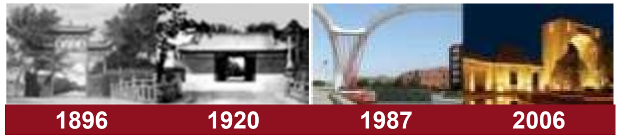


# Introduction to Marine Hydrodynamics

## (NA235)

Department of Naval Architecture and Ocean Engineering

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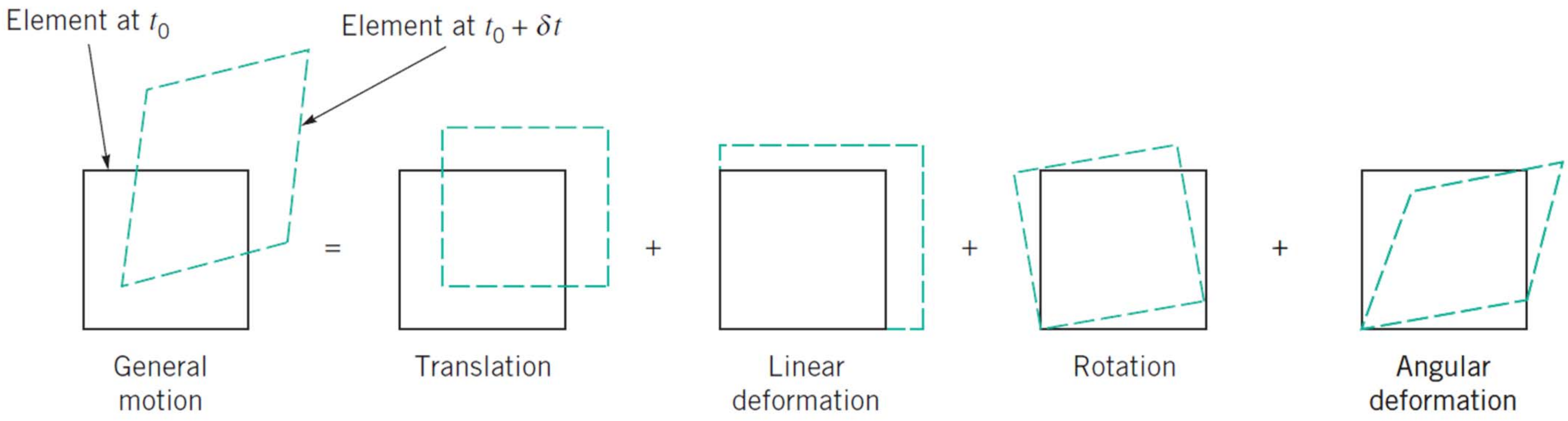
# Review

- **Pathline equation:** $$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}=dt$$

- **Streamline equation:** $$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

# Review

- **Motion of fluid elements**

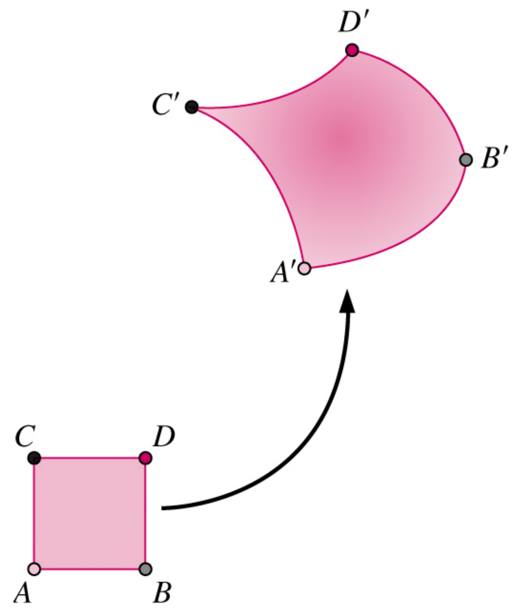


# Review

$$\mathbf{V}_{M'} = [\underbrace{\underline{u}}_{(1)} + \underbrace{\omega_z dy + \omega_y dz}_{(2)} + \underbrace{\varepsilon_{xx} dx}_{(3)} + \underbrace{\varepsilon_{xy} dy + \varepsilon_{xz} dz}_{(4)}]\mathbf{i}$$

$$+(\cdots)\mathbf{j} + (\cdots)\mathbf{k}$$

General motion = 
- Translation (1) + Rotation (2) (rigid body motion)
- + Dilatation (3) (change in volume)
- + Angular deformation (4) (change in shape)



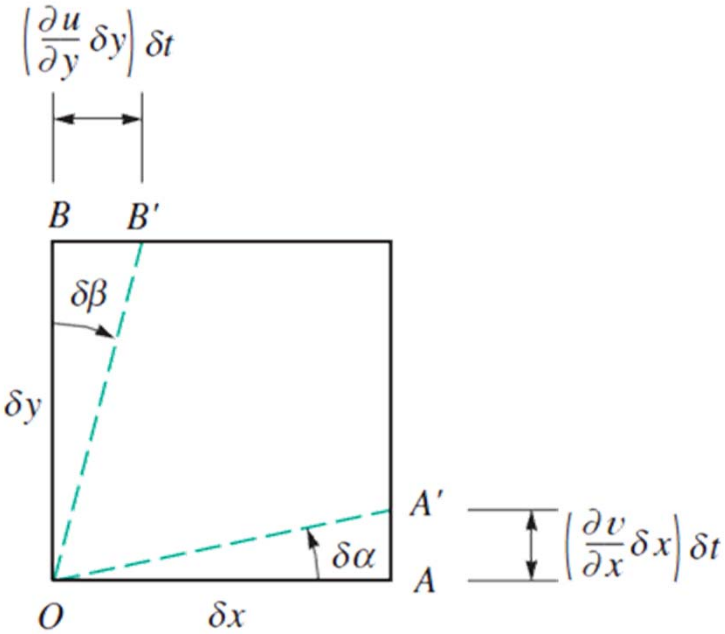
# Review

## Shear strain rate – angular deformation

$$\varepsilon_{xy}=\frac{1}{2}(\frac{d\alpha}{dt}+\frac{d\beta}{dt})=\frac{1}{2}(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})$$

**$\varepsilon_{xy}$ >0, AOB decreases**

$$\varepsilon_{yz}=\frac{1}{2}(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z});\quad \varepsilon_{zx}=\frac{1}{2}(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x})$$




## Rate of rotation

$$\omega_z=\frac{1}{2}\left(\frac{d\alpha}{dt}-\frac{d\beta}{dt}\right)=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

**$\omega_z$ >0, counterclockwise rotation**

$$\omega_x=\frac{1}{2}(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z});\quad \omega_y=\frac{1}{2}(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x})$$

