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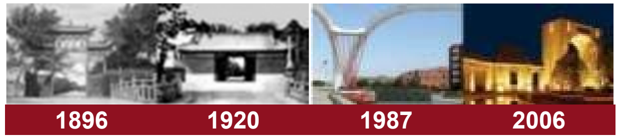


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Review

- **Momentum equation (conservation of momentum)**

$$\iiint_{MV} \rho \frac{d\mathbf{V}}{dt} d\mathcal{V} = \iiint_{MV} \left(\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \right) d\mathcal{V}$$

Integral form

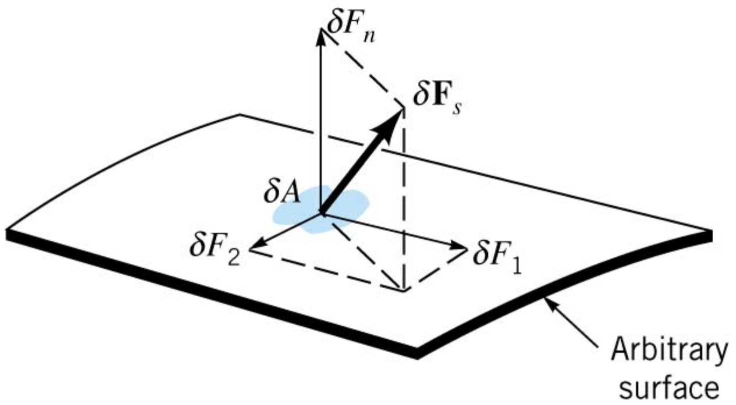
$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}$$

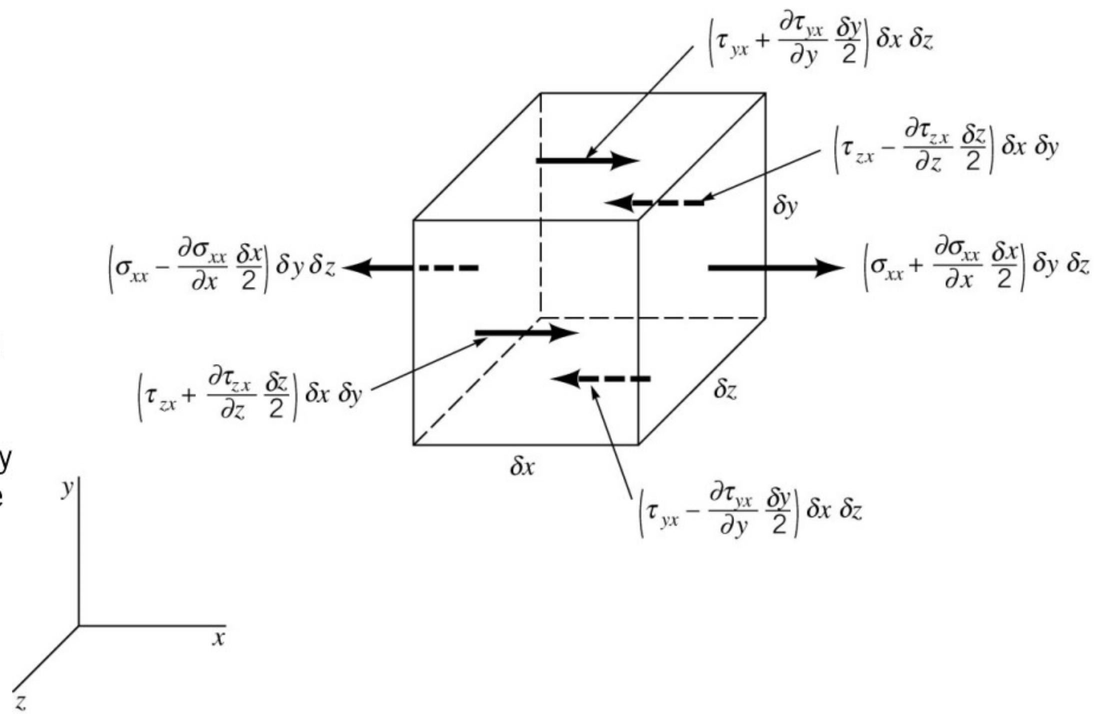
Differential form

Review

For Newtonian fluids, we can set up the relation between surface stress and strain-rate (constitutive equation):

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} = \left(-p + \lambda \frac{\partial u_l}{\partial x_l} \right) \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$





Review

- **Relation between surface stress and strain-rate of incompressible fluids (constitutive equation):**

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- **Momentum equation of incompressible flows (conservation of momentum)**

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p + \mu\nabla^2\mathbf{V} + \rho\mathbf{g}$$

- **Governing equation of incompressible flows (Navier-Stokes equations)**

$$\begin{cases} \nabla\cdot\mathbf{V} = 0 \\ \dfrac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\dfrac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V} + \mathbf{g} \end{cases}$$

Review

The physical explanation of each item in NS momentum equation:

(I) local acceleration;

(II) convective acceleration (inertia, convection, nonlinear term of the equation);

(III) pressure gradient;

(IV) volume force or gravity;

(V) viscous diffusion of momentum due to molecular viscosity of the fluid.

Review

- **Euler equation for ideal fluids:**

$$\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}=-\frac{1}{\rho}\nabla p+\mathbf{f}$$

- **Lamb equation**

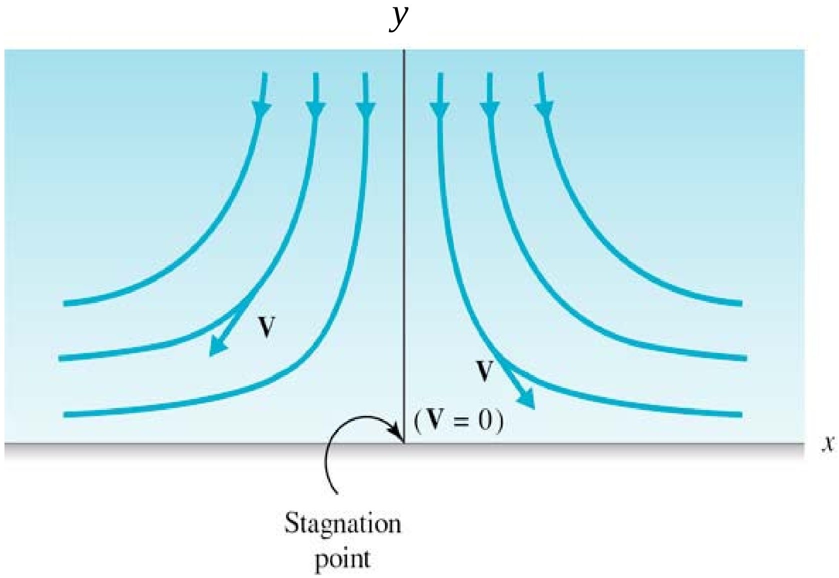
$$\frac{\partial \mathbf{V}}{\partial t}+\nabla\left(\frac{V^2}{2}\right)-2\mathbf{V}\times\boldsymbol{\omega}=-\frac{1}{\rho}\nabla p+\mathbf{f}$$

