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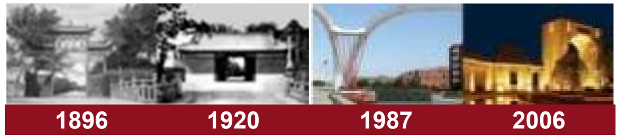


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Third Assignment

- The assignment can be downloaded from following website:

  Website： ftp://public.sjtu.edu.cn

  Username： dcwan

  Password： 2015mhydro

  Directory： IntroMHydro2015-Assignments

- Eight problems

- Submit the assignment on March 30th (in English, written on paper)

Review

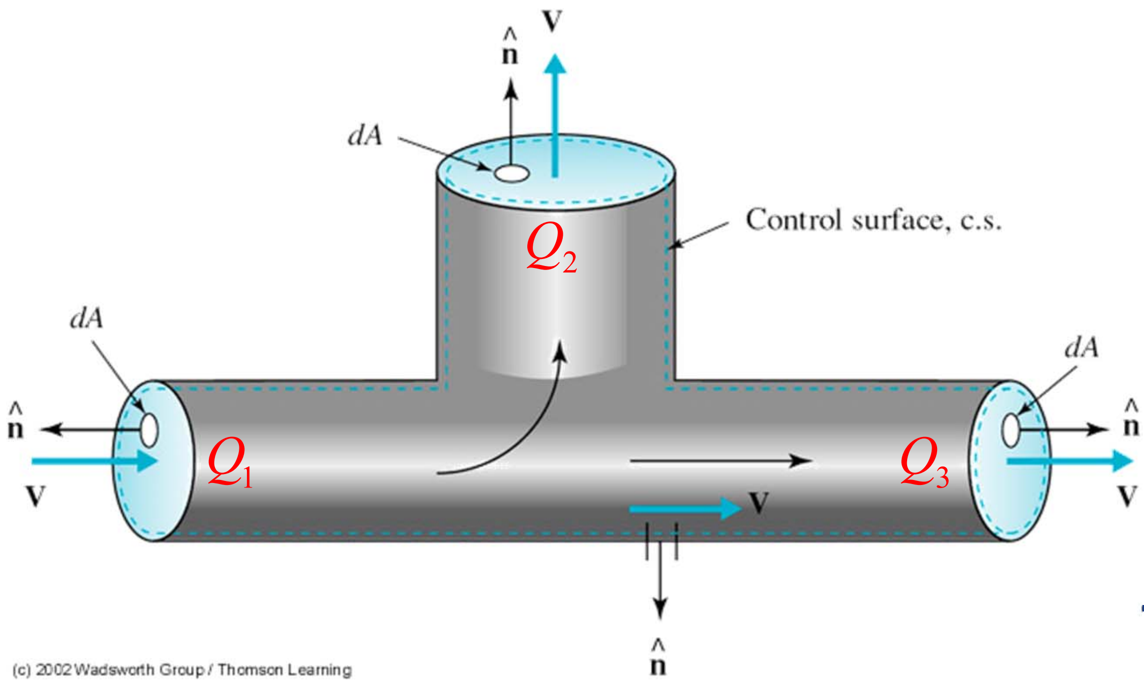
- **Continuity equation (equation of mass conservation)**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \qquad \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0$$

Incompressible flow: $\nabla \cdot \mathbf{V} = 0$

$$\oiint_{CS} \rho \mathbf{V} \cdot \mathbf{n} \, dA = 0$$

$$Q_1 = Q_2 + Q_3$$



Review

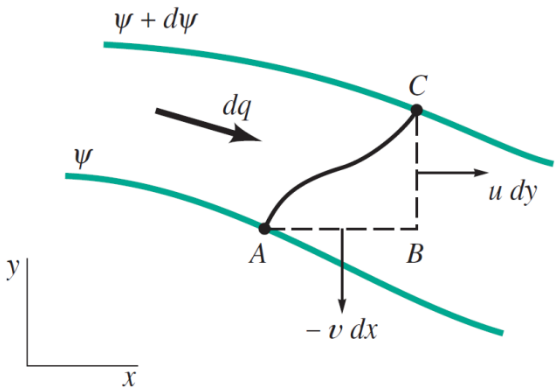
- **Stream function: incompressible 2D flow**

$$\psi = \int -v dx + u dy \quad \longleftrightarrow \quad \begin{cases} \dfrac{\partial \psi}{\partial x} = -v \\ \dfrac{\partial \psi}{\partial y} = u \end{cases}$$

Review

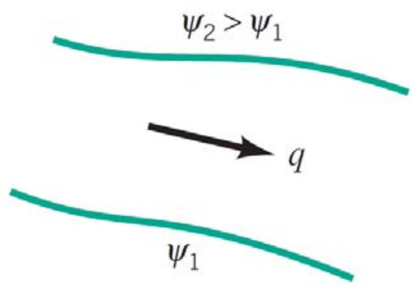
Relationship between stream function Ψ and volumetric flux Q:

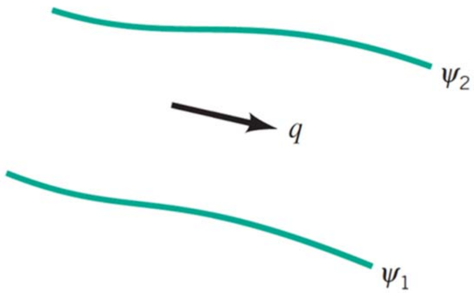
The difference in the value of stream function from one streamline to another is equal to the volume flow rate per unit width between the two streamlines (i.e., $Q_{AC} = \psi_C - \psi_A$,Ψ is single-valued function).