*Rotation with/w.o. deformation?*

# 2.3 Deformation and Rotation of Fluid Elements

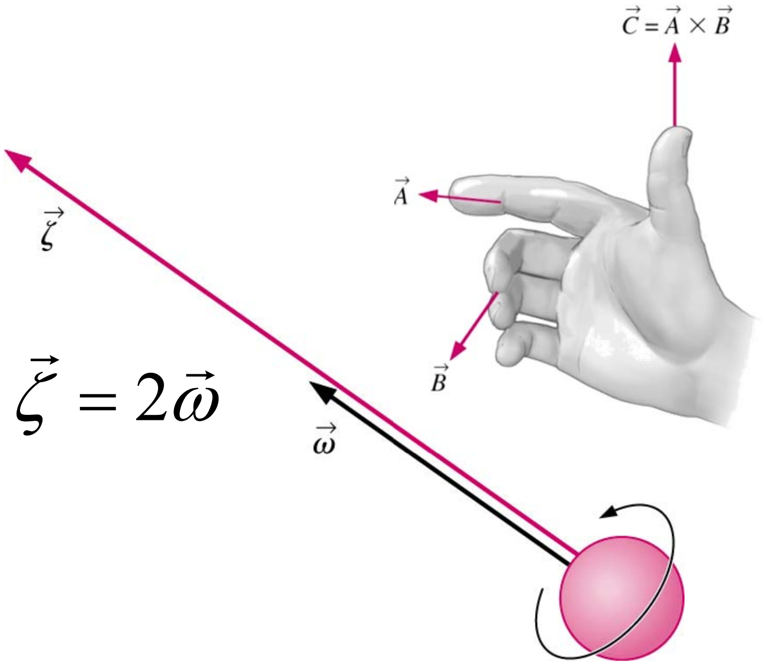
shear rate tensor $\mathbf{E} = \varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$

rate of rotation $\boldsymbol{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} = \dfrac{1}{2}\zeta$ (vorticity)

vorticity $\zeta = \mathbf{\Omega} = \nabla \times \mathbf{V}$ (curl of velocity)

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{2\omega_x}\vec{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{2\omega_y}\vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{2\omega_z}\vec{k}$$




$\vec{\zeta} = 2\vec{\omega}$

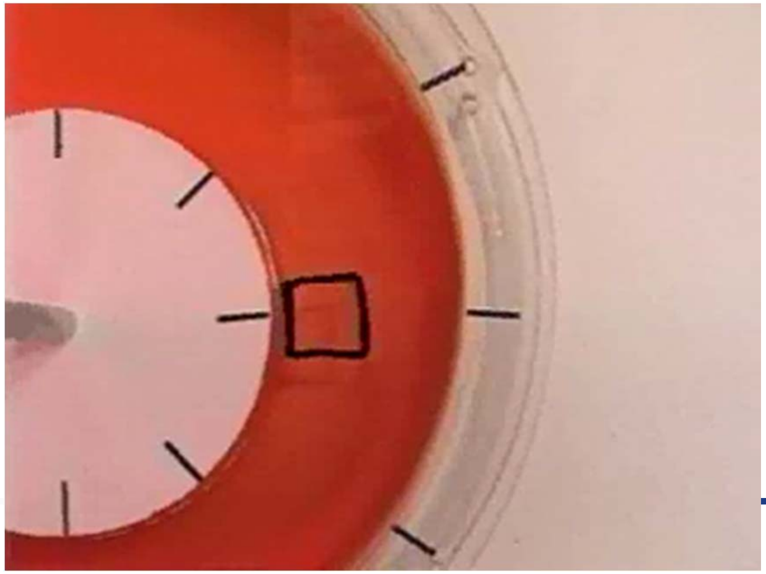
# 2.3 Deformation and Rotation of Fluid Elements

Helmholtz decomposition can be written in tensor form:

Let $\delta\mathbf{r} = \mathrm{d}x\,\mathbf{i} + \mathrm{d}y\,\mathbf{j} + \mathrm{d}z\,\mathbf{k}$

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{E}\cdot\delta\mathbf{r} + \boldsymbol{\omega}\times\delta\mathbf{r}$$

**Rate of translation** ($\mathbf{V}_0$) **Strain rate** ($\mathbf{E}\cdot\delta\mathbf{r}$) **Rate of rotation/angular velocity** ($\boldsymbol{\omega}\times\delta\mathbf{r}$)

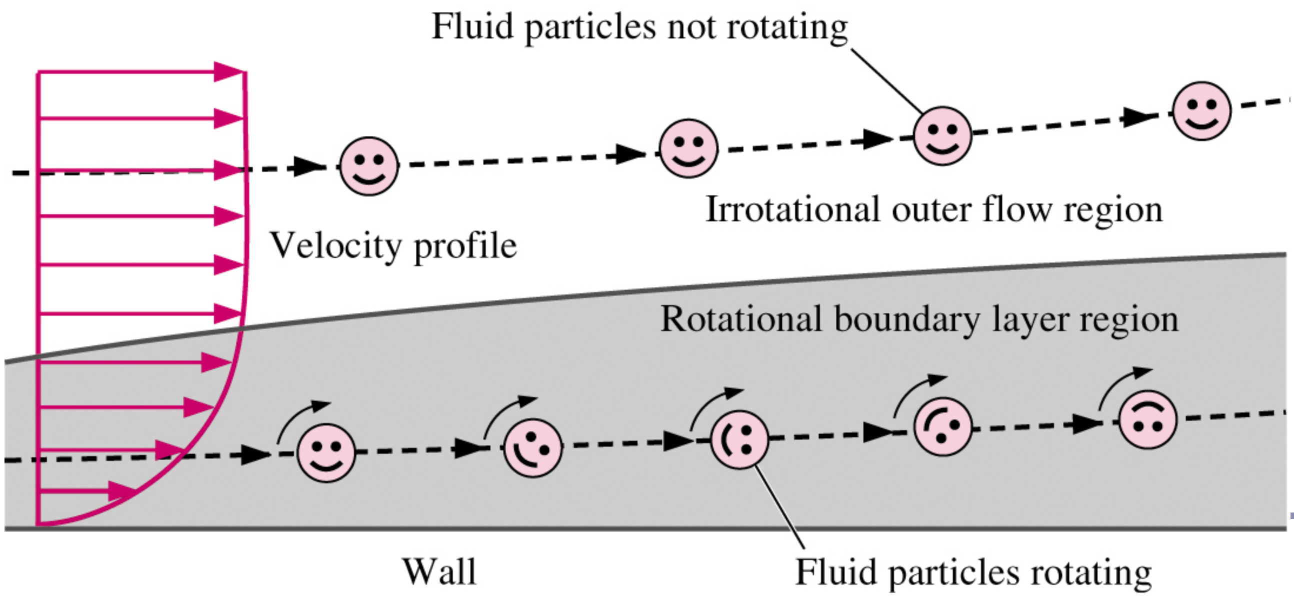


# 2.4 Rotational and Irrotational Flows

**Definition:** If the vorticity in a flow field is zero, $\vec{\Omega}=0$ , the fluid particles are not rotating, the flow in that region is called **irrotational.** Otherwise, the flow in that region is called **rotational.**

$$\Omega=\left(\Omega_x,\ \Omega_y,\ \Omega_z\right)=0$$

$$\Omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0,\quad \Omega_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0,\quad \Omega_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0$$



## Application 1

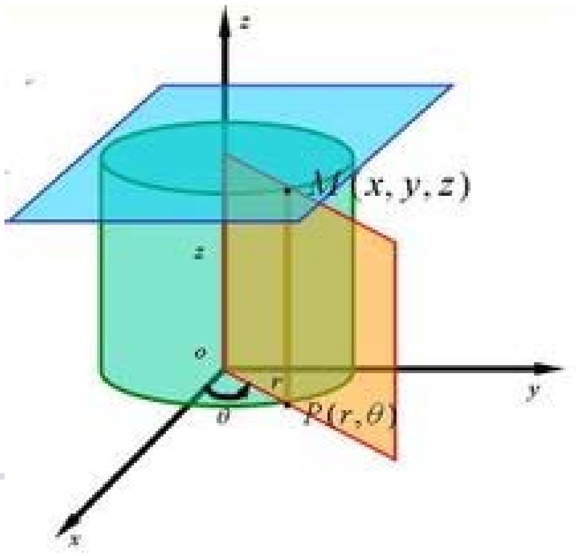
**The irrotational flow condition in the *xy*-plane in Cartesian coordinates is:** $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$