3.6 Governing Equations of Fluid Motion

Application: **A water column flows against a plate below.**

The velocity potential is: $\phi = -k\left(x^2 - y^2\right)$ (k=const)

ρ is the density of the water and the viscosity is neglected. Verify that the pressure distribution along the plate is: $\dfrac{\partial p}{\partial x} = -4\rho k^2 x$



Verification: Velocity components can be determined from the velocity potential function

$$u = \frac{\partial \phi}{\partial x} = -2kx, \quad v = \frac{\partial \phi}{\partial y} = 2ky \qquad (1)$$

As this is a 2D steady flow of an ideal fluid, from Euler equation, then:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} \qquad (2)$$

3.6 Governing Equations of Fluid Motion

Along the plate surface, $y = 0, v = 0$. Substituting into the equation (2) above:

$$\frac{\partial p}{\partial x} = -\rho u \frac{\partial u}{\partial x} \qquad (3)$$

From equation (1), we get: $\frac{\partial u}{\partial x} = -2k$

Substitute it into equation (3):

$$\frac{\partial p}{\partial x} = -\rho(-2kx)(-2k) = -4\rho k^2 x$$

3.7 Bernoulli Equation

Consider:

- ideal fluid, inviscid flow
- constant density, incompressible flow
- steady flow
- body force is gravity

Lamb equation can be rewritten as:

$$\frac{\partial \mathbf{V}}{\partial t}+\nabla\left(\frac{V^2}{2}\right)-2\mathbf{V}\times\boldsymbol{\omega}=-\frac{1}{\rho}\nabla p+\mathbf{f}$$

⇓

$$\nabla\left(\frac{V^2}{2}\right)+\nabla\left(\frac{p}{\rho}\right)+\nabla\left(gz\right)=2\mathbf{V}\times\boldsymbol{\omega}$$

3.7 Bernoulli Equation

$$\nabla\left(\frac{V^2}{2}\right)+\nabla\left(\frac{p}{\rho}\right)+\nabla(gz)=2\mathbf{V}\times\boldsymbol{\omega}$$

⇩

$$\nabla\left(\frac{V^2}{2}+\frac{p}{\rho}+gz\right)=2\mathbf{V}\times\boldsymbol{\omega}$$

⇩

$$\nabla H=2\mathbf{V}\times\boldsymbol{\omega},\quad H=\frac{V^2}{2}+\frac{p}{\rho}+gz$$

H **is called Bernoulli function**

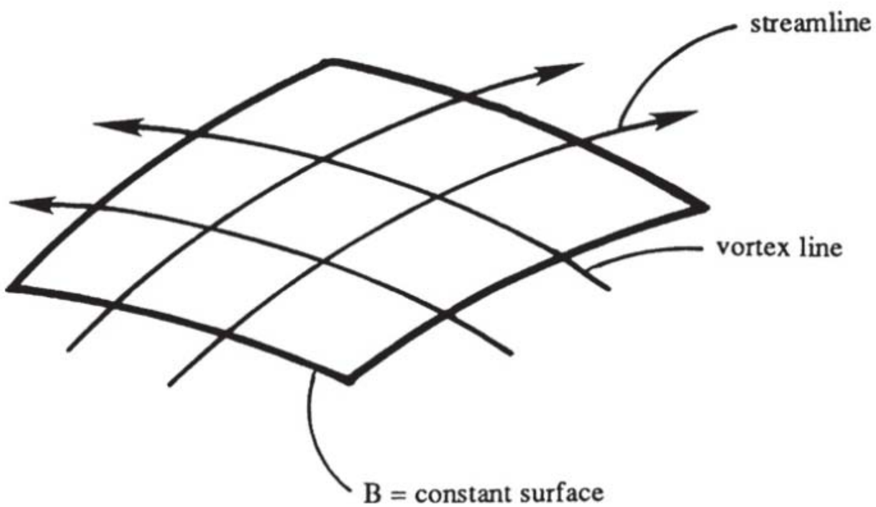
3.7 Bernoulli Equation

Take a small segment d*l* from a streamline (*V*) or vortex line (*ω*), the dot product of the equation above is:

$$\nabla H \cdot dl = (2\mathbf{V} \times \boldsymbol{\omega}) \cdot dl$$

$$\left.\begin{aligned} \nabla H \cdot dl &= dH \\ (2\mathbf{V} \times \boldsymbol{\omega}) \cdot dl &= 0 \end{aligned}\right\} \Rightarrow dH = 0$$

$$H = \frac{V^2}{2} + \frac{p}{\rho} + gz = C_l = \text{const}$$



Therefore, Bernoulli function along the same streamline or vortex line is a constant. The equation above is called **Bernoulli equation**.

3.7 Bernoulli Equation

The physical explanation of each item in Bernoulli equation:

$$\frac{V^2}{2} + \frac{p}{\rho} + gz = C_l = \text{const}$$

It can be measured by units of length:

$$\frac{V^2}{2g} + \frac{p}{\rho g} + z = C_l = \text{const}$$

- $\frac{V^2}{2g}$: velocity head, representing kinetic energy
- $\frac{p}{\rho g}$: pressure head, representing pressure energy
- z: elevation head, representing potential energy