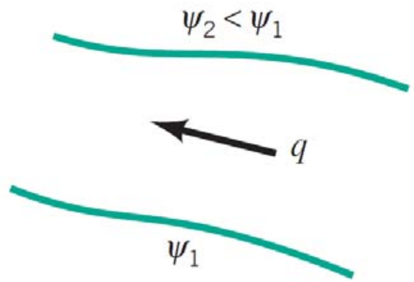
$$dq = udy - vdx$$

$$= \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$

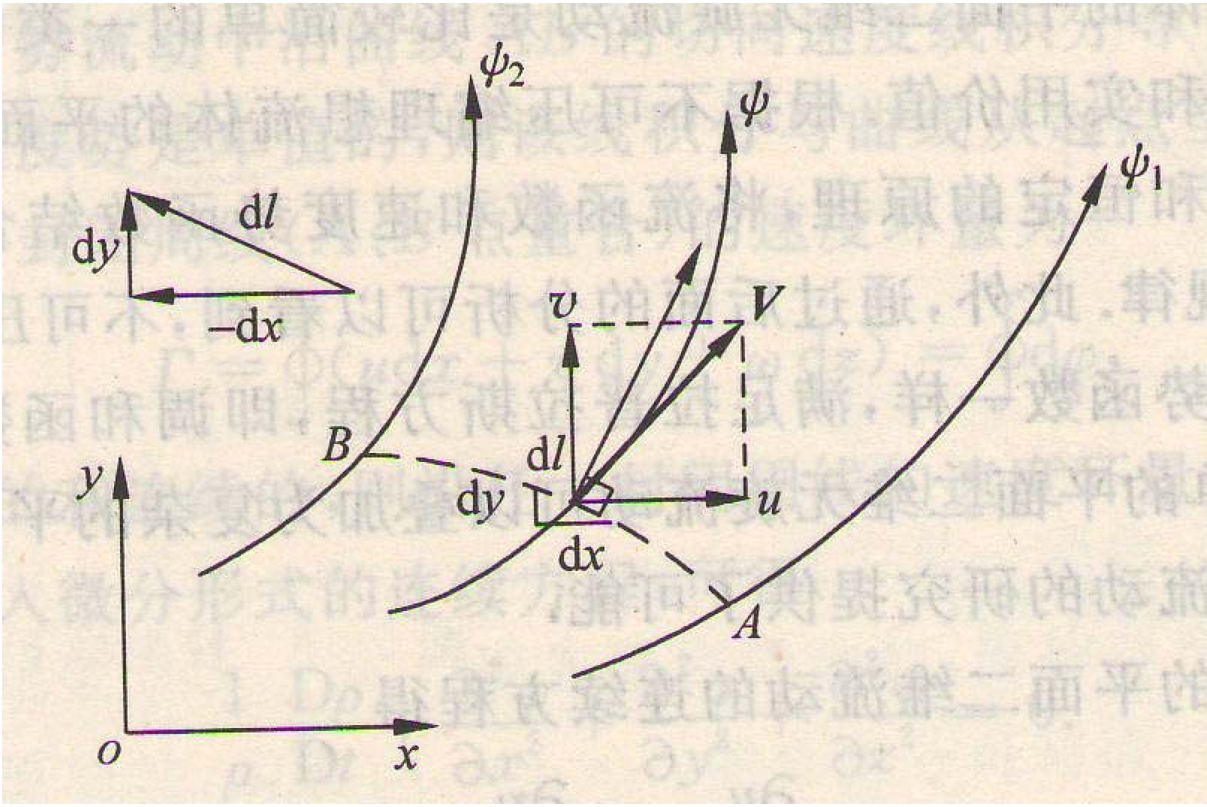




$$\therefore \quad q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$



Review



$$q=\int_{A}^{B} v_n dl=\int_{A}^{B}\left[u\cos(n,y)+v\cos(n,x)\right]dl$$

$$=\int_{A}^{B}\left[u\frac{dy}{dl}+v\frac{-dx}{dl}\right]dl=\int_{A}^{B}\left[udy-vdx\right]dl=\int_{A}^{B}d\psi=\psi_2-\psi_1$$

3.4 Stream Function

Problem 1: **Assume the velocity profile is as follows, determine *Ψ*.**

$$u = \frac{m}{2\pi}\frac{x}{x^2+y^2}, \quad v = \frac{m}{2\pi}\frac{y}{x^2+y^2}$$

Solution: First, verify that if Ψ exists, i.e., $\nabla \cdot \mathbf{V} = 0$

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{m}{2\pi}\cdot\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0$$

Thus, Ψ exists, and:

$$\psi = \int -v dx + u dy = \int -\frac{m}{2\pi}\frac{y}{x^2+y^2}dx + \frac{m}{2\pi}\frac{x}{x^2+y^2}dy$$

$$= \frac{m}{2\pi}\int\frac{xdy - ydx}{x^2+y^2} = \frac{m}{2\pi}\int\frac{d(y/x)}{1+(\frac{y}{x})^2}$$

$$= \frac{m}{2\pi}tg^{-1}\frac{y}{x} + c$$

3.4 Stream Function

Another solution:

$$\because \frac{\partial \psi}{\partial y} = u = \frac{m}{2\pi}\frac{x}{x^2+y^2}$$

$$\therefore \psi = \int u\,dy = \frac{m}{2\pi}\int \frac{x}{x^2+y^2}dy$$

$$= \frac{m}{2\pi}\int \frac{d\frac{y}{x}}{1+(y/x)^2} = \frac{m}{2\pi}tg^{-1}(y/x) + f(x)$$

