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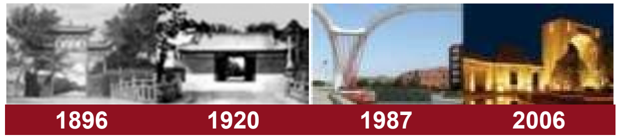
SHANGHAI JIAO TONG UNIVERSITY



Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

About 1st Assignment

- Main problems in the assignment:

  No. 2, 3, 6

- Who did well:

  吴华坚

About 1st Assignment

◆ Problem No. 2

Determine the force *F* to pull the plate

Solution:

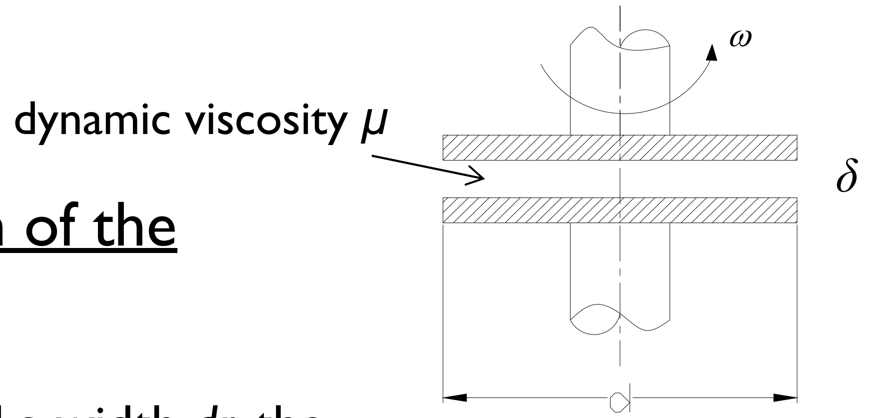
F equals the shear force *T*:

$$T = \mu \cdot A \cdot du / dy$$

$$= 0.86 \times (2 \times 0.5 \times 0.5) \times \frac{1}{0.02} = 21.5N$$

About 1st Assignment

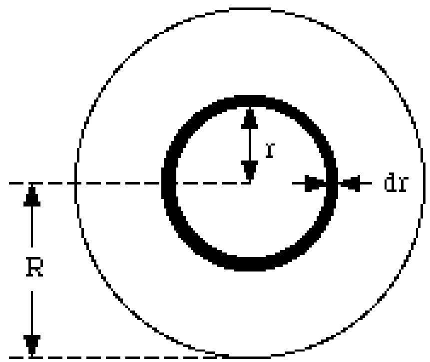
◆ Problem No. 3



<u>Write the expression of the moment *M*</u>

Select a circle of the disk with a radius *r* and a width *dr*, the shear stress on it can be written as:

$$\tau(r) = \mu \frac{\omega r}{\delta}$$



The moment is (=force×arm of force):

$$dM = \underline{\tau(r)\cdot 2\pi r dr} \cdot \underline{r} = \frac{2\pi\mu\omega}{\delta} r^3 dr$$

The total moment is the integral of *dM*:

$$M = \int dM = \frac{2\pi\mu\omega}{\delta}\int_0^{d/2} r^3 dr = \frac{\pi\mu\omega d^4}{32\delta}$$

About 1st Assignment

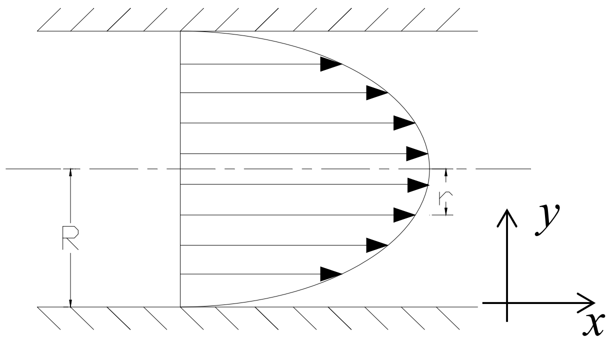
- **Problem No. 6** Determine the shear stress

$$\tau = \mu \frac{du}{dy}$$

Velocity gradient:

Let y=R-r, $u(y) = C(1 - \frac{(R-y)^2}{R^2}) = C \cdot \frac{2Ry - y^2}{R^2}$

$$\tau = \mu \frac{du}{dy} = C\mu \cdot \frac{2R - 2y}{R^2} = \frac{2C\mu}{R^2} r$$



$$u = C(1 - \frac{r^2}{R^2})$$

Review: Chapter 2

- **Rotational and irrotational flows**

$$\text{rate of rotation } \boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{V} = \frac{1}{2}\boldsymbol{\Omega}$$

$$\text{Vorticity } \boldsymbol{\Omega} = \nabla \times \mathbf{V} = 2\boldsymbol{\omega}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{2\omega_x}\vec{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{2\omega_y}\vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{2\omega_z}\vec{k}$$

$$\Omega = \left(\Omega_x,\ \Omega_y,\ \Omega_z\right) = 0$$

Review: Chapter 2

- **Velocity potential**

Irrotational $\Leftrightarrow \nabla \times \mathrm{V} = 0 \Leftrightarrow \phi \Leftrightarrow$ Potential flow