**Derive the expression of irrotational flow condition in plane polar coordinates.**

**Solution:** the relationship between Cartesian coordinates $(x, y)$ and polar coordinates $(r,\theta)$ is:

$$x = r\cos\theta, \quad y = r\sin\theta$$

where $r=\sqrt{x^2+y^2}$ $\quad \theta=\arctan(\frac{y}{x})$




The relationships between the velocities in the two coordinates system are:

$$u = v_r\cos\theta - v_\theta\sin\theta$$

$$v = v_r\sin\theta + v_\theta\cos\theta$$

# 2.4 Rotational and Irrotational Flows

The relationships between the differential operators are:

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\frac{\partial y}{\partial r}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta}=\frac{\partial}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}$$

Thus:

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\sin\theta\frac{\partial}{r\partial \theta}$$

$$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{\partial}{r\partial \theta}$$

# 2.4 Rotational and Irrotational Flows

The **irrotational flow condition** in Cartesian coordinates is:

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

Transformed to polar coordinates:

$$\frac{\partial v}{\partial x}=(\cos\theta\frac{\partial}{\partial r}-\sin\theta\frac{\partial}{r\partial\theta})(v_r\sin\theta+v_\theta\cos\theta)$$

$$=\sin\theta\cos\theta\frac{\partial v_r}{\partial r}+v_r\cos\theta\cdot 0-\sin^2\theta\frac{\partial v_r}{r\partial\theta}-\frac{v_r}{r}\sin\theta\cos\theta$$

$$+\cos^2\theta\frac{\partial v_\theta}{\partial r}+v_\theta\cos\theta\cdot 0-\sin\theta\cos\theta\frac{\partial v_\theta}{r\partial\theta}+\frac{v_\theta}{r}\sin^2\theta$$

$$\frac{\partial u}{\partial y}=(\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{\partial}{r\partial\theta})(v_r\cos\theta-v_\theta\sin\theta)$$

$$=\sin\theta\cos\theta\frac{\partial v_r}{\partial r}+v_r\sin\theta\cdot 0+\cos^2\theta\frac{\partial v_r}{r\partial\theta}-\frac{v_r}{r}\sin\theta\cos\theta$$

$$-\sin^2\theta\frac{\partial v_\theta}{\partial r}-v_\theta\sin\theta\cdot 0-\sin\theta\cos\theta\frac{\partial v_\theta}{r\partial\theta}-\frac{v_\theta}{r}\cos^2\theta$$

# 2.4 Rotational and Irrotational Flows

⇒

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=(\cos^2\theta-\sin^2\theta)\frac{\partial v_\theta}{\partial r}+(\sin^2\theta-\cos^2\theta)\frac{v_\theta}{r}-(\sin^2\theta+\cos^2\theta)\frac{\partial v_r}{r\partial\theta}=0$$

i.e.,

$$\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{\partial v_r}{r\partial\theta}=0$$

Or,

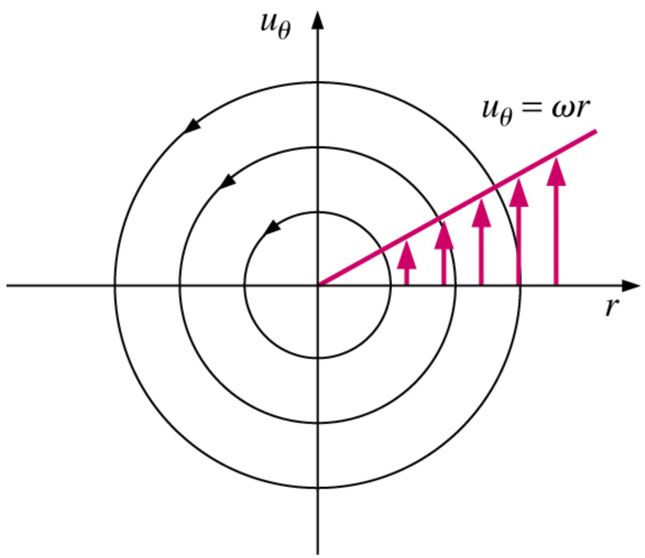
$$\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_r}{\partial\theta}=0$$

## Application 2

For a two-dimensional flow field, the velocity is given as:

$$u = -\omega y, \;\; v = \omega x$$

Is this flow rotational or irrotational? Will the fluid element deform or not?




**Solution:**

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega, \quad \Omega_x = 0, \quad \Omega_y = 0$$

<u>The flow is rotational</u>

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0$$

$$\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

<u>There is no deformation of the fluid element.</u> This is used to describe the flow <u>*in*</u> the core region of a tornado.

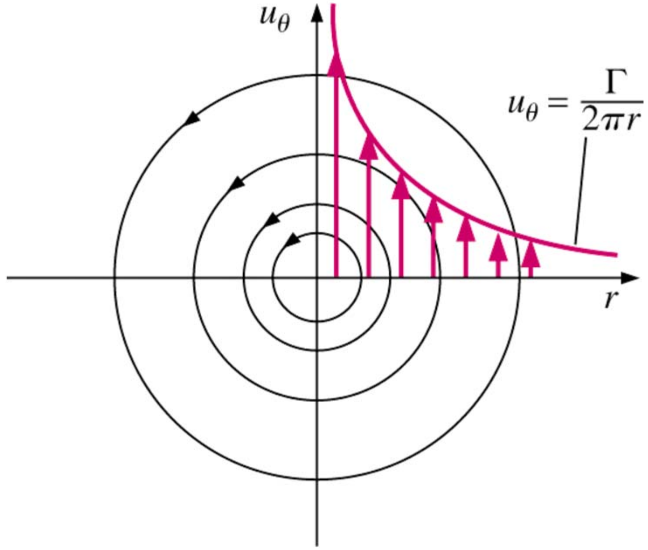
# 2.4 Rotational and Irrotational Flows

## Application 3

A two-dimensional flow field the velocity is given as: ,

$$u = -\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}, \quad v = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$$

Is this flow rotational or irrotational? Will the fluid element deform or not?




**Solution:**

$$\omega_x = \omega_y = 0, \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

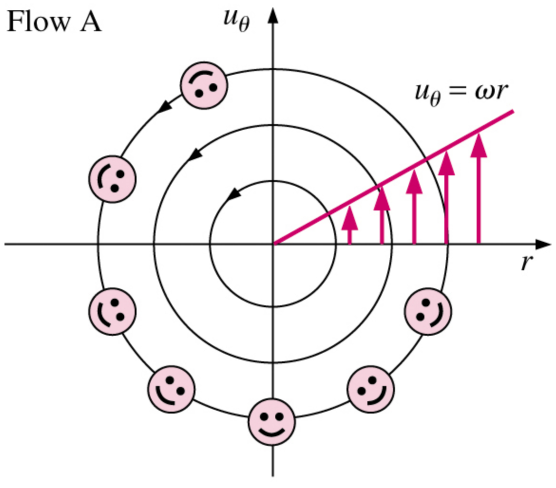
$$\varepsilon_{xx} = \frac{\Gamma}{2\pi}\frac{xy}{(x^2+y^2)} = -\varepsilon_{yy} \neq 0, \quad \varepsilon_{zz} \neq 0$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \varepsilon_{zx} = 0$$

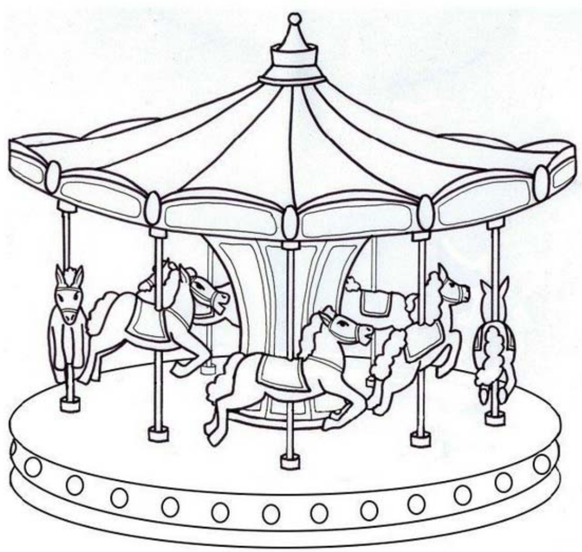
The flow is irrotational but deformed, can be used to describe the flow <u>*out*</u> of the core region of a tornado.