And $\frac{\partial \psi}{\partial x} = -v$, i.e., $\frac{m}{2\pi}\frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} + f'(x) = -\frac{m}{2\pi}\frac{y}{x^2+y^2}$

$$f'(x) = 0, \quad \text{i.e.,} \quad f(x) = c \quad \Rightarrow \quad \psi = \frac{\mathrm{m}}{2\pi}tg^{-1}(\frac{y}{x}) + c$$

3.4 Stream Function

Problem 2: **Consider a given velocity potential of a flow field: $\phi = 4xy$. Solve its stream function.**

Solution: The velocity can be determined from the velocity potential

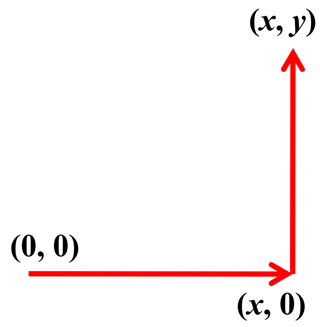
$$u = \frac{\partial \phi}{\partial x} = 4y, \qquad v = \frac{\partial \phi}{\partial y} = 4x$$

From the equation above, we get: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, so there exists the stream function.

By definition of stream function:

$$\psi = \int -v dx + u dy = \int -4x dx + 4y dy$$

$$= \int_{(0,0)}^{(x,0)} + \int_{(x,0)}^{(x,y)} = -2x^2 + 2y^2 + C$$



Where C is a constant.

3.4 Stream Function

Problem 3: The velocity distribution of a two dimensional flow is given as: $u = 2xy, \quad v = x^2 - y^2$. Determine velocity potential function and stream function.

Solution: Because $\Omega_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 2x - 2x = 0$, there is velocity potential.

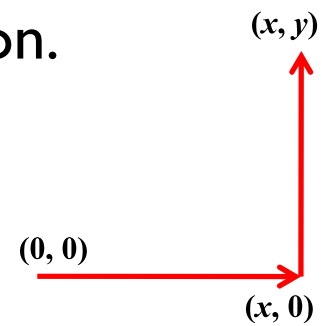
$$\phi = \int u\,\mathrm{d}x + v\,\mathrm{d}y = \int 2xy\,\mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y$$

$$= \int_{(0,0)}^{(x,0)} + \int_{(x,0)}^{(x,y)} = 0 + \int_{(x,0)}^{(x,y)} \left(x^2 - y^2\right)\mathrm{d}y = x^2 y - \frac{1}{3}y^3$$

And because $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 2y - 2y = 0$, there is stream function.

$$\psi = \int -v\,dx + u\,dy = \int -\left(x^2 - y^2\right)dx + 2xy\,dy$$

$$= \int_{(0,0)}^{(x,0)} + \int_{(x,0)}^{(x,y)} = -\frac{x^3}{3} + xy^2 + C$$



3.4 Stream Function

Problem 4: The velocity potential of an irrotational flow of an incompressible fluid is given as below, determine the stream function.

$$(1)\ \ \phi = x/(x^2+y^2) \qquad\qquad (2)\ \ \phi = \frac{m}{2\pi}\ln r \quad (m = \text{const})$$

Solution: (1)

$$u = \frac{\partial \phi}{\partial x} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad v = \frac{\partial \phi}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{-2x(x^2+y^2)^2-(y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \frac{2x^5-4x^3y^2-6xy^4}{(x^2+y^2)^4}$$

$$\frac{\partial v}{\partial y} = \frac{-2x(x^2+y^2)^2+2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \frac{-2x^5+4x^3y^2+6xy^4}{(x^2+y^2)^4}$$

Because $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, there is stream function Ψ.

3.4 Stream Function

From $v = -\dfrac{\partial \psi}{\partial x}$:

$$\psi = \int \frac{2xy}{(x^2+y^2)^2}dx + f(y) = y\int \frac{1}{(x^2+y^2)^2}d(x^2+y^2) + f(y) = -\frac{y}{x^2+y^2} + f(y)$$

As $u = \dfrac{\partial \psi}{\partial y} = -\dfrac{x^2+y^2-2y^2}{(x^2+y^2)^2} + f'(y) = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, so:

$$f'(y) = 0 \qquad \Rightarrow \qquad f(y) = C$$

Thus, the stream function is:

$$\psi = -\frac{y}{x^2+y^2} + C$$

3.4 Stream Function

More convenient with polar coordinates. The relations between polar coordinates and Cartesian coordinates are:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{\partial \phi}{r\partial \theta}$$

$$\nabla \cdot \mathbf{V} = \frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta}$$

Then:

$$\phi(x, y) = \phi(r, \theta) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{\cos\theta}{r^2}, \quad v_\theta = \frac{\partial \phi}{r\partial \theta} = -\frac{\sin\theta}{r^2}$$

$$\frac{\partial (r v_r)}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{\cos\theta}{r}\right) = \frac{\cos\theta}{r^2}, \quad \frac{\partial v_\theta}{\partial \theta} = -\frac{\cos\theta}{r^2}$$

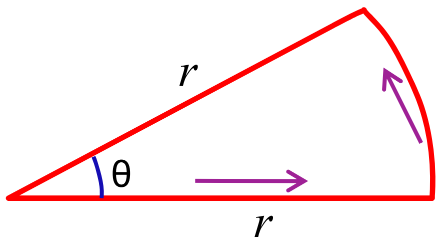
As $\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0$, there is stream function Ψ.