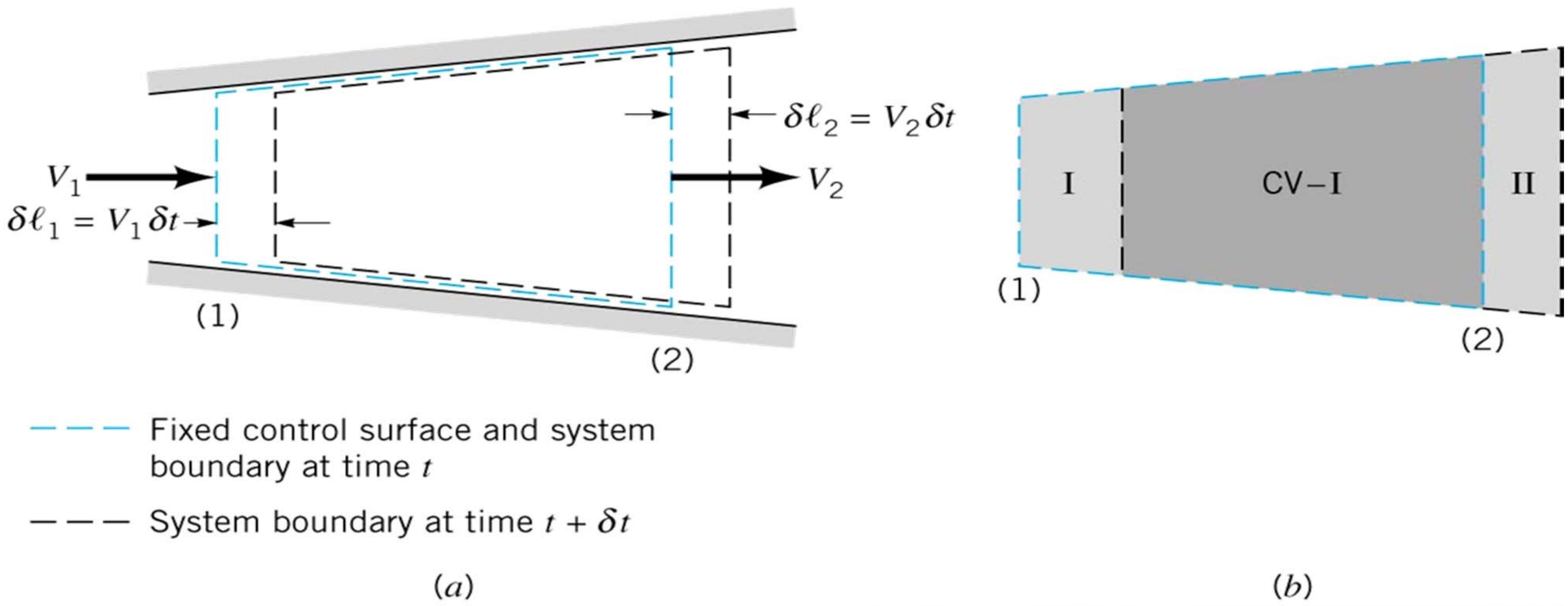
$$\phi(x, y, z; t) = \int u dx + v dy + w dz$$

$$\mathbf{V} = \nabla \phi, \qquad \left.\begin{aligned} \frac{\partial \phi}{\partial x} &= u \\ \frac{\partial \phi}{\partial y} &= v \\ \frac{\partial \phi}{\partial z} &= w \end{aligned}\right\}$$

Review: Chapter 3

- **Fluid dynamics:** laws governing fluid motion (e.g., conservation of mass, Newton's laws of motion)

- **System and Control volume**

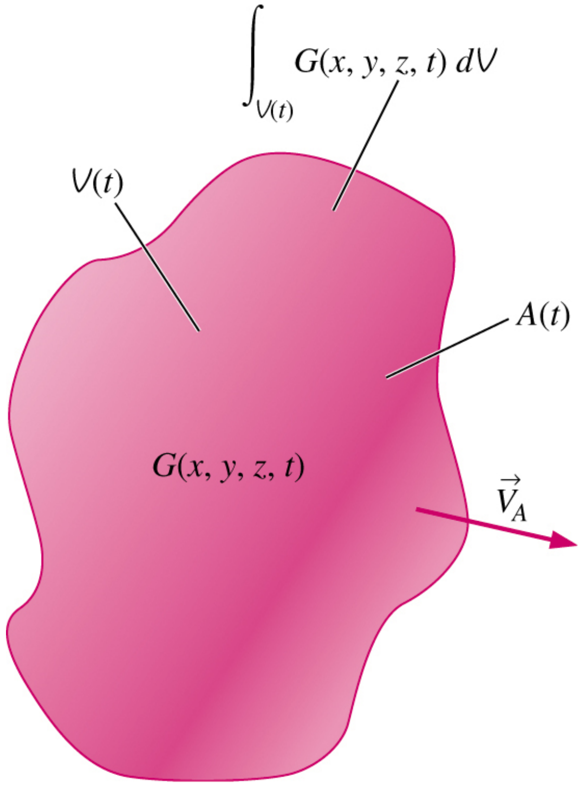


Review: Chapter 3

System: a collection of matter of fixed identity (always the same fluid particles), which may move, flow, and interact with its surroundings.

