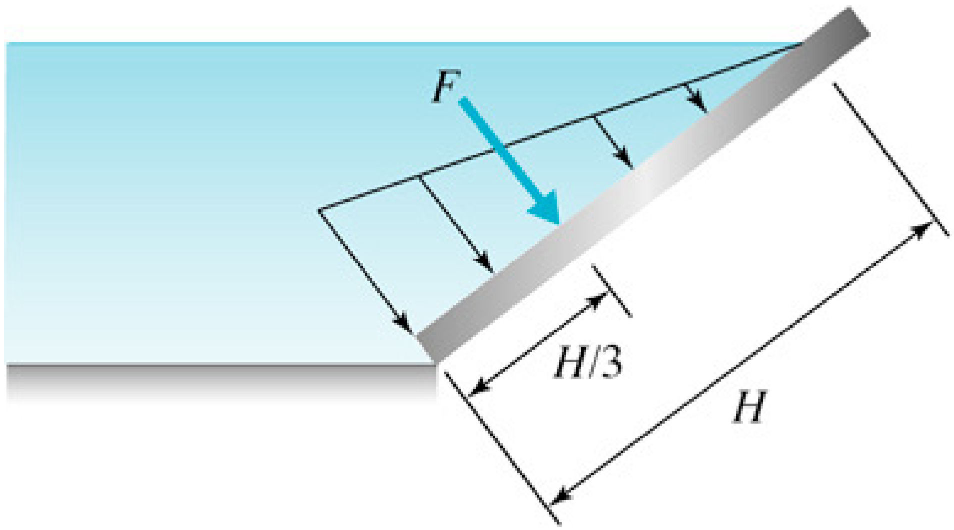
GG is an axis passing through the centroid and parallel to the base of the figure.

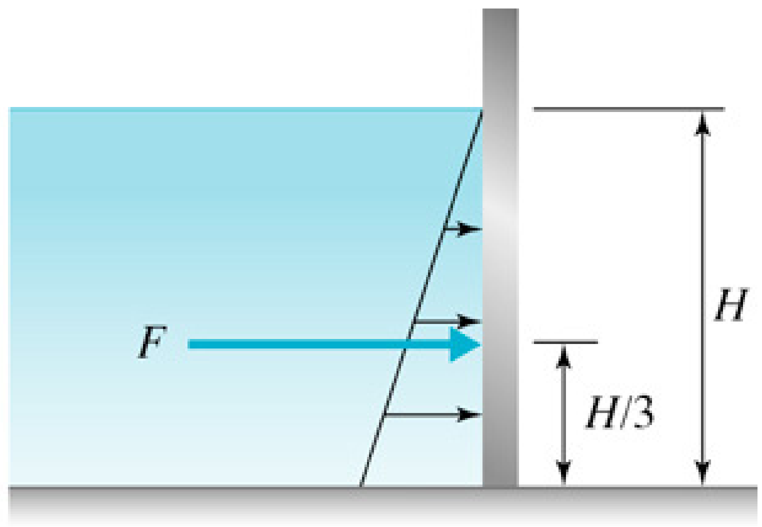
Shape	Dimensions	Area	(moment of inertia about GG) I_c
Rectangle		bd	$\frac{bd^3}{12}$
Triangle		$\frac{bh}{2}$	$\frac{bh^3}{36}$
Circle		πR^2	$\frac{\pi R^4}{4}$
Semi-circle		$\frac{\pi R^2}{2}$	$0.11R^4$

4.8 Hydrostatic Force on a Plane Surface

3) For a flat surface that pierces through the free surface, and hence triangular pressure distribution:

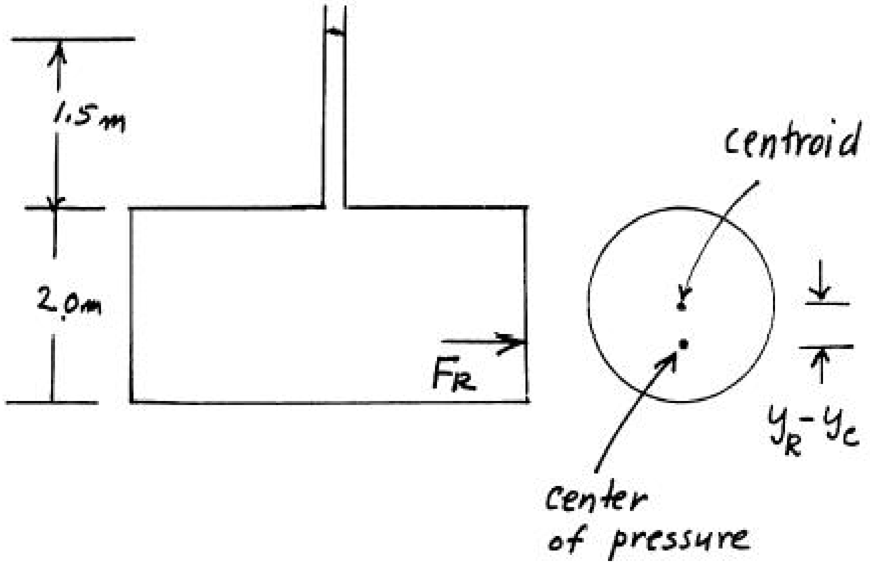
$$A = HB, \quad h_c = \frac{1}{2}H\sin\theta, \quad h_R = \frac{2}{3}H\sin\theta, \quad F = \frac{1}{2}\rho g H^2 B\sin\theta$$





4.8 Hydrostatic Force on a Plane Surface

Application 1: A horizontal cylindrical bucket with 2.0m-diameter and 4.0m-length, as shown in the figure. A 0.1m-diameter small pipe is connected on the top of the cylindrical bucket. A liquid with a specific weight 7.74 kN/m^3 is filled in the cylindrical bucket and the pipe, the liquid height is up to 1.5m above the bucket. Determine the acting force and point on the top face of the bucket.



Solution: The total pressure on the top of the cylindrical bucket is:

$$F_R = \gamma h_c A, \quad \text{where} \quad h_c = 1.5\text{m} + 1.0\text{m} = 2.5\text{m}$$

Then:

$$F_R = \gamma h_c A = \left(7.74\frac{\text{kN}}{\text{m}^3}\right)(2.5\text{m})\left(\frac{\pi}{4}\right)(2.0\text{m})^2 = 60.8\,\text{kN}$$