3.4 Stream Function

The stream function in polar coordinates is:

$$\psi = \int v_r r d\theta - v_\theta dr = \int -\frac{\cos\theta}{r} d\theta + \frac{\sin\theta}{r^2} dr = -\frac{\sin\theta}{r}$$

Select the integral path as: $(0,0) \rightarrow (r,0) \rightarrow (r,\theta)$



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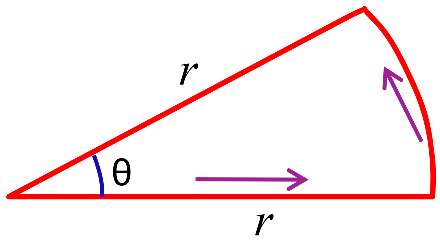
3.4 Stream Function

(2) $$v_r = \frac{\partial \phi}{\partial r} = \frac{m}{2\pi r}, \qquad v_\theta = \frac{\partial \phi}{r\partial \theta} = 0$$

Because $\frac{\partial (rv_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0$, there is stream function Ψ.

Select the integral path as: $(0,0) \rightarrow (r,0) \rightarrow (r,\theta)$

$$\psi = \int v_r r d\theta - v_\theta dr = \int \frac{m}{2\pi r} r d\theta = \frac{m}{2\pi}\theta + C$$



3.5 Momentum Equation

First of all, to derive the equation

Let $G=\rho\mathbf{V}$, the application of the Reynolds transport theorem gives:

$$\frac{d}{dt}\iiint_{MV}\rho\mathbf{V}\,d\mathcal{V}=\iiint_{CV}\frac{\partial\rho\mathbf{V}}{\partial t}d\mathcal{V}+\iint_{CS}\rho\mathbf{V}\mathbf{V}\cdot\mathbf{n}\,dS$$

$$=\iiint_{CV}\left[\frac{\partial\rho}{\partial t}\mathbf{V}+\rho\frac{\partial\mathbf{V}}{\partial t}+\nabla\cdot(\rho\mathbf{V}\mathbf{V})\right]d\mathcal{V}$$

$$=\iiint_{CV}\left\{\mathbf{V}\left[\cancel{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})}\right]+\rho\left[\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right]\right\}d\mathcal{V}$$

$$=\iiint_{CV}\rho\frac{D\mathbf{V}}{Dt}d\mathcal{V}=\iiint_{MV}\rho\frac{d\mathbf{V}}{dt}d\mathcal{V}$$

i.e.,

$$\boxed{\frac{d}{dt}\iiint_{MV}\rho\mathbf{V}\,d\mathcal{V}=\iiint_{MV}\rho\frac{d\mathbf{V}}{dt}d\mathcal{V}=\iiint_{CV}\rho\frac{D\mathbf{V}}{Dt}d\mathcal{V}}$$

3.5 Momentum Equation

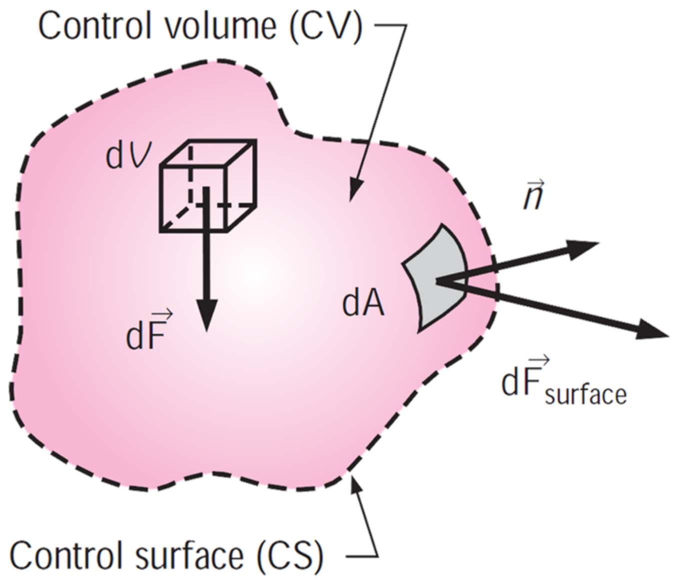
Conservation of momentum (Newton's 2nd law of motion): time rate of change of the momentum of a body is equal to the net force acting on it

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\left(m\vec{V}\right)}{dt}$$

3.5 Momentum Equation

Applying conservation of momentum to a control volume:

$$\underbrace{\frac{d}{dt}\iiint_{MV}\rho\mathbf{V}\,d\mathcal{V}}_{\text{rate of change of momentum}} = \underbrace{\iint_{MS}\mathbf{s}\,dA}_{\text{surface force}} + \underbrace{\iiint_{MV}\rho\mathbf{g}\,d\mathcal{V}}_{\text{body force}}$$



First term on the right side: $\iint_{MS}\mathbf{s}\,dA = \iint_{MS}\boldsymbol{\sigma}\cdot\mathbf{n}\,dA = \iiint_{MV}\nabla\cdot\boldsymbol{\sigma}\,d\mathcal{V}$

Thus: $\frac{d}{dt}\iiint_{MV}\rho\mathbf{V}\,d\mathcal{V} = \iiint_{MV}\rho\frac{d\mathbf{V}}{dt}\,d\mathcal{V} = \iiint_{MV}(\rho\mathbf{g}+\nabla\cdot\boldsymbol{\sigma})\,d\mathcal{V}$

3.5 Momentum Equation

Since MV(CV) is arbitrary, thus:

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}$$

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \quad \text{or} \quad \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

?