Material volume: a volume that contains the same fluid as it moves and deforms following the motion of the fluid.

Material surface: an enclosing surface of a material volume; no fluid particles can cross it.

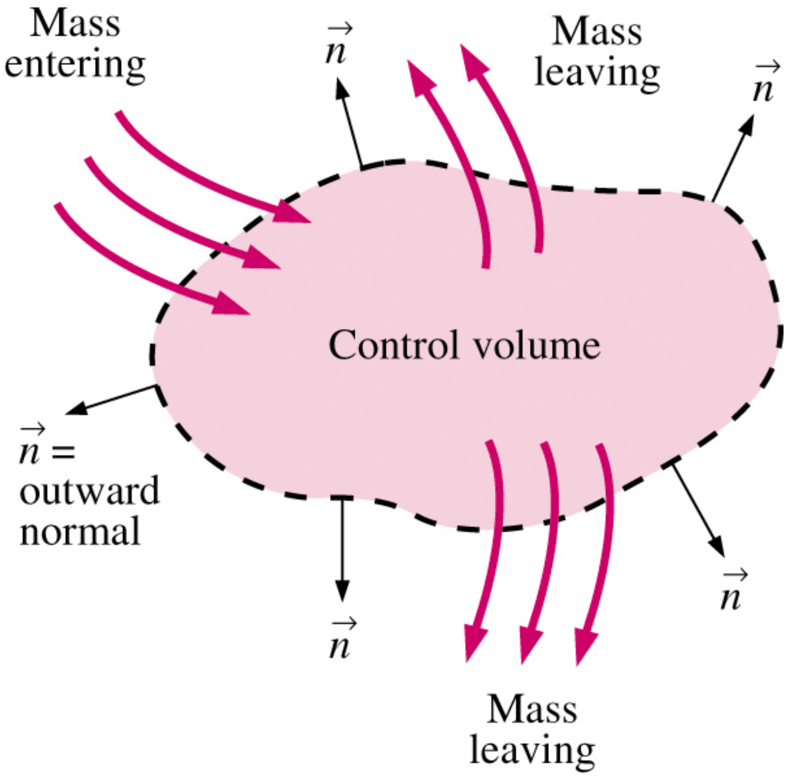


Review: Chapter 3

Control volume: a volume in space (a geometric entity, independent of mass) through which fluid may flow.

Control volume: a volume of fluid in a flow field, usually fixed in space, to be occupied by different fluid particles at different times.

Control surface: imaginary or physical enclosing surface of a control volume, fluid particles can cross it.



$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA$$

Review

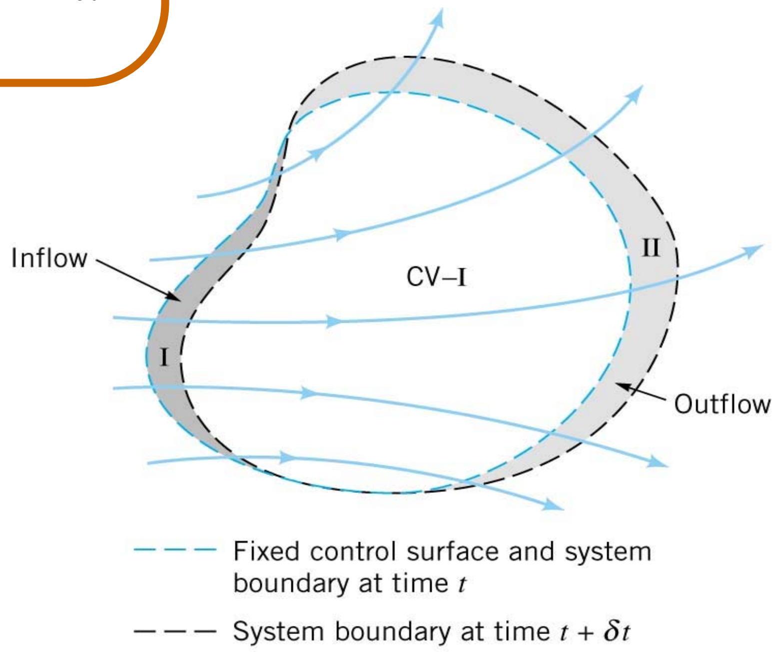
- **Reynolds Transport Theorem (RTT)**

$$\frac{d}{dt}\iiint_{MV} G\,d\mathcal{V} = \frac{\partial}{\partial t}\iiint_{CV} G\,d\mathcal{V} + \iint_{CS} G\mathbf{V}\cdot\mathbf{n}\,dA$$