# 2.4 Rotational and Irrotational Flows

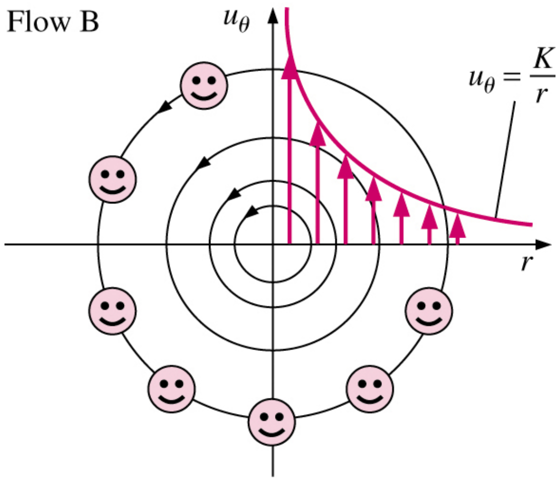
## Not all flows with circular streamlines are rotational




(a)



n: rotational, no
o describe the flow
a tornado.

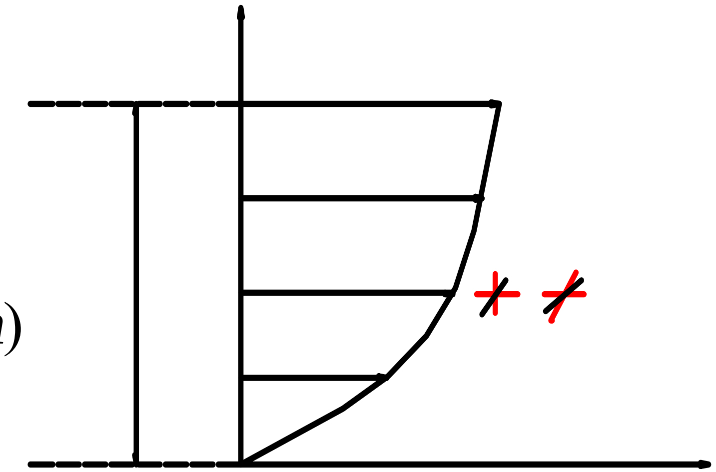



(b)

tational, deformation;
the flow out of the
rnado.

## Application 4

Consider a linear flow: $u = \dfrac{v_{max}}{h}(2y - y^2/h), \quad v = 0$

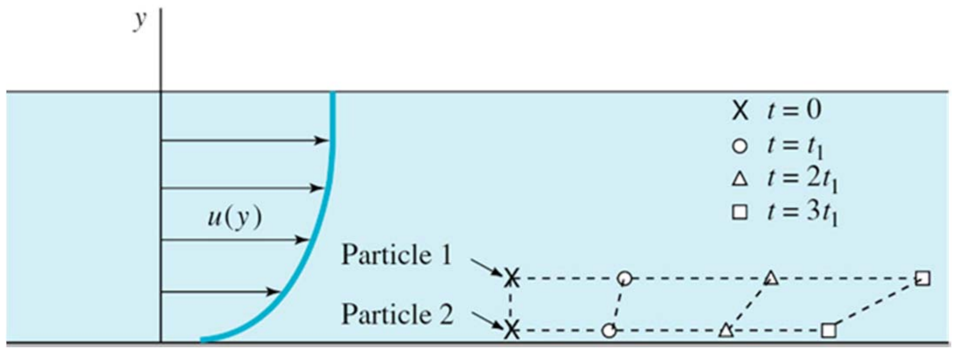


**Solution:**

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0$$

$$\omega_x = \omega_y = 0, \quad \omega_z = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = \frac{v_{max}}{h}(1 - y/h)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = \frac{v_{max}}{h}\left(1 - \frac{y}{h}\right) \neq 0$$




This is a rotational flow with deformation.

# Application 5

Consider a uniform linear motion of a fluid element:

$$u = v_0 = const, \quad v = 0$$

Solution:

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0$$

$$\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

$$\omega_x = \omega_y = \omega_z = 0$$

This is a irrotational flow without any deformation.

# 2.5 Velocity Potential

- *Velocity potential*

  —— **for irrotational flow**

- *Stream function*

  —— **for incompressible flow**

# 2.5 Velocity Potential

According to Green's theorem: let *P, Q, R* be functions of $(x,y,z)$ and have continuous first-order partial derivatives

$$\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial z}, \frac{\partial Q}{\partial x}, \frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \quad \cdots$$

in a simple closed curve and its bounded region. If the following equations satisfied in the region:

$$\left.\begin{aligned} \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y} \\ \frac{\partial R}{\partial x} &= \frac{\partial P}{\partial z} \\ \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \end{aligned}\right\}$$

⇒ Then there must be a function defined by the line integral below. This function is potential, and is called **potential function**:

***F(x,y,z)***

⇒ $$F(x,y,z) = \int Pdx + Qdy + Rdz$$

# 2.5 Velocity Potential

The integral is not path-dependent, and has following relationships:

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= P \\ \frac{\partial F}{\partial y} &= Q \\ \frac{\partial F}{\partial z} &= R \end{aligned}\right\}$$

$$F(x, y, z) = \int \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

# 2.5 Velocity Potential

For an *irrotational flow*, $\omega$=0, thus:

$$\left.\begin{aligned} \frac{\partial v}{\partial z} &= \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} &= \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} \end{aligned}\right\}$$

comparing

$$\left.\begin{aligned} \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y} \\ \frac{\partial R}{\partial x} &= \frac{\partial P}{\partial z} \\ \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \end{aligned}\right\}$$

$$u \rightarrow P$$

$$v \rightarrow Q$$

$$w \rightarrow R$$

# 2.5 Velocity Potential

Thus, there is a function:

$$\phi(x, y, z; t) = \int u dx + v dy + w dz$$

With the relationships:

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= u \\ \frac{\partial \phi}{\partial y} &= v \\ \frac{\partial \phi}{\partial z} &= w \end{aligned}\right\} \quad \text{i.e,} \quad \mathbf{V} = \nabla \phi$$

$\phi$ is called the **velocity potential**. A flow has the velocity potential is called **potential flow**.