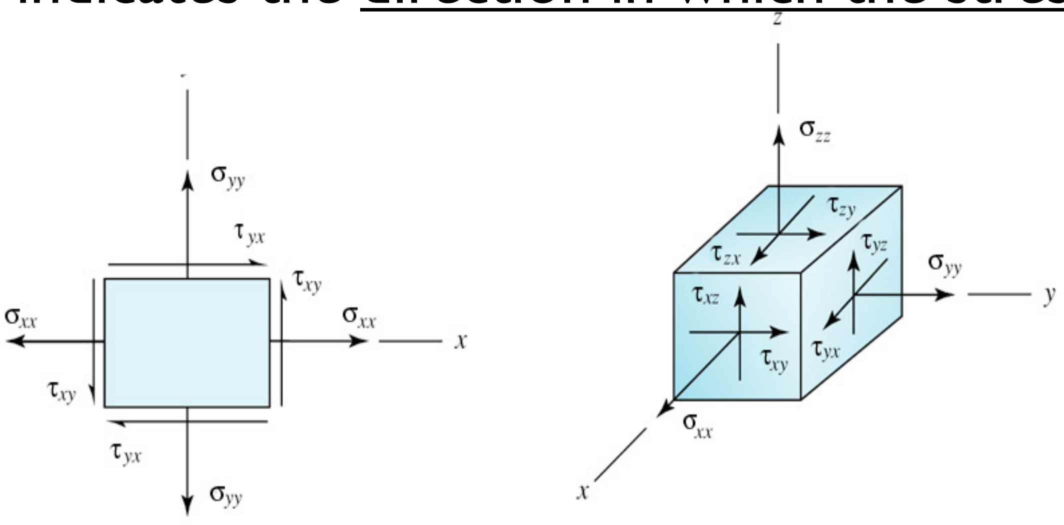
The equation above is the **momentum equation**

3.5 Momentum Equation

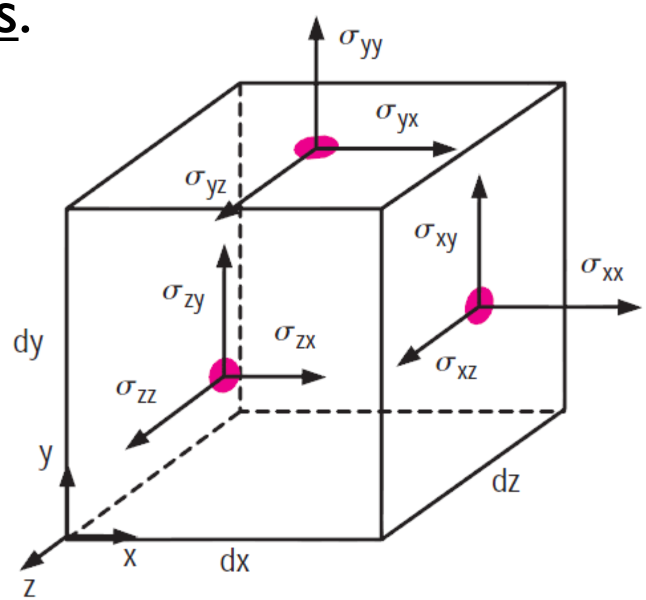
Expression of the surface stresses: 2nd-order tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \sigma_{ij} \quad (i, j = 1, 2, 3)$$

The first subscript indicates the direction of the normal to the surface on which the stress is considered; the second subscript indicates the direction in which the stress acts.



(a) (b)

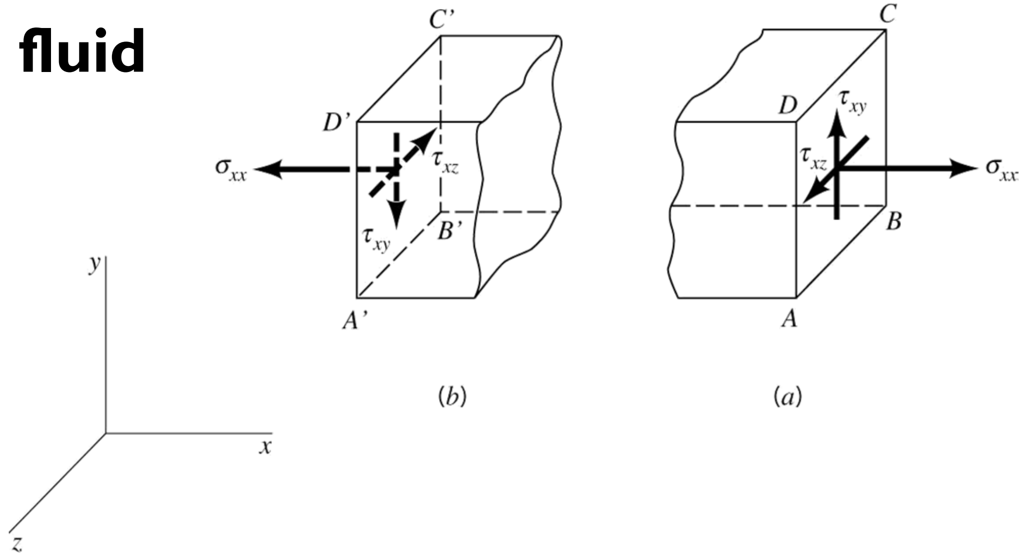


3.5 Momentum Equation

Consider the balance of the fluid element:

$$\tau_{ij} = \tau_{ji} \quad (i \neq j)$$

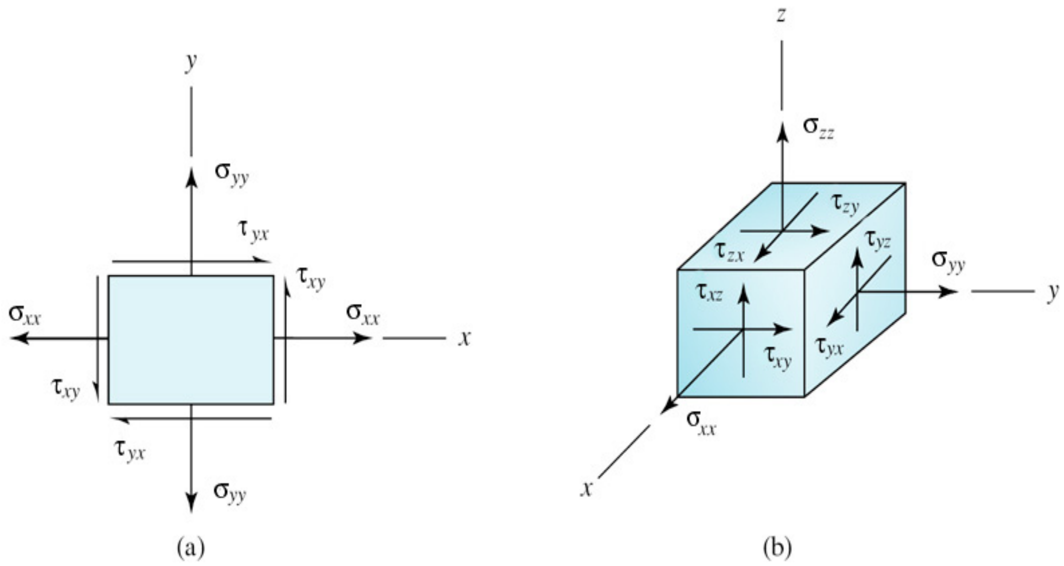
Thus, the surface stress tensor is a symmetric tensor.



Because the normal stress is pressure, then surface stresses is rewritten as:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$



3.5 Momentum Equation

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial \sigma_{ij}}{\partial x_j}+\rho g_i \Rightarrow \rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial(-p\delta_{ij}+\tau_{ij})}{\partial x_j}+\rho g_i$$

There are seven unknown variables in the momentum equation: 3 u_i, 1 p, and 3 τ_{ij}. However, the number of governing equations is only four: momentum equation (in three directions) and a continuity equation.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

To close the equation, it is necessary to build up the relations between surface stresses and kinematics, i.e., relations between stress and strain-rate, which is called Constitutive Equation.

3.5 Momentum Equation

Relation between stress and strain-rate:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Consider Newtonian fluids, from Newton's law of viscosity, the shear stress is proportional to the velocity gradient. For a small fluid element, a general Newton's law of viscosity can be derived:

$$\tau_{ij} = C_{ijlm} \frac{\partial u_l}{\partial x_m}$$

Where C_{ijlm} is a forth-order tensor coefficient, i.e., with $3^4 = 81$ coefficients. From tensor theory, forth-order tensor consists of second-order tensors, i.e., :

$$C_{ijlm} = \lambda \delta_{ij}\delta_{lm} + \mu\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)$$

81 coefficients are reduced to two: λ and μ

3.5 Momentum Equation

$$C_{ijlm} = \lambda \delta_{ij} \delta_{lm} + \mu \left(\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl} \right)$$

Substitute this coefficient into the shear stress equation:

$$\tau_{ij} = C_{ijlm} \frac{\partial u_l}{\partial x_m} = \lambda \frac{\partial u_l}{\partial x_l} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Relation between surface stresses and strain-rate:

$$\sigma_{ij} = -p \delta_{ij} + \tau_{ij} = \left(-p + \lambda \frac{\partial u_l}{\partial x_l} \right) \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

3.5 Momentum Equation

For incompressible fluids: $\nabla \cdot \mathbf{V} = \frac{\partial u_l}{\partial x_l} = 0$

We get: $\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

Relation between surface stresses and strain-rate for incompressible fluids is:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

3.5 Momentum Equation

Substitute into the momentum equation:

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Momentum equation: $\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$

$$\Rightarrow \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial\left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)}{\partial x_j} + \rho g_i$$

$$= -\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i}\right) + \rho g_i$$

$$= -\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right) + \rho g_i$$

3.5 Momentum Equation

Momentum equation for incompressible flows:

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\rho g_i+\mu\frac{\partial^2 u_i}{\partial x_j^2}\quad (i,j=1,2,3)$$

or

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+g_i+\nu\frac{\partial^2 u_i}{\partial x_j^2}$$

(I) (II) (III) (IV) (V)

where $\nu=\mu/\rho$ is the kinematic viscosity

Or, in tensor form: $$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=-\nabla p+\rho\mathbf{g}+\mu\nabla^2\mathbf{V}$$