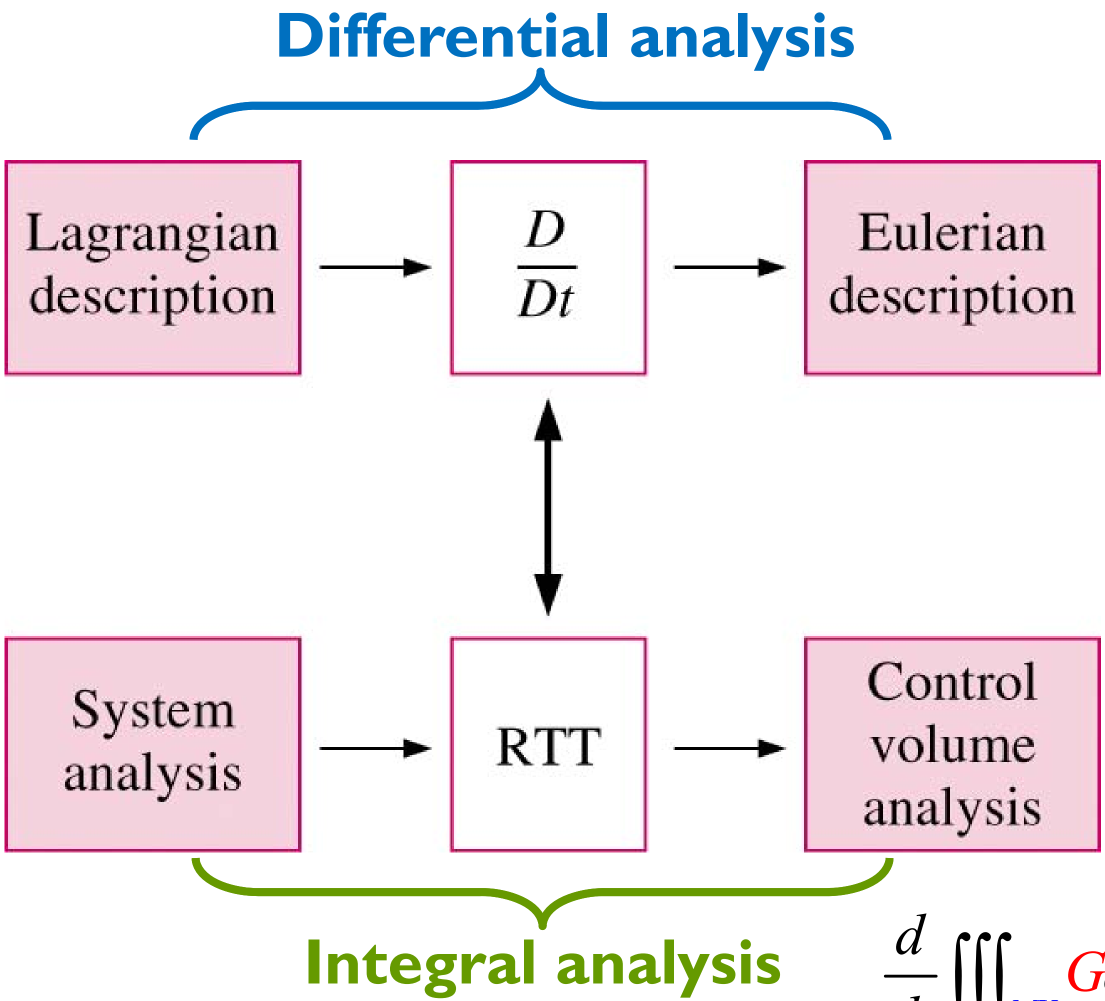
time rate of change of property G within the MV

rate of change of G within the CV as the fluid flows through it

net flow rate (flux) of G across the entire CS



3.2 Reynolds Transport Theorem



$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{V} \bullet \nabla) f$$

$$\frac{d}{dt}\iiint_{MV} G\, d\mathcal{V} = \frac{\partial}{\partial t}\iiint_{CV} G\, d\mathcal{V} + \iint_{CS} G\mathbf{V}\cdot\mathbf{n}\, dA$$

3.3 Continuity Equation

Continuity Equation: conservation of mass (mass of a system is neither be created nor destroyed)

In RTT equation, if the quantity is mass, i.e., $G=\rho$ **, then:**

L.H.S. $$\frac{d}{dt}\iiint_{MV}\rho d\forall=\frac{d}{dt}(\text{mass in } MV)=0$$

(From the definition of MV: it always contains the same fluids)