Irrotational $\Leftrightarrow$ $\nabla \times V = 0$ $\Leftrightarrow$ $\phi$ $\Leftrightarrow$ Potential flow

# 2.5 Velocity Potential

## Significance of velocity potential:

**Velocity potential $\phi$ is a scalar quantity with one component only:**

$$\phi(x, y, .z; t) = \int u dx + v dy + w dz$$

**Velocity V is a vector quantity with three components:**

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= u \\ \frac{\partial \phi}{\partial y} &= v \\ \frac{\partial \phi}{\partial z} &= w \end{aligned}\right\}$$

Irrotational velocity field $\Leftrightarrow$ Potential flow

# 2.5 Velocity Potential

## Application

The velocity field of a rotational flow is given by:

$$u = 2(x-a)y,\ v = (x+a)^2 - y^2$$

where *a* is a constant. Consider an irrotational flow, its linear strain rate and shear strain rate are identical to those in rotational flow. The velocity at the origin is 0, i.e., when $x = y = 0$, $u = v = 0$. Determine the velocity expression and the velocity potential of this irrotational flow.

**Solution**: for the given rotational flow

linear strain rate: $\varepsilon_{xx} = \dfrac{\partial u}{\partial x} = 2y,\quad \varepsilon_{yy} = \dfrac{\partial v}{\partial y} = -2y$ (a)

shear strain rate: $\varepsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) = 2x \Rightarrow \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 4x$ (b)

The linear strain rate and shear strain rate of the irrotational flow are the same, and the irrotational flow has to satisfy:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{(c)}$$

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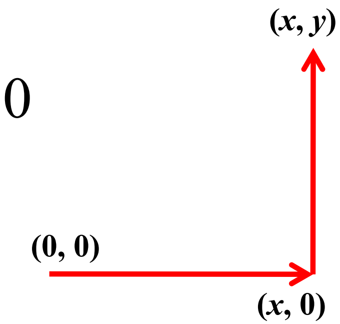
# 2.5 Velocity Potential

From (c)+(b) : $\dfrac{\partial v}{\partial x} = 2x$ i.e., $v = x^2 + f(y)$

Substituting it into (a): $f'(y) = -2y$, i.e., $f(y) = -y^2 + C_1$

From (c)-(b): $\dfrac{\partial u}{\partial y} = 2x$ i.e., $u = 2xy + g(x)$

Substituting it into (a): $g'(x) = 0$, i.e., $g(x) = C_2$




$C_1$ and $C_2$ are integral constants. When $x = y = 0$ , $u = v = 0$

$$\Rightarrow \quad C_1 = C_2 = 0$$

Thus, the velocity is: $u = 2xy, \quad v = x^2 - y^2$

The velocity potential is: $\phi = \int u\,dx + v\,dy = \int 2xy\,dx + (x^2 - y^2)\,dy$

$$= \int_{(0,0)}^{(x,0)} + \int_{(x,0)}^{(x,y)} = 0 + \int_{(x,0)}^{(x,y)} \left(x^2 - y^2\right) dy = x^2 y - \frac{1}{3} y^3$$

# Review: Key Points in Chapter 2

- **Two ways of describing a fluid flow:**

  **Lagrangian description, Eulerian description**

- **Material derivative:** $\dfrac{D(\ )}{Dt} = \dfrac{\partial(\ )}{\partial t} + (\mathbf{V} \bullet \nabla)(\ )$

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{particle}, y_{particle}, z_{particle}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{particle}}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y_{particle}}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z_{particle}}\frac{dz_{particle}}{dt}$$

$$\Rightarrow \underbrace{\vec{a}_{particle}(x, y, z, t)}_{\text{Lagrangian}} = \frac{d\vec{V}}{dt} = \underbrace{\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{Eulerian}}$$

# Review: Key Points in Chapter 2

- **Pathline equation:** $$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}=dt$$

- **Streamline equation:** $$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

**Pathlines and streamlines are identical in steady flows**

# Review: Key Points in Chapter 2

- **Motion or deformation of fluid elements**

$$\omega = \frac{1}{2}\nabla \times \mathbf{V} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

- **Helmholtz decomposition**

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{E}\cdot\delta\mathbf{r} + \boldsymbol{\omega}\times\delta\mathbf{r} \qquad \mathbf{E} = \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- **Rotational and irrotational flows**

$$\Omega = \left(\Omega_x,\ \Omega_y,\ \Omega_z\right) = 0$$

- **Velocity potential: irrotational flows**

$$\mathbf{V} = \nabla\phi, \qquad \phi(x, y, z; t) = \int u\,dx + v\,dy + w\,dz$$

# Chapter 3
# Fluid Dynamics

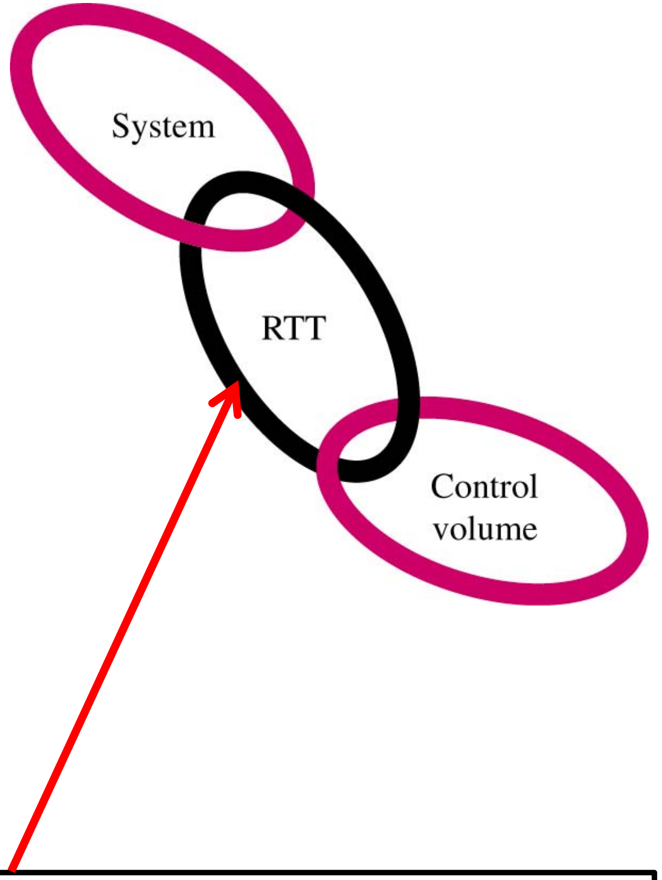
***Fluid dynamics***: study of the motion of fluids considering the forces and moments that cause the motion.

# 3.1 Representation of Fluid Flow

Two approaches in fluid dynamics: **system approach** and **control volume approach**

(1) **System** (material volume) approach: follows the fluid as it moves and deforms; no mass crosses the boundary (Lagrangian description)

(2) **Control volume** approach: considers the changes in a certain fixed volume; mass can cross the boundary (Eulerian description)



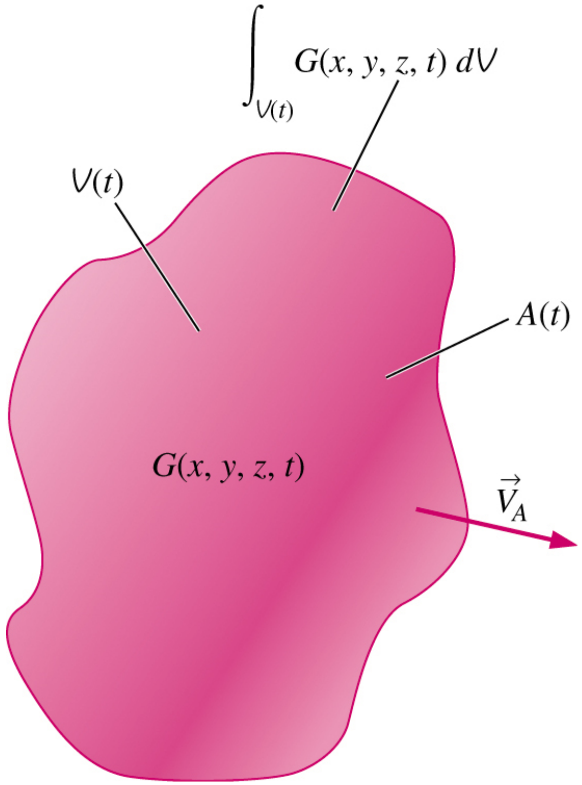

**Reynolds Transport Theorem (RTT)**

# 3.1 Representation of Fluid Flow

System: a collection of matter of fixed identity (always the same fluid particles), which may move, flow, and interact with its surroundings.