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3.5 Momentum Equation

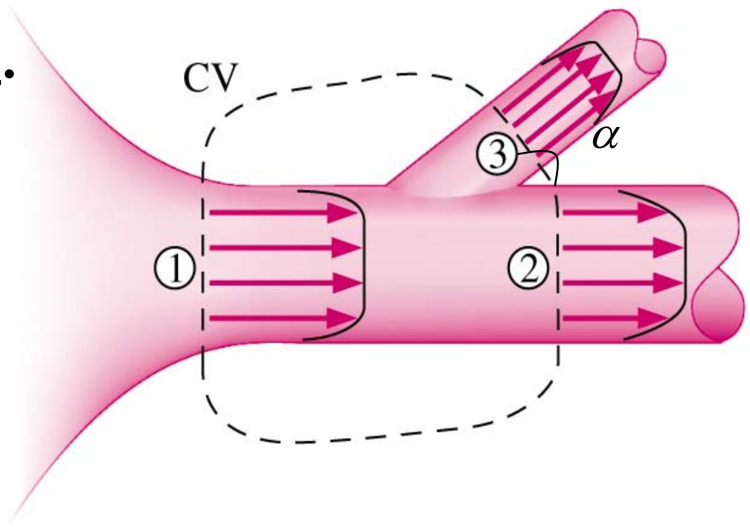
Physical interpretation of each item:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=-\nabla p+\mu\nabla^{2}\mathbf{V}+\rho\mathbf{g}$$

- **Local acceleration** — $\frac{\partial \mathbf{V}}{\partial t}$
- **Convective acceleration (inertia, nonlinear item)** — $\mathbf{V}\cdot\nabla\mathbf{V}$
- **Pressure gradient** — $-\nabla p$
- **Viscous diffusion due to molecular viscosity of the fluid** — $\mu\nabla^{2}\mathbf{V}$
- **Gravity (body force)** — $\rho\mathbf{g}$

3.5 Momentum Equation

<u>Momentum equation can be derived from the practical engineering problems</u>.



(Pipe flows as shown in the figure)
According to the conservation of momentum: time rate of change of the momentum of a CV is equal to the net force acting on it:

$$F_x = -Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \cos\alpha$$
$$F_y = Q_3 V_3 \sin\alpha$$

***Q*: flux**
***V*: velocity**
***S*: cross-sectional area**
***F*: forces acting on CV**

From conservation of mass:

$$Q_1 = Q_2 + Q_3 \quad \Rightarrow \quad \rho S_1 V_1 = \rho S_2 V_2 + \rho S_3 V_3$$

Exp.

3.6 Governing Equations of Fluid Motion

Continuity equation and momentum equation form the basic governing equations of fluid flow.

For incompressible Newtonian fluids, the basic governing equation is:

Continuity equation:

$$\nabla \cdot \mathbf{V} = 0$$

Momentum equation:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{V}$$

This system of equations has four unknowns: three u_i and one p. The number of the governing equations is four, so the system of equations is closed.

This is so-called Navier-Stokes equations, NS equations for short.

3.6 Governing Equations of Fluid Motion

In vector form:

Continuity: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

x-component: $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x$$

y-component: $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+g_y$$

z-component: $$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+g_z$$

If the gravity is the body force, then $g_x = g_y = 0$, $g_z = -g$

3.6 Governing Equations of Fluid Motion

Or denoted by Einstein notation:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

(I) (II) (III) (IV) (V)

Or:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

The physical explanation of each item in NS momentum equation:

(I) local acceleration;

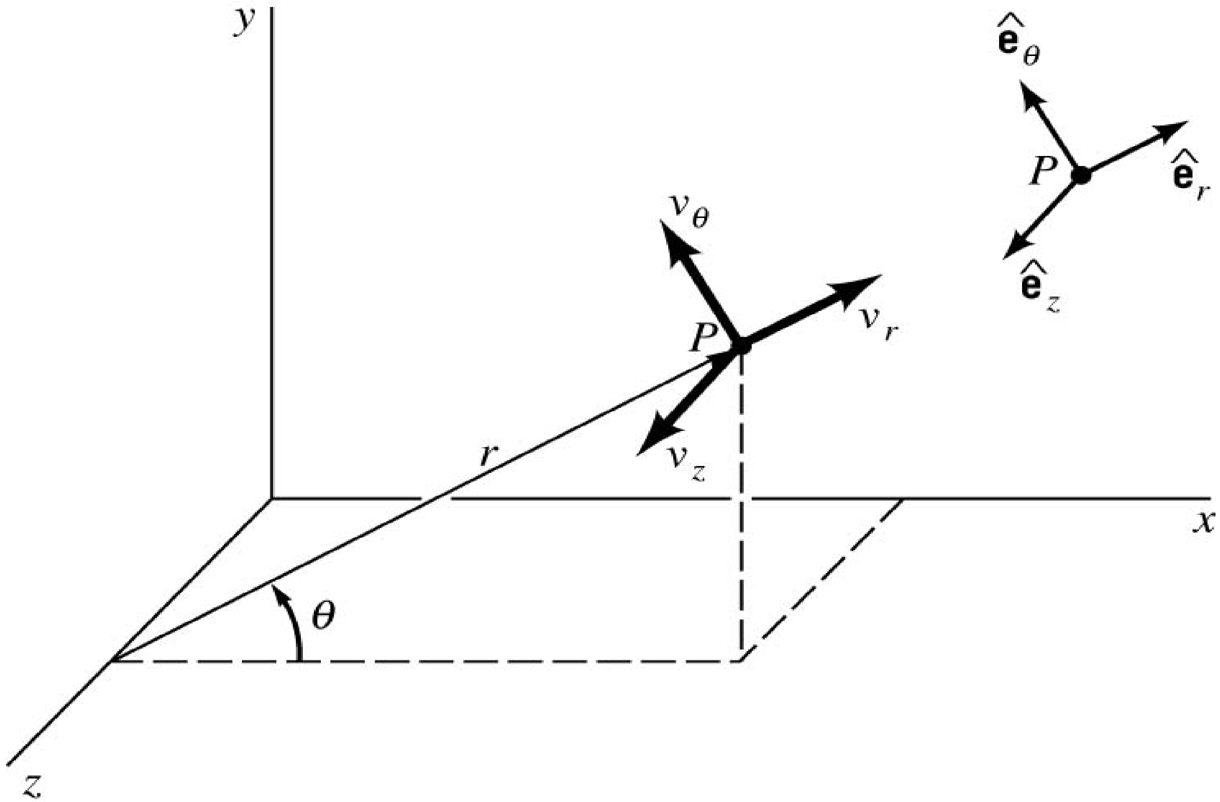
(II) convective acceleration (inertia, convection, nonlinear term of the equation);