R.H.S. $$\frac{\partial}{\partial t}\iiint_{CV}\rho d\forall+\iint_{CS}\rho \boldsymbol{V}\cdot\boldsymbol{n}dA$$

$$=\underbrace{\iiint_{CV}\frac{\partial\rho}{\partial t}d\forall}_{CV\text{ is stationary}}+\underbrace{\iiint_{CV}\nabla\cdot(\rho\boldsymbol{V})d\forall}_{\text{by Gauss theorem}}$$

3.3 Continuity Equation

Because the control volume (CV) is arbitrary, then:

$$\iiint_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \iiint_{CV} \nabla \cdot \left(\rho V \right) d\mathcal{V} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{V} \right) = 0}$$

This is the continuity equation

Further, substituting $\nabla \cdot \left(\rho \mathbf{V} \right) = \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V}$ **into the equation**

we get:

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

3.3 Continuity Equation

For an incompressible fluid:

the density of a fluid particle is invariant with time $\Leftrightarrow \dfrac{D\rho}{Dt} = 0$

Thus, its continuity equation is:

$$\nabla \cdot \mathbf{V} = 0$$

(i.e., divergence of velocity is zero for incompressible flow)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

3.3 Continuity Equation

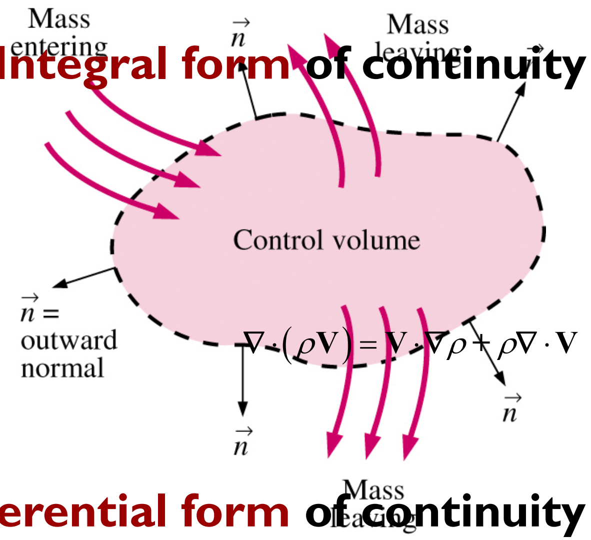
The continuity equation can <u>also</u> be derived from conservation of mass. Take a region Ω from the flow field, its surface is *S*. The conservation of mass is expressed as:

Increase (decrease) of the fluid mass = Mass flux flows into (out of) S per unit time

$$\frac{\partial}{\partial t}\iiint_{\Omega}\rho\, d\Omega = -\oiint_{S}\rho(\mathbf{V}\cdot\mathbf{n})ds = -\iiint_{\Omega}\nabla\cdot(\rho\mathbf{V})d\Omega$$

$$\Rightarrow \quad \iiint_{\Omega}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})\right]d\Omega = 0$$

Integral form of continuity equation



Because Ω is arbitrary, thus:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V}) = 0$$

$$\nabla\cdot(\rho\mathbf{V}) = \mathbf{V}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{V}$$

$$\Downarrow$$

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\mathbf{V} = 0$$

Differential form of continuity equation

$$\dot{B}_{\text{net}} = \dot{B}_{\text{out}} - \dot{B}_{\text{in}} = \int_{\text{CS}}\rho b\vec{V}\cdot\vec{n}\, dA$$

3.3 Continuity Equation

Derived from conservation of mass, the continuity equation is also called equation of mass conservation. Both real and ideal fluids should satisfy continuity equation.

Several expressions:

Compressible, unsteady flow: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

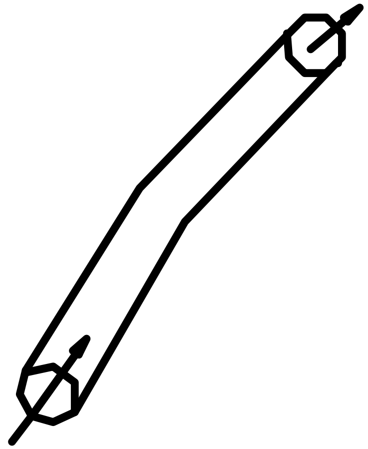
Compressible, steady flow: $$\nabla \cdot (\rho \mathbf{V}) = 0$$

