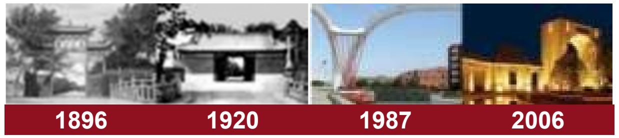


Introduction to Marine Hydrodynamics

(NA235)

Department of Naval Architecture and Ocean Engineering

School of Naval Architecture, Ocean & Civil Engineering

Shanghai Jiao Tong University

Second Assignment

- The assignment can be downloaded from following website:

  Website： ftp://public.sjtu.edu.cn

  Username： dcwan

  Password： 2015mhydro

  Directory： IntroMHydro2015-Assignments

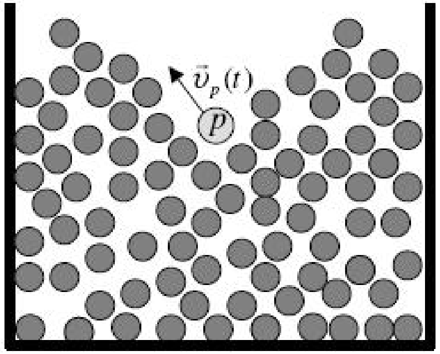
- Eight problems

- Submit the assignment on March 19th (in English, written on paper)

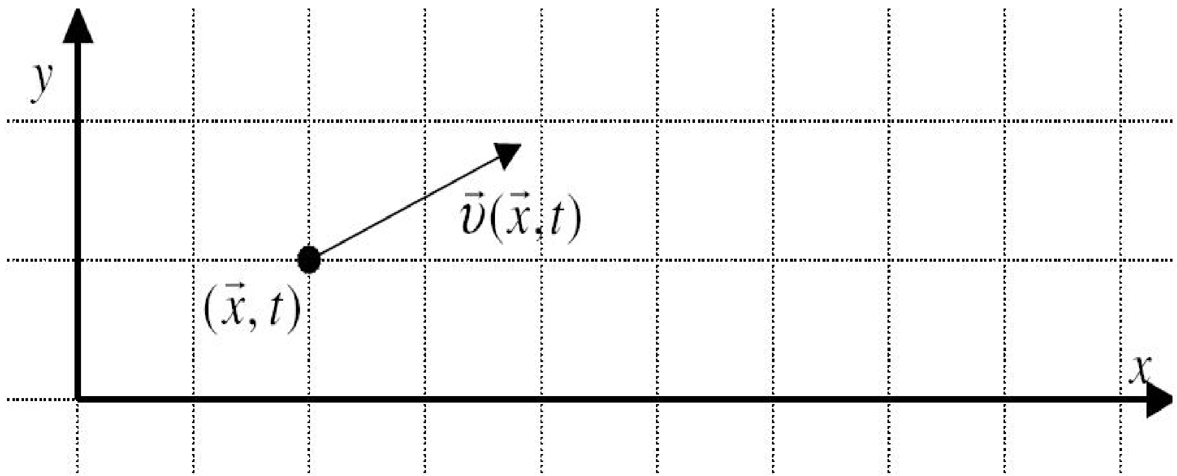
Review

- **Two ways of describing a fluid flow:**

Lagrangian description, Eulerian description



Lagrangian description; snapshot



Eulerian description; Cartesian grid

Differences:

Particle positions *x*, *y*, *z* in Lagrangian description;

while spatial point (*x*, *y*, *z*) in Eulerian description.

Spatial point (x, y, z) in Eulerian description is **independent** variable of *t*, however, particle positions *x*, *y*, *z* in Lagrangian description are **functions** of *t*.