(III) pressure gradient;

(IV) volume force or gravity;

(V) viscous diffusion of momentum due to molecular viscosity of the fluid.

3.6 Governing Equations of Fluid Motion



Cartesian/Rectangular Coordinates (x, y, z)

Cylindrical Coordinates (r,θ, z)

NS equations in Cylindrical Coordinates (r, θ, z):

Continuity: $$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

r-component: $$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+g_r$$

θ-component: $$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$
$$+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+g_\theta$$

z-component: $$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+g_z$$

3.6 Governing Equations of Fluid Motion

NS equations specifies the motion of real fluids. To simplify the problem, we consider the ideal fluids first. Namely, the fluids have no viscosity and their viscosity coefficients are 0, $\mu = \nu = 0$. Then, NS equations can be simplified as:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \mathbf{f}, \quad \nabla \cdot \mathbf{V} = 0$$

This governing equation is called Euler equation for ideal fluids.

Because: $\mathbf{V} \cdot \nabla \mathbf{V} = \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times (\nabla \times \mathbf{V}) = \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times \mathbf{\Omega} = \nabla\left(\frac{V^2}{2}\right) - 2\mathbf{V} \times \boldsymbol{\omega}$

Then, Euler equation can be rewritten as:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) - 2\mathbf{V} \times \boldsymbol{\omega} = -\frac{1}{\rho} \nabla p + \mathbf{f}$$

This form of Euler equation is called Lamb equation.

3.6 Governing Equations of Fluid Motion

$$\mathbf{V}\times(\nabla\times\mathbf{V})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u & v & w\\ \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} & \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} & \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\end{vmatrix}=\left[v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right]\vec{i}+\left[w\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)-u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]\vec{j}+\left[u\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-v\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\right]\vec{k}$$

$$=\left[\frac{1}{2}\left(\frac{\partial v^2}{\partial x}+\frac{\partial w^2}{\partial x}\right)-\left(v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]\vec{i}+\left[\frac{1}{2}\left(\frac{\partial w^2}{\partial y}+\frac{\partial u^2}{\partial y}\right)-\left(u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial z}\right)\right]\vec{j}+\left[\frac{1}{2}\left(\frac{\partial u^2}{\partial z}+\frac{\partial v^2}{\partial z}\right)-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}\right)\right]\vec{k}$$

$$=\left[\frac{1}{2}\frac{\partial\left(u^2+v^2+w^2\right)}{\partial x}-\left(\frac{1}{2}\frac{\partial u^2}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]\vec{i}+\left[\frac{1}{2}\frac{\partial\left(u^2+v^2+w^2\right)}{\partial y}-\left(u\frac{\partial v}{\partial x}+\frac{1}{2}\frac{\partial v^2}{\partial y}+w\frac{\partial v}{\partial z}\right)\right]\vec{j}+\left[\frac{1}{2}\frac{\partial\left(u^2+v^2+w^2\right)}{\partial z}-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{1}{2}\frac{\partial w^2}{\partial z}\right)\right]\vec{k}$$

$$=\left[\frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}\right)\vec{i}+\frac{\partial}{\partial y}\left(\frac{u^2+v^2+w^2}{2}\right)\vec{j}+\frac{\partial}{\partial z}\left(\frac{u^2+v^2+w^2}{2}\right)\vec{k}\right]-\left[\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\vec{i}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\vec{j}+\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\vec{k}\right]$$

$$=\nabla\left(\frac{u^2+v^2+w^2}{2}\right)-\left[\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)u\vec{i}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)v\vec{j}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)w\vec{k}\right]$$

$$=\nabla\left(\frac{V^2}{2}\right)-(\mathbf{V}\cdot\nabla)\begin{pmatrix}u\vec{i}\\ v\vec{j}\\ w\vec{k}\end{pmatrix}=\nabla\left(\frac{V^2}{2}\right)-\mathbf{V}\cdot\nabla\mathbf{V}$$

$$\boxed{\mathbf{V}\cdot\nabla\mathbf{V}=\nabla\left(\frac{V^2}{2}\right)-\mathbf{V}\times(\nabla\times\mathbf{V})}$$

3.6 Governing Equations of Fluid Motion

Gradient

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_i} \qquad (i = 1, 2, 3)$$

Divergence

$$\nabla \cdot \mathbf{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (u, v, w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u_i}{\partial x_i}$$

Curl

$$\nabla \times \mathbf{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}\vec{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}\vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}\vec{k}$$

Convective acceleration

$$\mathbf{V} \cdot \nabla \mathbf{V} = \nabla \left(\frac{V^2}{2}\right) - \mathbf{V} \times (\nabla \times \mathbf{V}) = \nabla \left(\frac{V^2}{2}\right) - \mathbf{V} \times \mathbf{\Omega}$$