Material volume: a volume that contains the same fluid as it moves and deforms following the motion of the fluid.

Material surface: an enclosing surface of a material volume; no fluid particles can cross it.

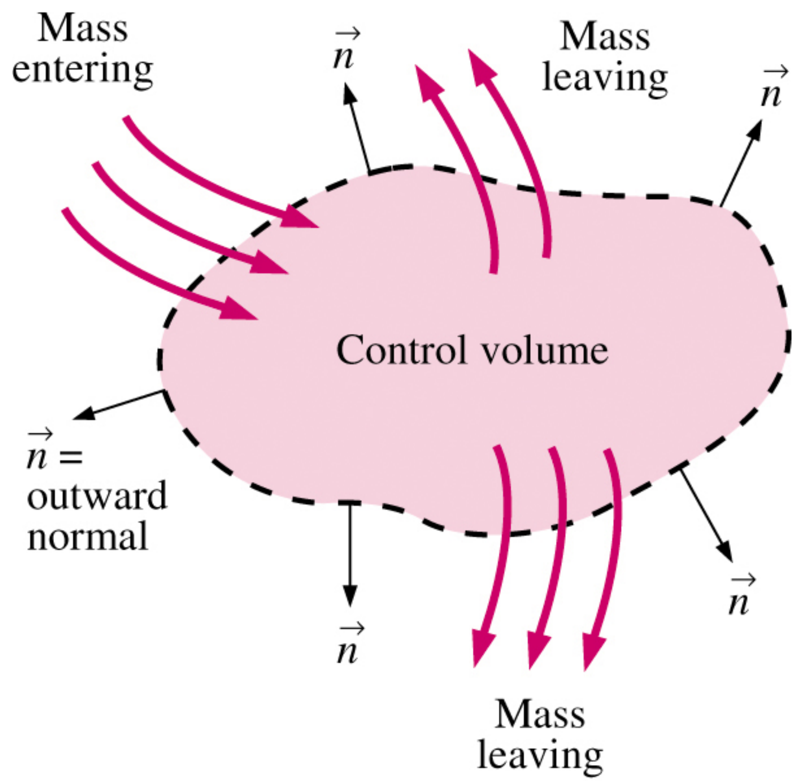


# 3.2 Reynolds Transport Theorem

**Control volume: a volume in space (a geometric entity, independent of mass) through which fluid may flow.**

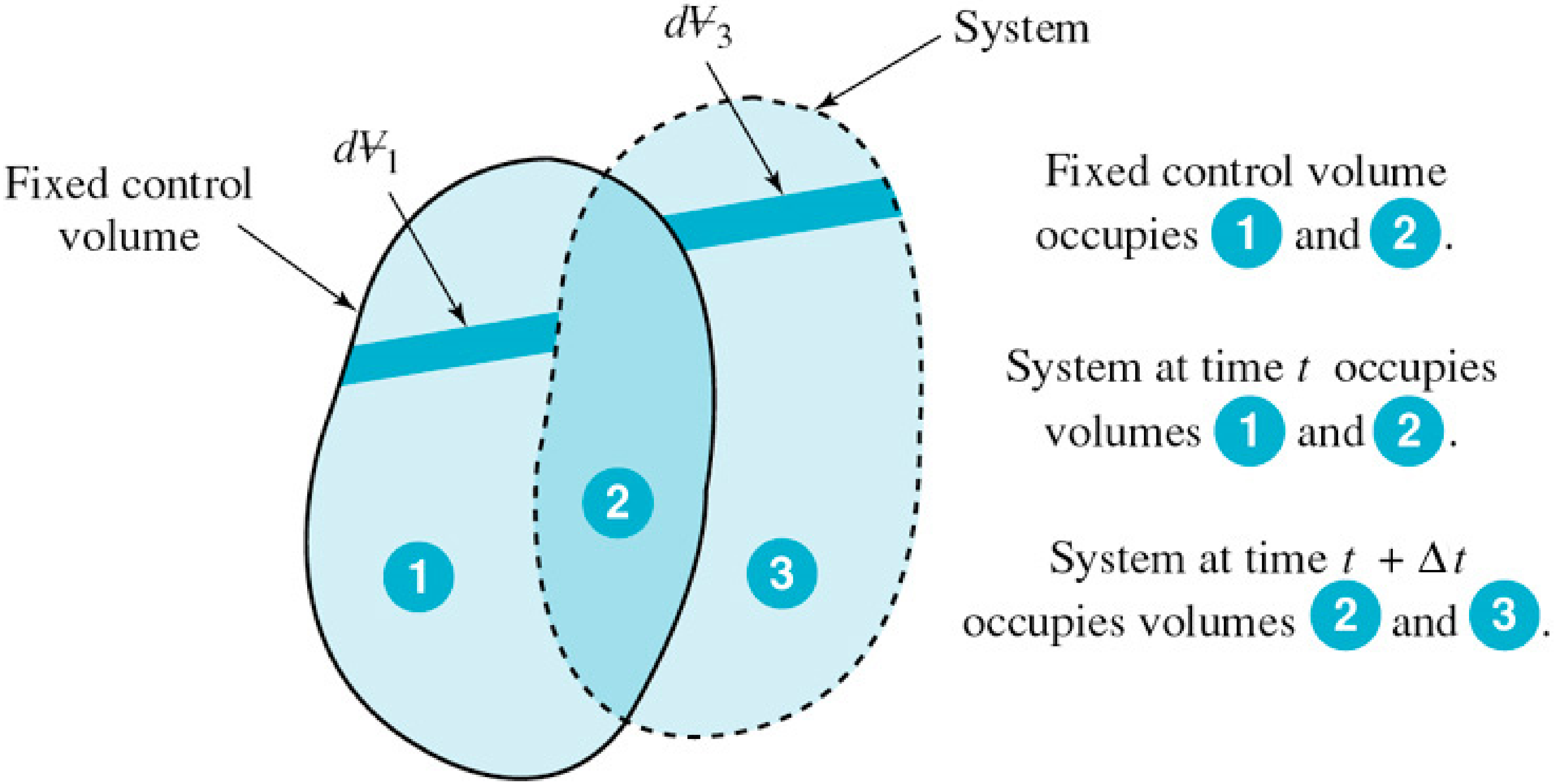
**Control volume**: a volume of fluid in a flow field, usually fixed in space, to be occupied by different fluid particles at different times.