Review

Total (material, substantial) derivative:

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V} \bullet \nabla)(\ )$$

Effects of the unsteadiness of the flow

Effects of the fluid particle moving (advecting or convecting) to a new location in the flow, where the velocity field is different

2.2 Pathlines and Streamlines

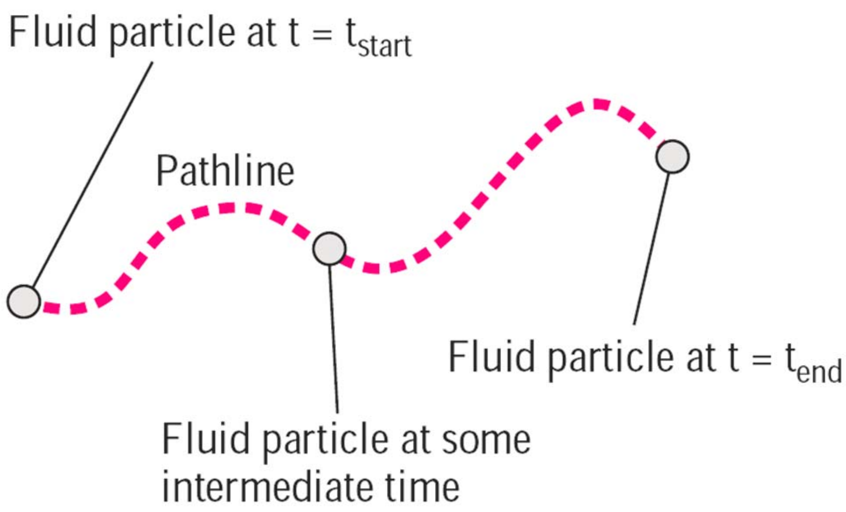
Various ways to visualize flow fields —

- **Pathlines**
- **Streamlines**

2.2.1 Pathlines

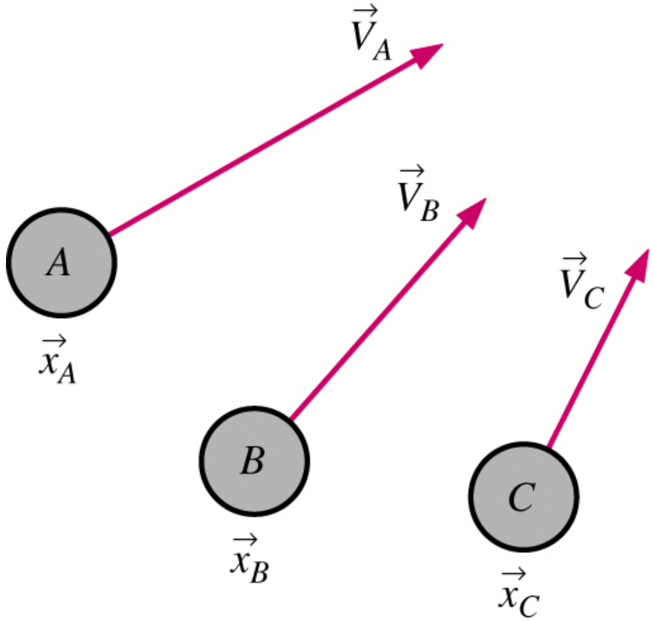
Definition: a pathline is the trajectory of a fluid particle of fixed identity over a period of time.

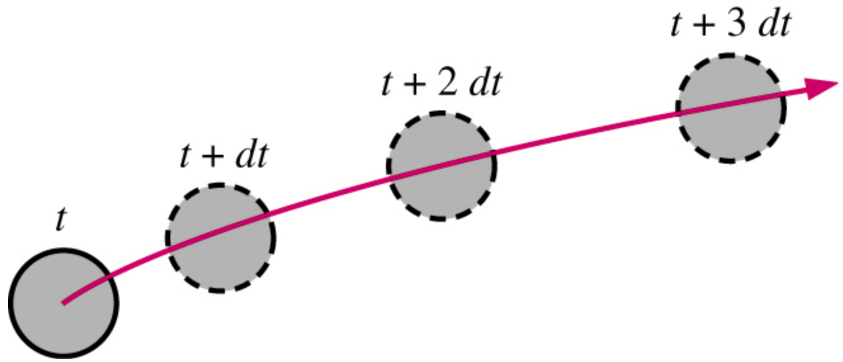
As pathlines are the actual paths traveled by individual fluid particles over some time period. In Lagrangian description, a pathline is the same as the fluid particle's position vector:

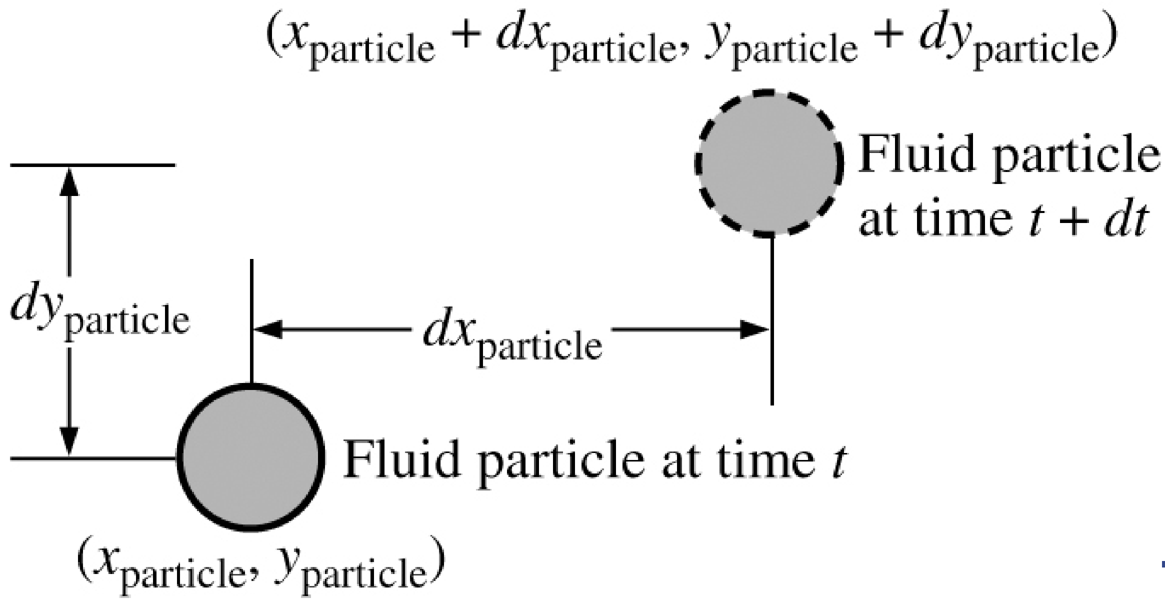


$$\begin{cases} x = x(a,b,c,t) \\ y = y(a,b,c,t) \\ z = z(a,b,c,t) \end{cases}$$

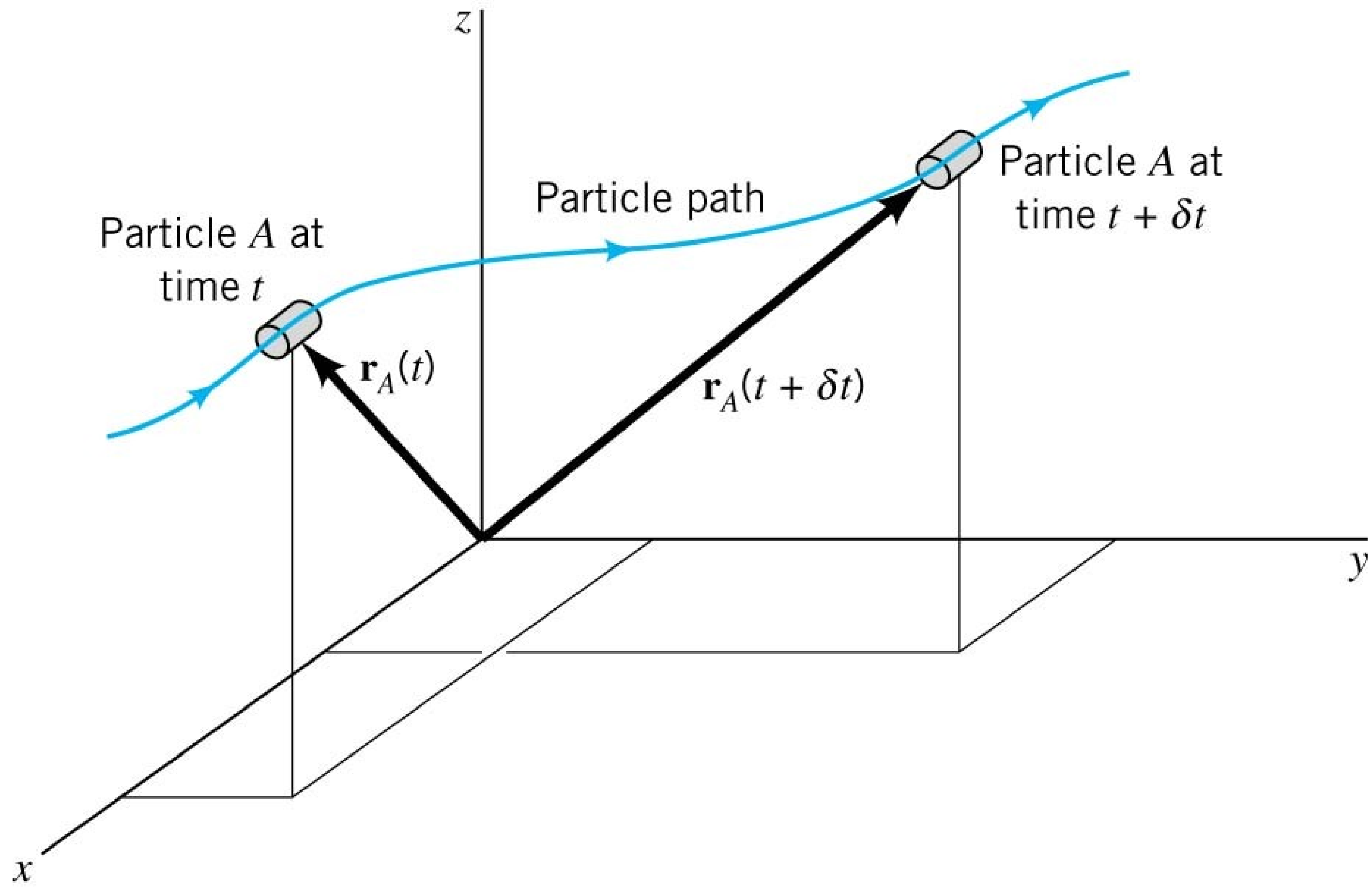
2.2.1 Pathlines



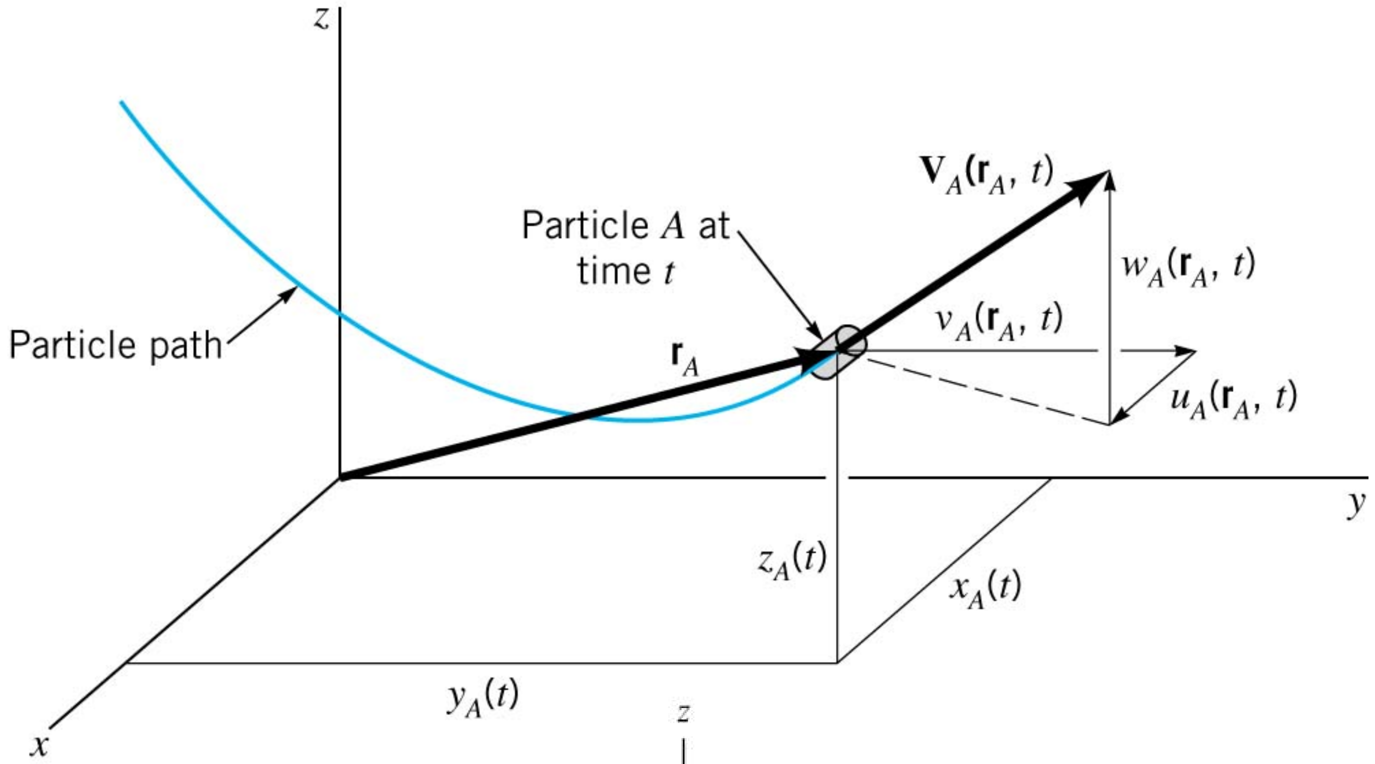


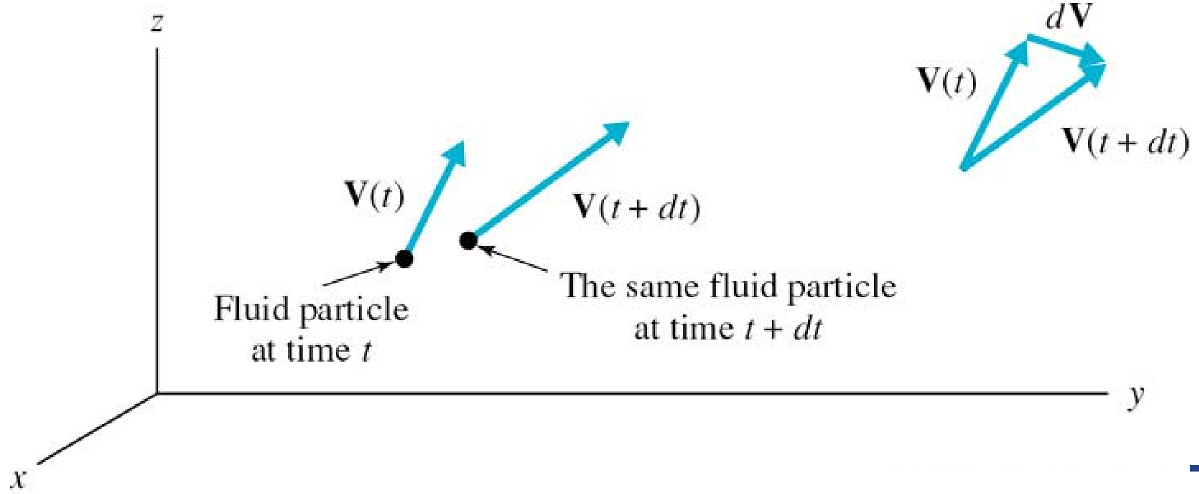


2.2.1 Pathlines



2.2.1 Pathlines





2.2.1 Pathlines

Normally, the velocity field is given by Eulerian description as:

$$\mathbf{V} = \mathbf{V}(\mathbf{r}, t) = \mathbf{V}(x, y, z, t)$$

A fluid particle moves from one spatial position **(*x*, *y*, *z*)** to another **(*x*+*u*d*t*, *y*+*v*d*t*, *z*+*w*d*t*)** over a period of time d*t*, i.e., it moves a distance of d**r**. The equation of the **pathline** for the fluid particle is:

$$\mathrm{d}\mathbf{r} = \mathbf{V}\,\mathrm{d}t$$

or written as →

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = v \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = w \end{cases}$$

x*, *y*, *z* are functions of *t

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt$$

Since the pathline is defined by integration of the relationship between velocity and displacement, to integrate (*u*, *v*, *w*) with respect to *t*, use initial condition (x_0, y_0, z_0, t_0) to determine the integral constants, then eliminate *t*.