Incompressible, unsteady flow: $$\nabla \cdot \mathbf{V} = 0$$

Incompressible, steady flow: $$\nabla \cdot \mathbf{V} = 0$$

3.3 Continuity Equation

Flows in a streamtube



S_1, S_2 are cross-sectional areas, V_1, V_2, ρ_1, ρ_2 are average velocities and densities of the cross sections. The continuity equation (equation of mass conservation) is:

$$\rho_1 \mathbf{V}_1 S_1 = \rho_2 \mathbf{V}_2 S_2$$

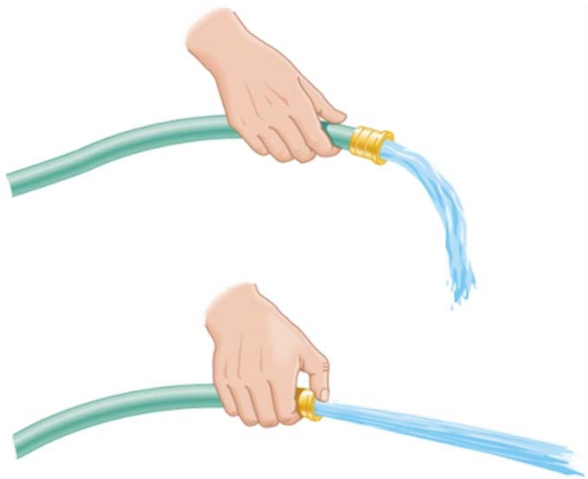
or

$$\rho \mathbf{V} S = const$$

For incompressible flow:

$$\mathbf{V}_1 S_1 = \mathbf{V}_2 S_2$$

Daily life



3.3 Continuity Equation

Application

The velocity field of a three-dimensional flow is:

$$u = x^2 y, \quad v = 4y^3 z, \quad w = 2z$$

Is this flow a real flow?

The real flow for an incompressible fluid must satisfy the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

From the given velocity field: $\frac{\partial u}{\partial x} = 2xy, \quad \frac{\partial v}{\partial y} = 12y^2 z, \quad \frac{\partial w}{\partial z} = 2$

Thus:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy + 12y^2 z + 2 \neq 0$$

So this flow is not a real flow

3.4 Stream Function

- *Velocity potential*

  —— **for irrotational flow**

- *Stream function*

  —— **for incompressible flow**

3.4 Stream Function

Definition

If functions $P(x,y)$, $Q(x,y)$ have continuous first-order partial derivatives in a closed region and its boundary, then the necessity and sufficiency for the line integral $\int P\,dx + Q\,dy$ which is path-independent is:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

According to the continuity equation, for an incompressible two-dimensional flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial y}(-v)$$

Let $P = -v$, $Q = u$, then there is a integral function:

$$\psi = \int P dx + Q dy = \int -v dx + u dy$$

which is path-independent. $\psi = (x, y, t)$ is called stream function.

3.4 Stream Function

$$\psi = \int -vdx + udy \quad \Rightarrow \quad \int d\psi = \int -vdx + udy$$

$$\Rightarrow \quad \int \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \int -vdx + udy$$

$$\Rightarrow \quad \begin{cases} \dfrac{\partial \psi}{\partial x} = -v \\ \dfrac{\partial \psi}{\partial y} = u \end{cases}$$

3.4 Stream Function

The conditions of stream function:

For ideal or real fluids, if they are **incompressible** and **two-dimensional** flow, then there are stream functions;

For **compressible two-dimensional** flows, only if the flow is **steady**, there also exists stream functions.

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0 \Longrightarrow \frac{\partial \rho u}{\partial x}=\frac{\partial}{\partial y}(-\rho v)$$

$$\Longrightarrow \psi=\int -\rho v dx+\rho u dy \Longrightarrow \begin{cases} \rho u=\dfrac{\partial \psi}{\partial y} \\ -\rho v=\dfrac{\partial \psi}{\partial x} \end{cases}$$

3.4 Stream Function

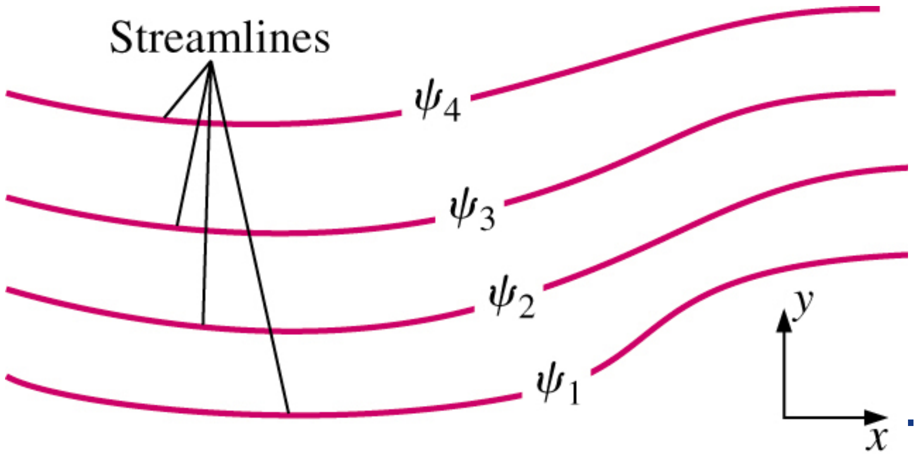
Properties of stream function:

1) The relationship between steam function and streamlines:
lines of constant stream function are streamlines of the flow, i.e., value of the stream function is constant along a streamline.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$