Velocity head + Pressure head + Elevation head = Const

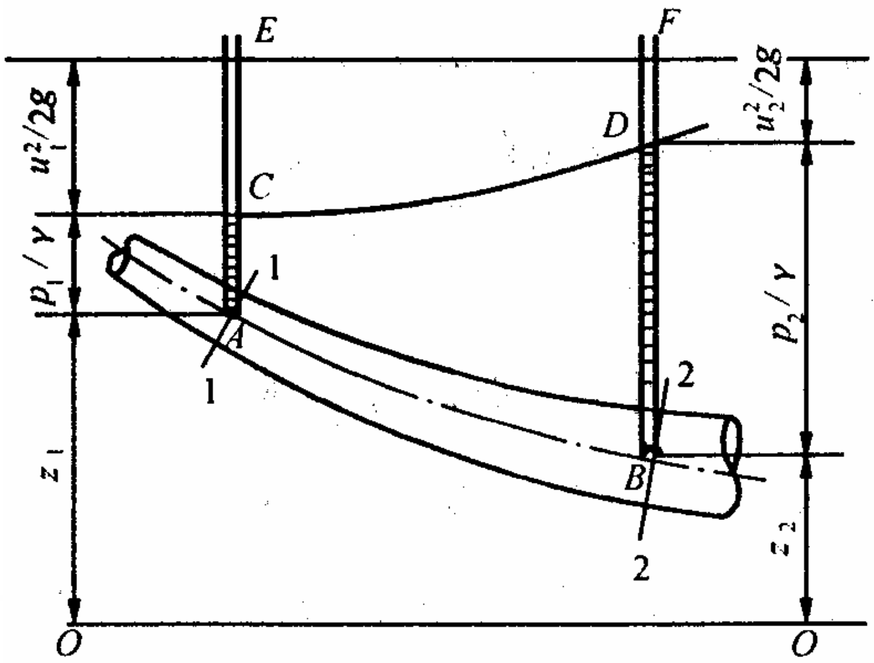
3.7 Bernoulli Equation

Bernoulli equation indicates that for the steady flow of an ideal fluid in a stream tube, the sum of kinetic energy, gravitational potential energy and pressure at a point in unit volume is a constant.

The scope of application of Bernoulli equation:

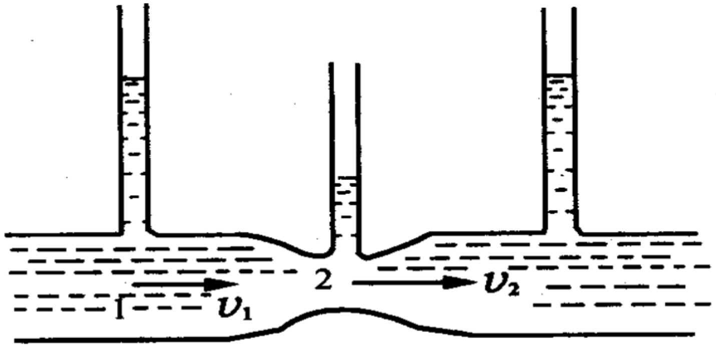
Ideal, incompressible fluids,

steady motion.



3.8 Applications of Bernoulli Equation and Momentum Equation

Application 1: **Water flows through a horizontal pipe as shown in the the picture. Which point (1 or 2) has larger pressure?**



Solution: The cross-sectional area of 1 is larger than that of 2, from the continuity equation, small cross-sectional area has larger velocity, so $v_1 < v_2$. As the pipe is horizontal, according to Bernoulli equation:

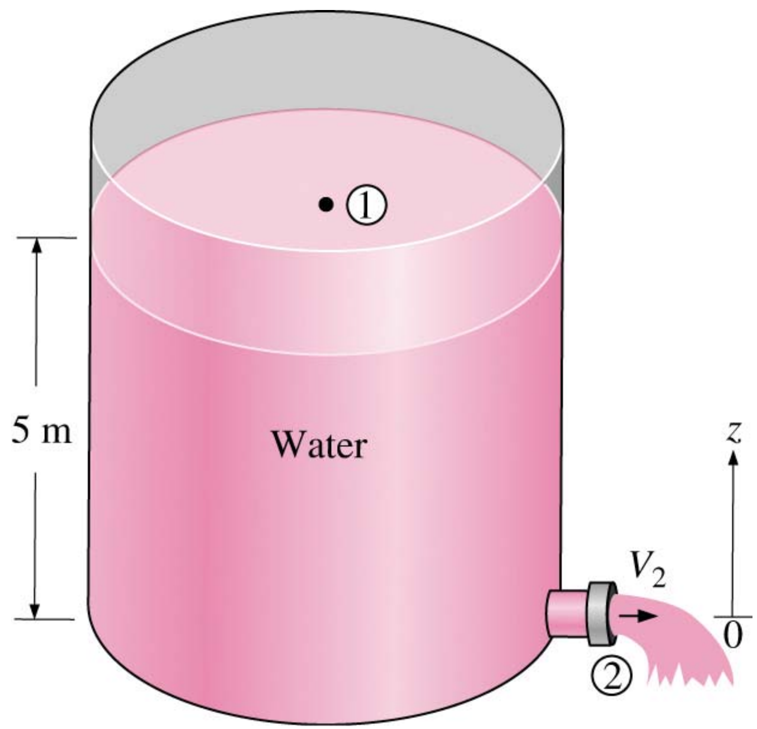
$$\frac{1}{2}\rho v_1^{\ 2} + P_1 = \frac{1}{2}\rho v_2^{\ 2} + P_2$$

Conclusion: for the flow in a horizontal pipe, the place with small velocity has larger pressure, while larger velocity indicates smaller pressure. Therefore, pressure at point 1 is larger.

Application 2: A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap. A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

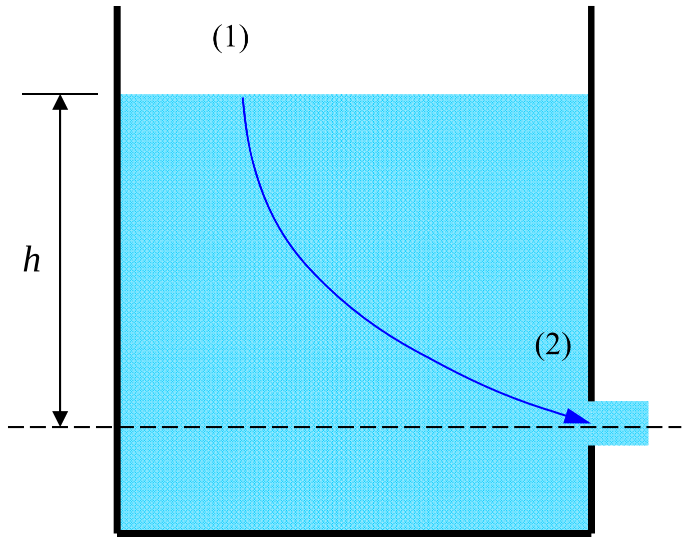
Solution: Take point 1 to be at the free surface and point 2 at the outlet, the reference level is at the center of the outlet. $p_1 = p_{atm}$, $V_1 \approx 0$, $z_1 = 5$ m, $z_2 = 0$, $p_2 = p_{atm}$, from the Bernoulli equation:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$



3.8 Applications of Bernoulli Equation and Momentum Equation

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2$$



p_1 and p_2 are atmospheric pressure

Size of the tank (1) is larger than the tap (2), according to the continuity equation, $V1 \ll V2$. Point 2 locates at the outlet, $z_2=0$. Thus:

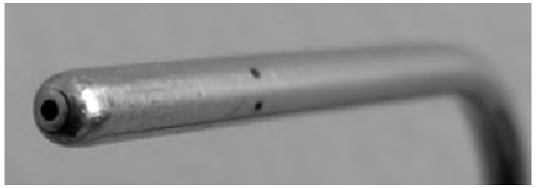
$$V_2^2=2gz_1$$

$$\Rightarrow\ V_2=\sqrt{2gz_1}$$

$$=\sqrt{2\left(9.8m/s^2\right)\left(5m\right)}=9.9\ m/s$$

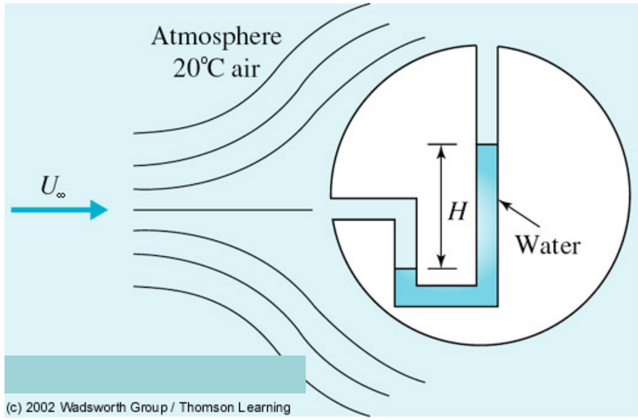
Velocity measurement by a Pitot probe (Pitot tube)

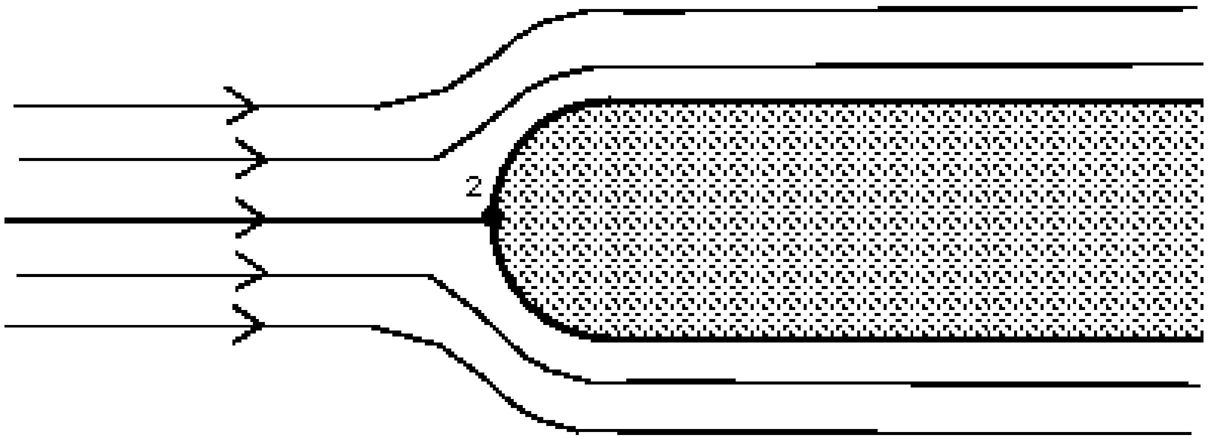
A Pitot probe is a tube with a pressure tap at the <u>stagnation</u> point to measure stagnation pressure and thus fluid flow velocity.

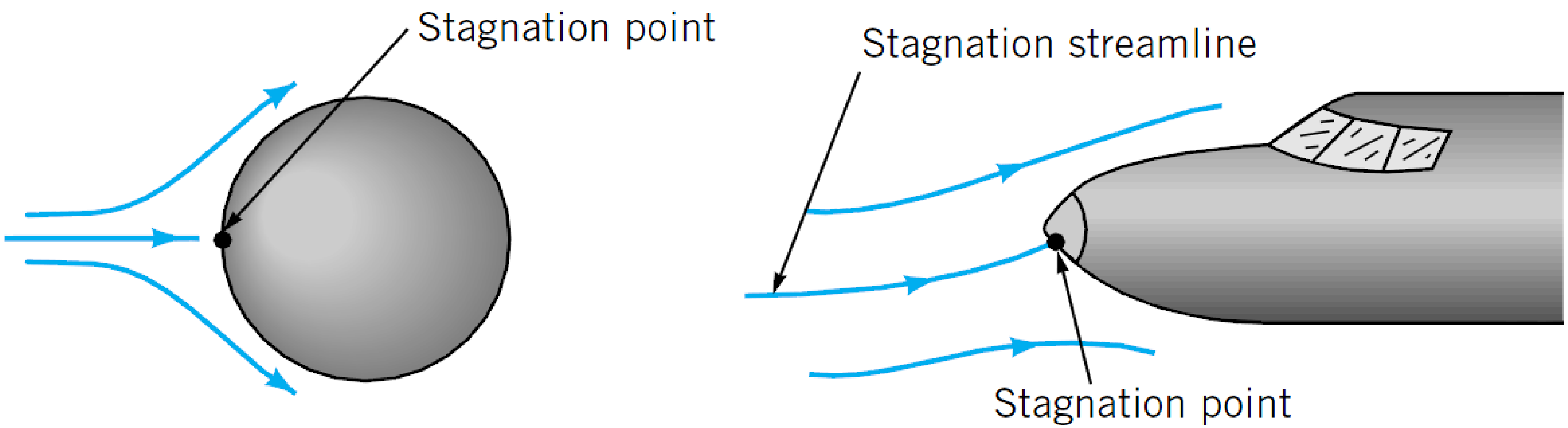


3.8 Applications of Bernoulli Equation and Momentum Equation

Stagnation point: a point in a flow field where the local velocity of the fluid is zero (when the fluid flows through a blunt body)

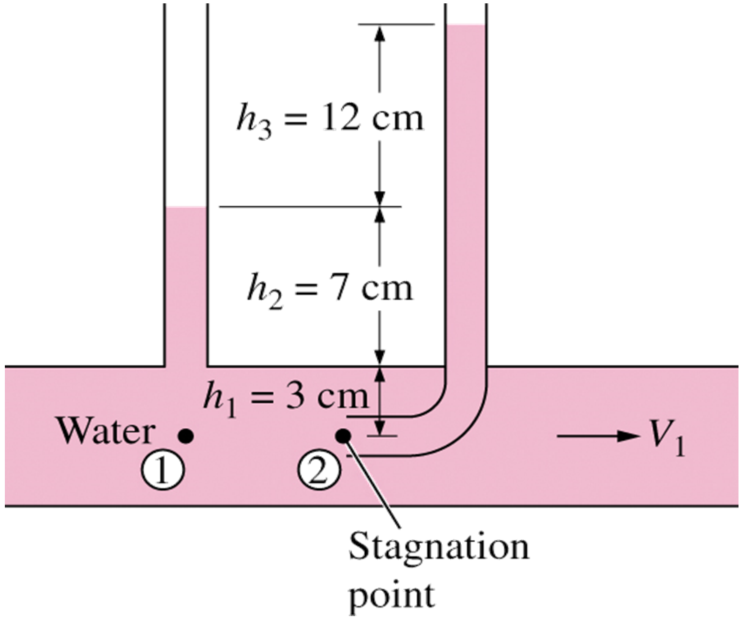






(a) (b)

Application 3: A Pitot tube is tapped into a horizontal water pipe, to measure pressures. For the indicated water column heights, determine the velocity at point 1.