**Control surface**: imaginary or physical enclosing surface of a control volume, fluid particles can cross it.

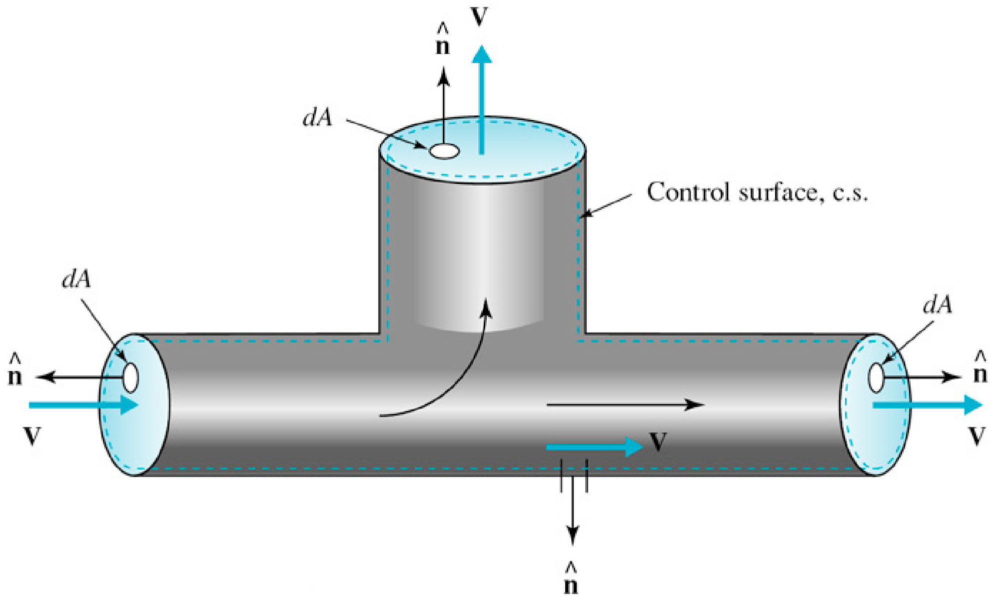


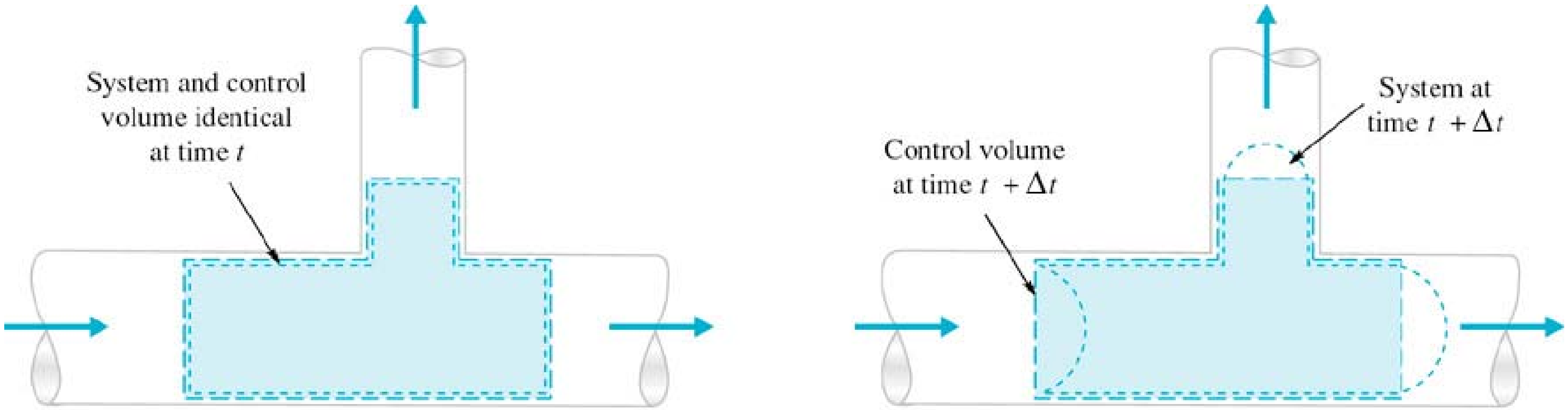

$$\dot{B}_{\text{net}} = \dot{B}_{\text{out}} - \dot{B}_{\text{in}} = \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA$$

# 3.2 Reynolds Transport Theorem



# 3.2 Reynolds Transport Theorem



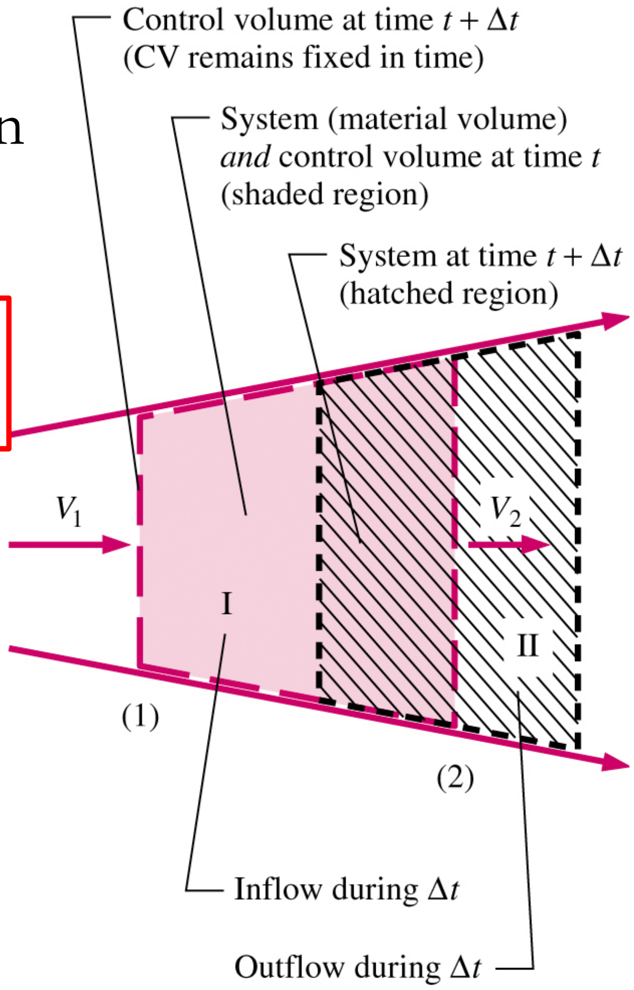

(a) (b)

# 3.2 Reynolds Transport Theorem

## Reynolds Transport Theorem

Any quantity ***G*** satisfies the relationship between **material volume** and **control volume**:

$$\underbrace{\frac{d}{dt}\iiint_{MV} G\,d\forall}_{\text{rate of change of the property G within the material volume}} = \underbrace{\frac{\partial}{\partial t}\iiint_{CV} G\,d\forall}_{\text{local rate of change of the property within the fixed control volume that happens to coincide with the material volume at that instant}} + \underbrace{\iint_{CS} G\mathbf{V}\cdot\mathbf{n}\,dA}_{\text{net out-flux of the property across the entire control surface}}$$




At time $t$: Sys = CV
At time $t+\Delta t$: Sys = CV − I + II

$\rho$ = density of fluid

$G$ = an intensive property of fluid

$MV$ = material volume that happens to coincide with $CV$ at time $t$

$CV$ = control volume (fixed in space)

$CS$ = control surface

$\mathbf{n}$ = unit outward normal to $CS$

# 3.2 Reynolds Transport Theorem

**For a material volume, after a time period *dt*:**

$$\left[\iiint_{MV} G(\mathbf{x},t)\,dV\right]_{t+dt} = \iiint_{MV(t+dt)} G(\mathbf{x},t+dt)\,dV = \iiint_{MV(t+dt)} \left[G(\mathbf{x},t)+\frac{\partial G}{\partial t}dt+O(dt)^2\right]dV$$