Thus, if Ψ = constant, $\frac{dx}{u} = \frac{dy}{v}$, which is the streamline equation

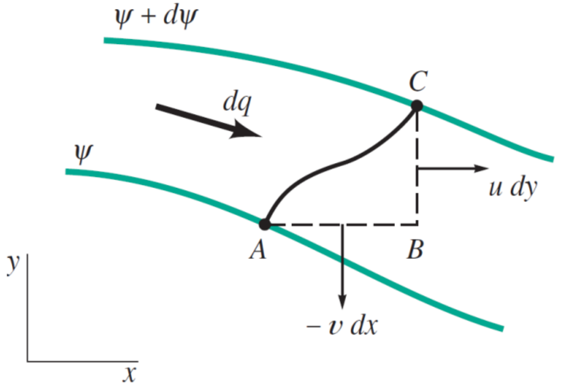
Notice! Stream function only exists in continuity equation of <u>two</u>-dimensional flows; <u>but</u> streamlines exist in all possible flows.



3.4 Stream Function

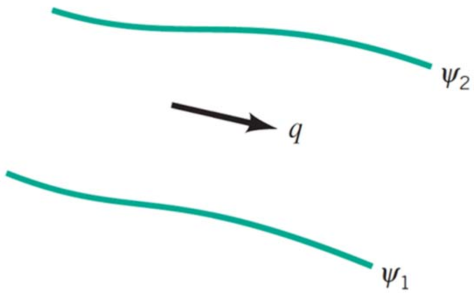
2) Relationship between stream function Ψ and volumetric flux Q:

The difference in the value of stream function from one streamline to another is equal to the volume flow rate per unit width between the two streamlines (i.e., $Q_{AC} = \psi_C - \psi_A$, Ψ is single-valued function).

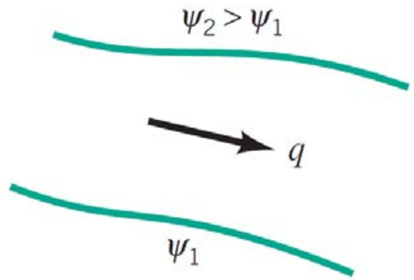


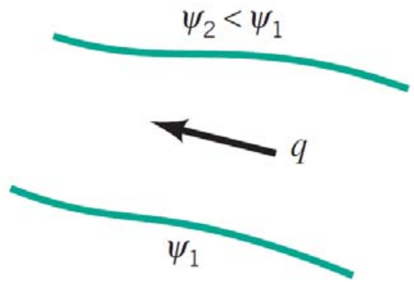
$$dq = udy - vdx$$

$$= \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$



$$\therefore \quad q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$





3.4 Stream Function

3) For incompressible, irrotational, two-dimensional potential flows, the velocity potential and the stream function are all harmonic functions, both satisfy Laplace's equation.

A. Incompressible two-(three-)dimensional potential flow (irrotational)

$$\left.\begin{array}{l}\textbf{irrotational} \Rightarrow \nabla \times \mathbf{V} = 0 \Rightarrow \mathbf{V} = \nabla \phi \\ \textbf{incompressible} \Rightarrow \nabla \cdot \mathbf{V} = 0\end{array}\right\} \Rightarrow \nabla^2 \phi = 0$$

B. Incompressible two-dimensional potential flow (irrotational)

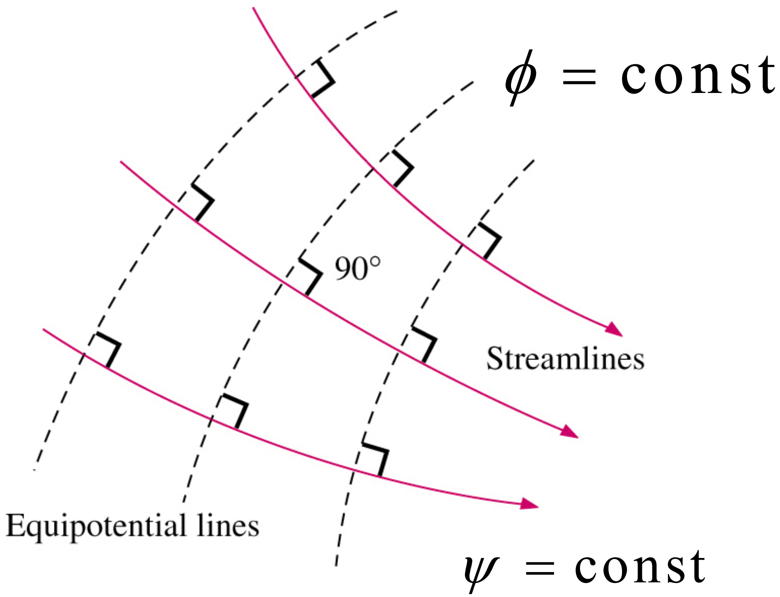
$$\left.\begin{array}{l}\textbf{2D incompressible} \Rightarrow u = \dfrac{\partial \psi}{\partial y},\ v = -\dfrac{\partial \psi}{\partial x} \\ \textbf{2D irrotational} \Rightarrow \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0\end{array}\right\} \Rightarrow \nabla^2 \psi = 0$$