Solution: Take point 2 at the tip of the Pitot tube. The gage pressures at points 1 and 2 can be expressed as:

$$p_1 = \rho g\left(h_1 + h_2\right), \quad p_2 = \rho g\left(h_1 + h_2 + h_3\right)$$

Points 1 and 2 are along the centerline of the pipe, thus $z_1=z_2=0$. Note that point 2 is a stagnation point and V_2 =0. According to Bernoulli equation, then:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

We get:

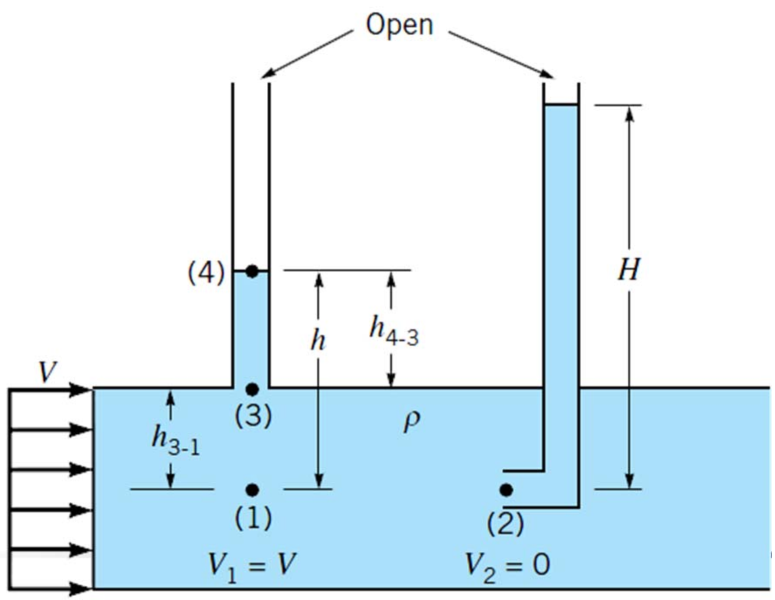
$$\frac{V_1^2}{2g} = \frac{p_2 - p_1}{\rho g} = \frac{\rho g\left(h_1 + h_2 + h_3\right) - \rho g\left(h_1 + h_2\right)}{\rho g} = h_3$$

$$V_1 = \sqrt{2gh_3} = \sqrt{2\left(9.81\,\mathrm{m/s^2}\right)\left(0.12\,\mathrm{m}\right)} = 1.53\,\mathrm{m/s}$$

Measuring flow velocity with a Pitot tube

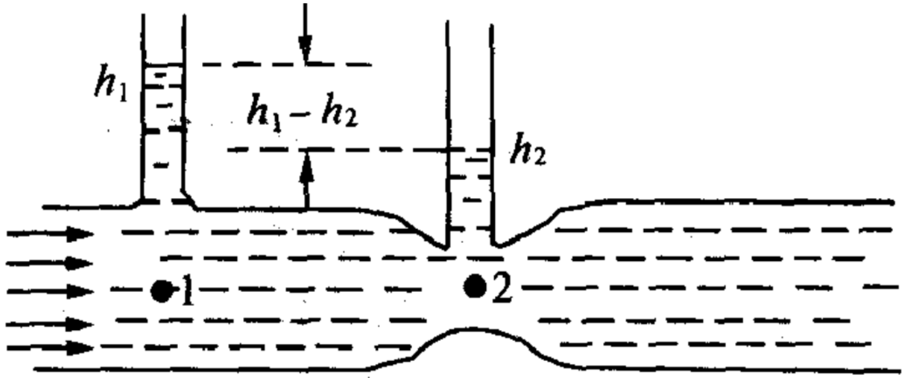
$$p_2 = p_1 + \frac{1}{2}\rho V_1^2 \quad \Rightarrow$$

$$\rho g H = \rho g h + \frac{1}{2}\rho V^2 \quad \Rightarrow \quad V = \sqrt{2g\left(H - h\right)}$$



3.8 Applications of Bernoulli Equation and Momentum Equation

Application 4: **Venturi meter** is used for flow rate measurement. For the indicated water column heights of the two vertical pipes, determine the rate of flow in the horizontal pipe.



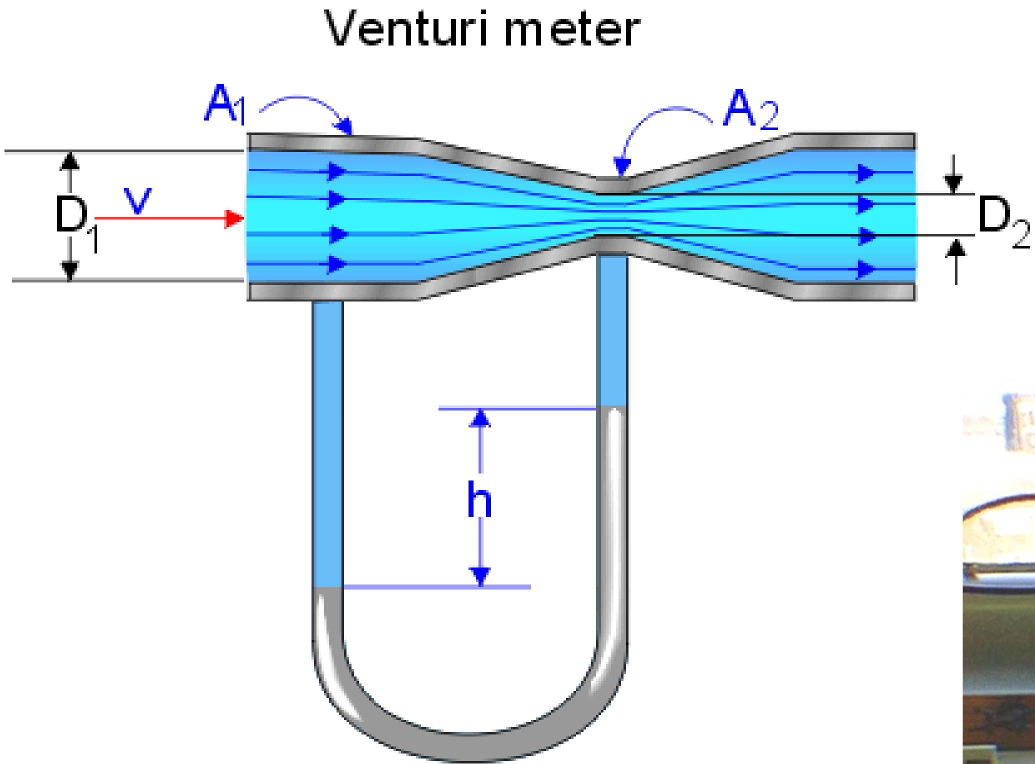
Solution: Let the cross-sectional area, pressure, velocity at point 1 and 2 be S_1, P_1, v_1 and S_2, P_2, v_2, respectively. The difference in the heights of water column in the vertical pipes is $h=h_1-h_2$. According to Bernoulli equation in the horizontal pipe:

$$P_1+\frac{1}{2}\rho\upsilon_1^2=P_2+\frac{1}{2}\rho\upsilon_2^2 \qquad P_1-P_2=\rho g\left(h_1-h_2\right)=\rho gh$$

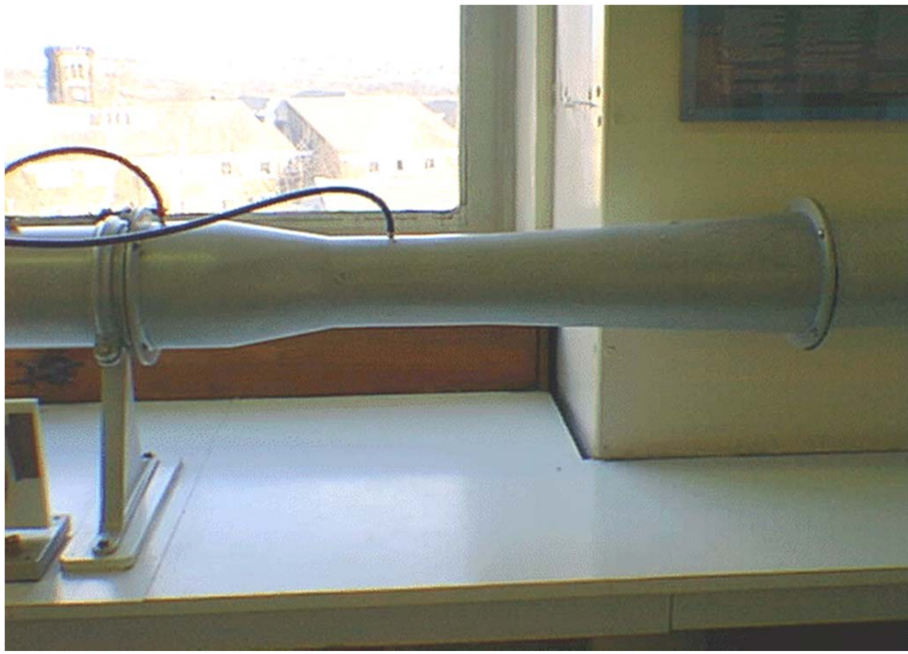
From continuity equation $S_1v_1=S_2v_2$, then

$$\upsilon_1=S_2\sqrt{\frac{2(P_1-P_2)}{\rho(S_1^2-S_2^2)}} \Rightarrow \text{rate of flow: } Q=S_1\upsilon_1=S_1S_2\sqrt{\frac{2gh}{S_1^2-S_2^2}}$$

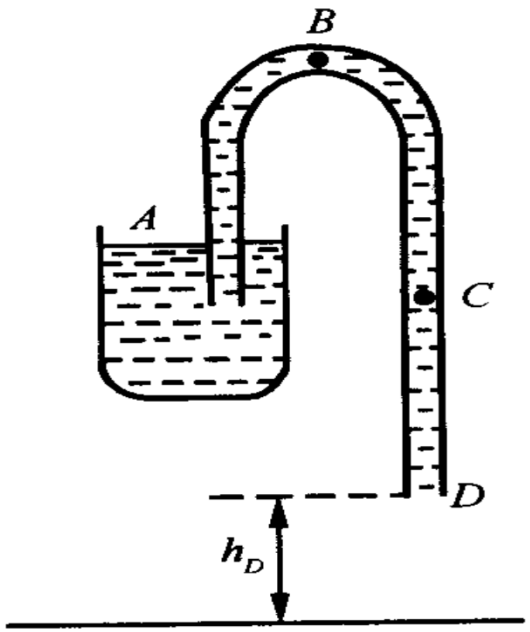
Flow rate measurement using Venturi meter



$$Q = A_1 A_2 \sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$



3.8 Applications of Bernoulli Equation and Momentum Equation



Application 5: **A siphon tube is used to transfer the fluid from one container to another. What is the condition for a siphon tube to work?**

Solution: (1) Velocity of the fluid

Assuming the fluid is ideal fluid and the diameter of the tube is uniform. For the free surface A in the container and the tip pf the tube, by Bernoulli equation:

$$\frac{1}{2}\rho\upsilon_A^2 + \rho g h_A + P_0 = \frac{1}{2}\rho\upsilon_D^2 + \rho g h_D + P_0$$

Because $S_A >> S_D$, from the continuity equation:

$$\upsilon_A^2 << \upsilon_D^2$$

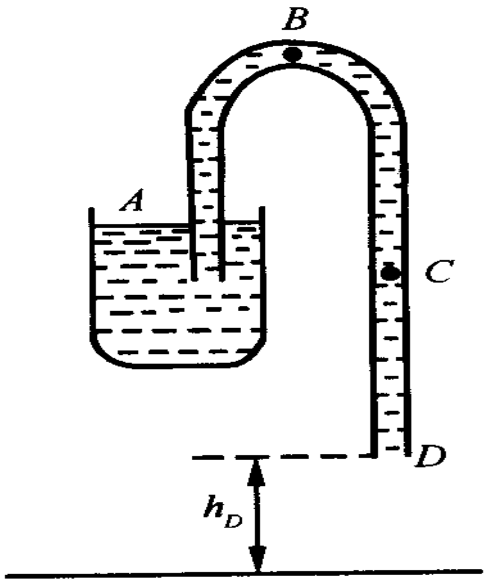
From the equation above, the velocity at the tip of the tube is:

$$\upsilon_D = \sqrt{2g(h_A - h_D)} = \sqrt{2gh_{AD}}$$

(2) Relationship between pressure and height

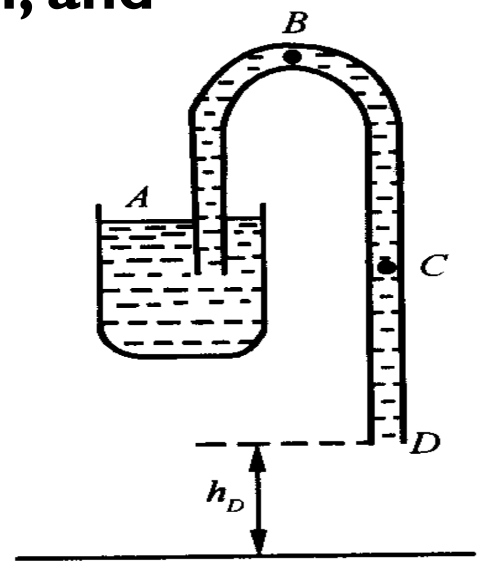
Because the diameter of the tube is uniform, from continuity equation, $v_B = v_C = v_D$. For points B and C, by Bernoulli equation:

$$\rho g h_B + P_B = \rho g h_c + P_c \Rightarrow \rho g h + P = C$$



Conclusion: for a uniform siphon tube, the pressure with larger height is smaller than that with lower height.

About points A and B: from Bernoulli equation, and because $\upsilon_A^2 << \upsilon_B^2$ **:**



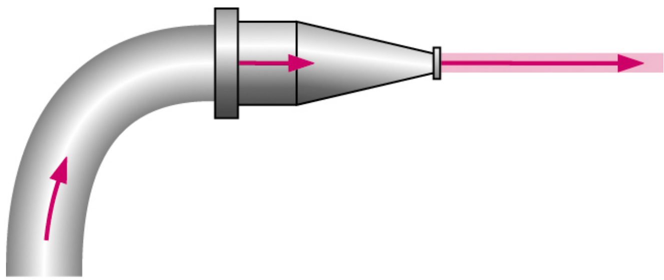
$$\rho g h_B + P_B + \tfrac{1}{2}\rho \upsilon_B^2 = \rho g h_A + P_0$$

$$h_A - h_B = \tfrac{1}{\rho g}(P_B - P_0) + \tfrac{1}{2g}\upsilon_B^2$$

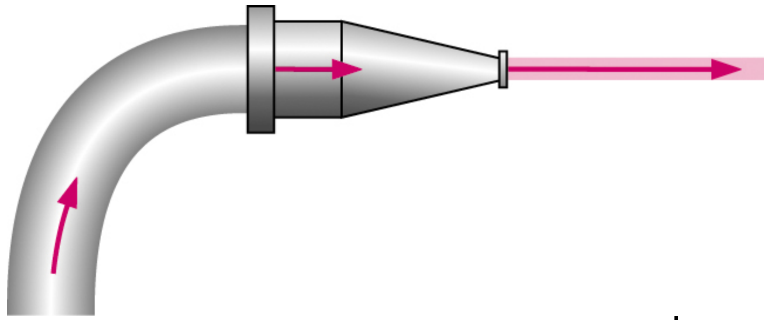
When P_B=0, (h_B-h_A) is the maximum, this is the condition for a siphon tube to work, i.e., the height between the highest point of the tube and the liquid level in the container should be smaller than:

$$h_B - h_A = \frac{P_0}{\rho g} - \frac{\upsilon_B^2}{2g} < \frac{P_0}{\rho g}$$

Application 6: **Firefighters are holding a nozzle at the end of a hose while trying to extinguish a fire. If the rate of flow of the nozzle is *Q*, and the cross-sectional area of the hose is A_1, the nozzle exit cross-sectional area is A_2. Determine the horizontal resistance force required of the firefighters to hold the nozzle.**

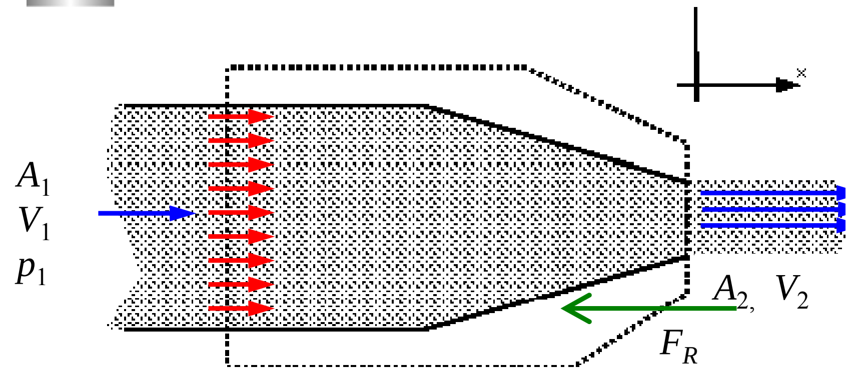


3.8 Applications of Bernoulli Equation and Momentum Equation



Solution: Select the control volume, according to the conservation of momentum, the total force acting on this control volume is:

$$\sum F = \rho Q\left(V_2 - V_1\right)$$



From the conservation of mass:

$$Q = A_1V_1 = A_2V_2 \quad \Rightarrow \quad \sum F = \rho Q^2\left(\frac{1}{A_2} - \frac{1}{A_1}\right)$$

For hose and exit of the nozzle, by Bernoulli equation:

$$\frac{p_1}{\rho g} + z_1 + \frac{V_1^{\,2}}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{V_2^{\,2}}{2g}$$

3.8 Applications of Bernoulli Equation and Momentum Equation

Because the nozzle is horizontal, the pressure at the exit of the nozzle equals the atmospheric pressure, i.e., p_2=0, thus:

$$p_1 = \frac{\rho}{2}\left(V_2^2 - V_1^2\right) = \frac{\rho Q^2}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)$$