**The change of material volume in *dt* is:**

$$\iiint_{MV(t+dt)}(\ ) = \iiint_{MV}(\ ) + \iiint_{\Delta V}(\ ) = \iiint_{CV}(\ ) + \iint_{CS}(\ )\,\mathbf{V}\cdot\mathbf{n}\,dS\,dt$$

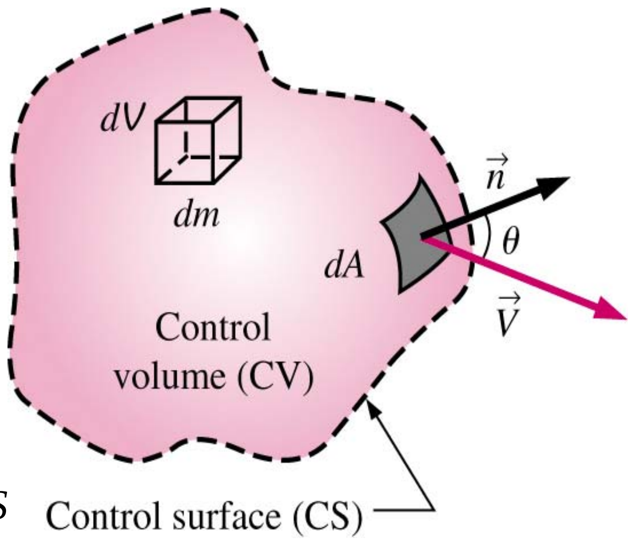
omit 2nd order quantities

**Thus:**

$$\left[\iiint_{MV} G(\mathbf{x},t)\,dV\right]_{t+dt} = \iiint_{MV(t+dt)} \left[G(\mathbf{x},t)+\frac{\partial G}{\partial t}dt\right]dV$$

$$= \iiint_{CV}\left[G(\mathbf{x},t)+\frac{\partial G}{\partial t}dt\right]dV + \iint_{CS}\left[G(\mathbf{x},t)+\frac{\partial G}{\partial t}dt\right]\mathbf{V}\cdot\mathbf{n}\,dS\,dt$$

$$= \iiint_{CV} G(\mathbf{x},t)\,dV + \left[\iiint_{CV}\frac{\partial G}{\partial t}dV + \iint_{CS} G(\mathbf{x},t)\,\mathbf{V}\cdot\mathbf{n}\,dS\right]dt$$




**RTT can be derived from the equation above:**

$$\frac{d}{dt}\iiint_{MV} G\,dV = \left\{\left[\iiint_{MV} G\,dV\right]_{t+dt} - \iiint_{MV} G\,dV\right\}\Big/dt = \iiint_{CV}\frac{\partial G}{\partial t}dV + \iint_{CS} G\,\mathbf{V}\cdot\mathbf{n}\,dS$$

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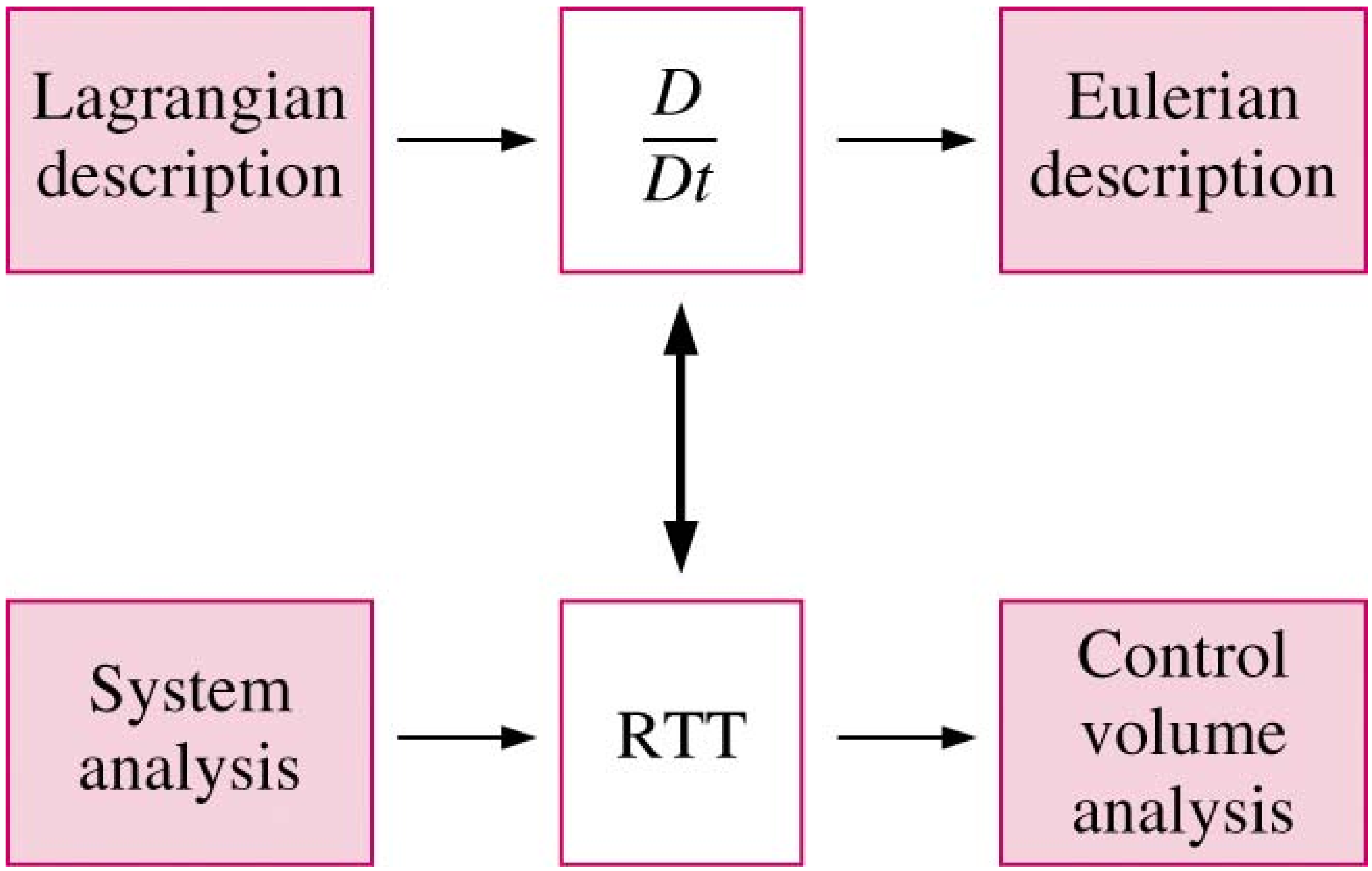

# 3.3 Continuity Equation

**Continuity Equation: conservation of mass (mass of a system is neither be created nor destroyed)**

**In RTT equation, if the quantity is mass, i.e., $G = \rho$ , then:**

L.H.S. $$\frac{d}{dt}\iiint_{MV} \rho d\forall = \frac{d}{dt}\left(\text{mass in } MV\right) = 0$$

(From the definition of MV: it always contains the same fluids)

R.H.S. $$\frac{\partial}{\partial t}\iiint_{CV} \rho d\forall + \iint_{CS} \rho \boldsymbol{V} \cdot \boldsymbol{n} dA$$

$$= \underbrace{\iiint_{CV} \frac{\partial \rho}{\partial t} d\forall}_{CV \text{ is stationary}} + \underbrace{\iiint_{CV} \nabla \cdot (\rho \boldsymbol{V}) d\forall}_{\text{by Gauss theorem}}$$