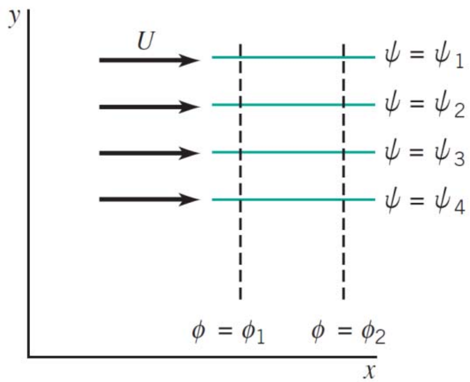
3.4 Stream Function

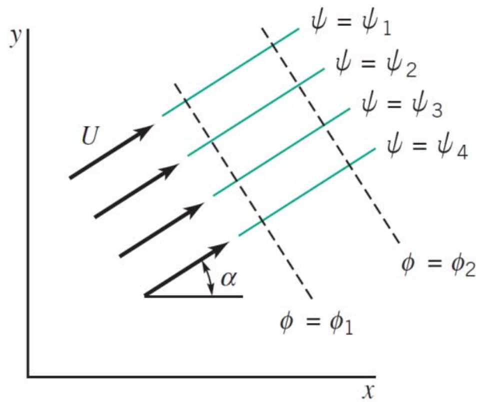
4) Flow net: consists of a family of streamlines and equipotential lines (lines of constant velocity potential) for any incompressible 2D potential flows.

$$\nabla\phi \equiv \boldsymbol{V} \quad \perp \quad \text{equipotential lines}$$

$$\Rightarrow \quad \text{streamlines} \quad \perp \quad \text{equipotential lines}$$







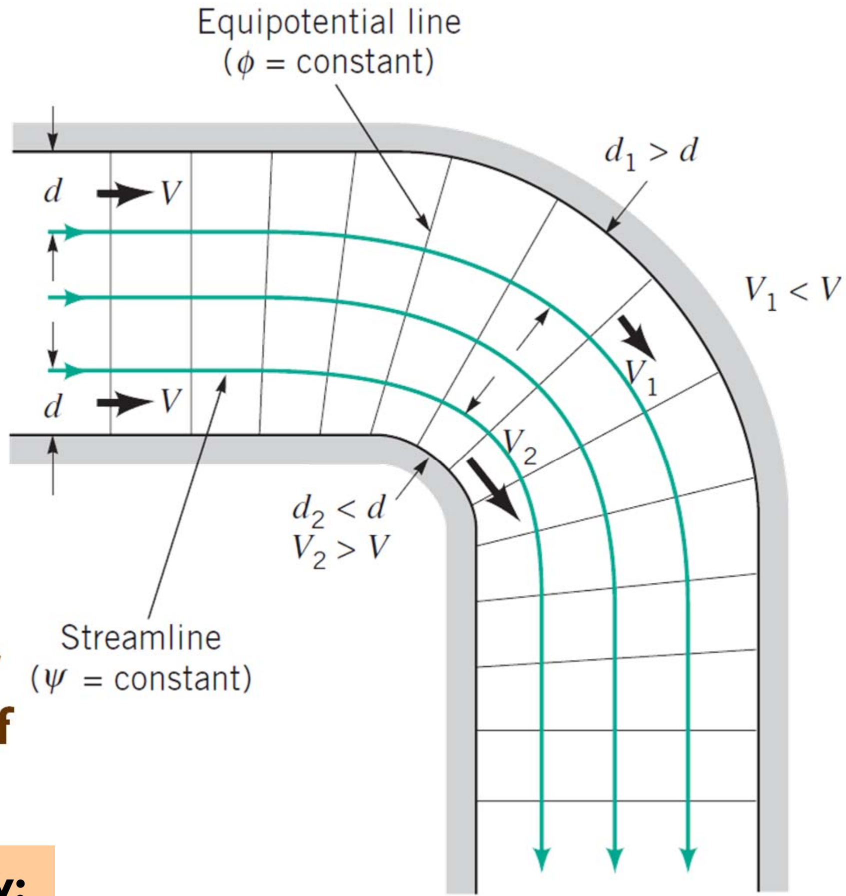
3.4 Stream Function

For a 90° bend (the velocity is inversely proportional to the streamline spacing):

$$V \approx \frac{\Delta\phi}{\Delta n} \approx \frac{\Delta\psi}{\Delta s}$$

Δn : distance between two adjacent equipotential lines

Δs : distance between two adjacent streamlines



Velocity near the inside corner is higher than the velocity along the outer part of the bend.

Flow net lines close together ⇒ high velocity;

Flow net lines far apart ⇒ low velocity.