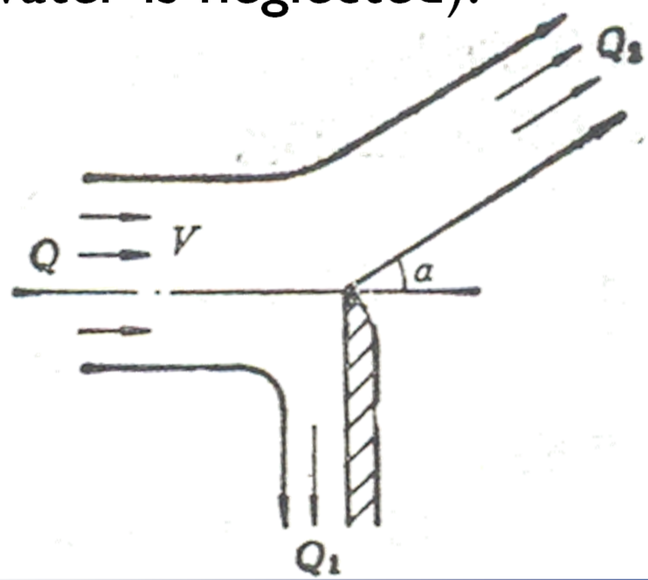
The force to hold the nozzle is F_R , then:

$$\sum F = -F_R + p_1 A_1 = \rho Q^2\left(\frac{1}{A_2} - \frac{1}{A_1}\right)$$

$$\Rightarrow F_R = \frac{\rho Q^2 A_1}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right) - \rho Q^2\left(\frac{1}{A_2} - \frac{1}{A_1}\right)$$

Application 7: Place a plate into a water column. The surface of the plate is perpendicular to the axis of the water column. The intercepted flow is as shown in the figure. Consider the flux of the water column is Q = 0.036 m^3/s, the inflow velocity is V = 30 m/s. If the intercepted flux is Q_1 = 0.012 m^3/s, determine the resultant force R acting on the plate by the water and the deflection angle α of the water (the weight and viscosity of the water is neglected).

Solution: Neglecting the gravity, pressures at inlet and outlet are all atmospheric pressure. According to Bernoulli equation: $V = V_1 = V_2 = 30$ m/s.



From conservation of mass, the flux at the outlet 2 is:

$$Q_2 = Q - Q_1 = 0.036 - 0.012 = 0.024\ \mathrm{m^3/s}$$

3.8 Applications of Bernoulli Equation and Momentum Equation

The force acting on the plate by the water flow is *R*, from conservation of momentum, the change of momentum equals the force on the fluid, i.e.,:

In *x* direction: $-\rho QV + \rho Q_2 V_2 \cos\alpha = -R$ (1)

In *y* direction: $-\rho Q_1 V_1 + \rho Q_2 V_2 \sin\alpha = 0$ (2)

From equation (2): $\sin\alpha = \dfrac{Q_1}{Q_2} = \dfrac{0.012}{0.024} = 0.5 \Rightarrow \alpha = 60^0$

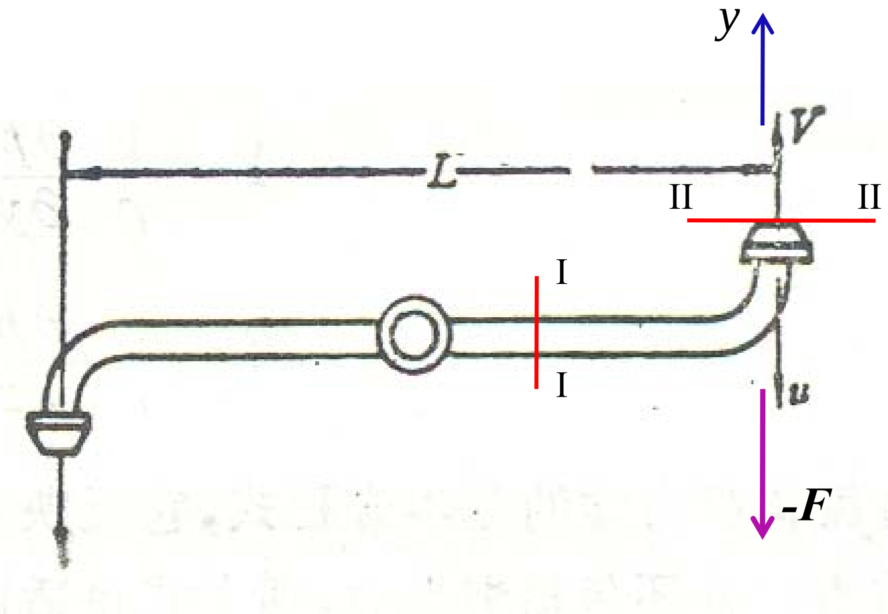
From equation (1):

$$R = \rho\left(QV - Q_2V_2\cos\alpha\right) = 1000\times\left(0.036\times30 - 0.024\times30\times0.867\right) = 455.76\,\text{N}$$

The force of water column acting on the plate is:

$$R' = -R \qquad R' = 0.456\ \text{kN}$$

Application 8: Water leaves the two ends of the nozzle (l=0.6m in length) as d = 12.5 mm diameter jets with velocity V = 6 m/s. The bearing point locates at the center of the nozzle. Determine (1) moment of rotation when the turning arm is fixed; (2) the power expression when the turning arm is rotating at a circumferential velocity u; (3) value of u corresponding to maximum power at V = 6 m/s.



Solution: (1) When turning arm is fixed

Take the control volume as the fluid in between the two sections I-II. From momentum equation, the acting force of the nozzle tip in y-direction is:

$$F=\left(\rho V\frac{\pi}{4}d^2\right)V=\frac{\pi}{4}\rho d^2V^2$$

The reacting force by the fluid on the nozzle tip is -F, the moment of rotation is:

$$T=-F\cdot l=-\frac{\pi}{4}\rho d^2V^2l=-\frac{\pi}{4}\times1000\times(0.0125)^2\times6^2\times0.6=-2.65\ \mathrm{N\cdot m}$$ — Clockwise

(2) When the turning arm is rotating at a circumferential velocity *u*, take the control volume as the fluid in between the two sections I-II again. The flux in and out of the control volume is still: $\rho V \frac{\pi}{4} d^2$, but the sprinkled absolute velocity is *V*-*u*. Based on momentum equation, the acting force of the nozzle tip in y-direction is:

$$F = \left(\rho V \frac{\pi}{4} d^2 \right)(V - u) = \frac{\pi}{4} \rho d^2 V (V - u)$$

The reacting force by the fluid on the nozzle tip is -*F*, the moment of rotation is:

$$T = -F \cdot l = -\frac{\pi}{4} \rho d^2 V (V - u) l \quad \text{if } V > u, \ T \text{ is clockwise}$$

Power: $N = T \cdot \omega = T \cdot \frac{u}{l/2} = -\frac{\pi}{2} \rho d^2 V (V - u) u$

(3) *u* value for maximum power:

$$\frac{dN}{du} = -\frac{\pi}{2} \rho d^2 V^2 + \pi \rho d^2 V u = 0 \quad \Rightarrow \quad u = \frac{V}{2} = 3\,\mathrm{m/s}$$