2.2.2 Streamlines

Definition: a streamline is a curve that is everywhere tangent to the velocity vector at a given instant.

At an instant of time, there is at every point a velocity vector with a definite direction. The instantaneous curves that are everywhere tangent to the direction field are called the streamlines of flow.

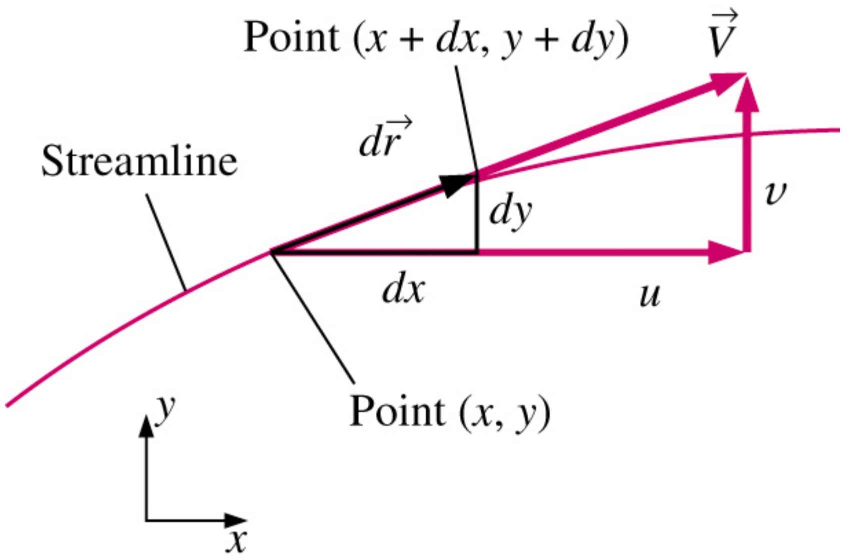
2.2.2 Streamlines

Two-dimensional flows

By definition, the local velocity vector **V** and the element of arc length along a streamline d**r** are locally parallel, thus the equation for a streamline is:

$$\mathbf{V} \times d\mathbf{r} = 0$$

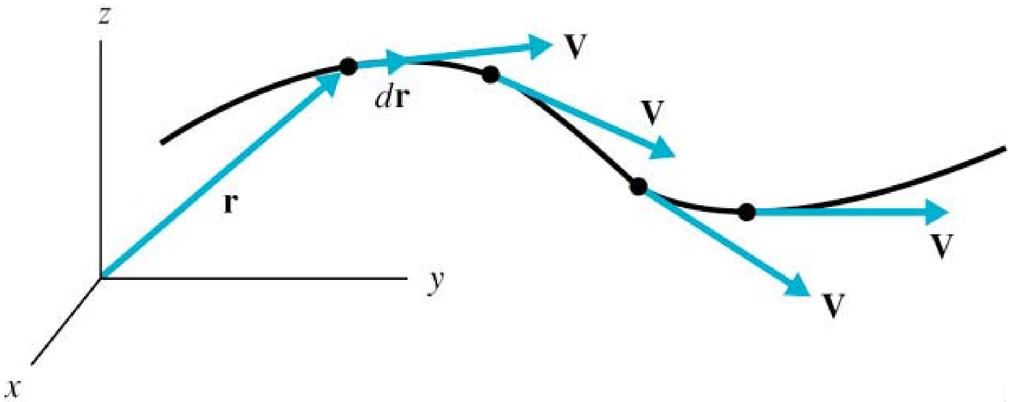
$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$



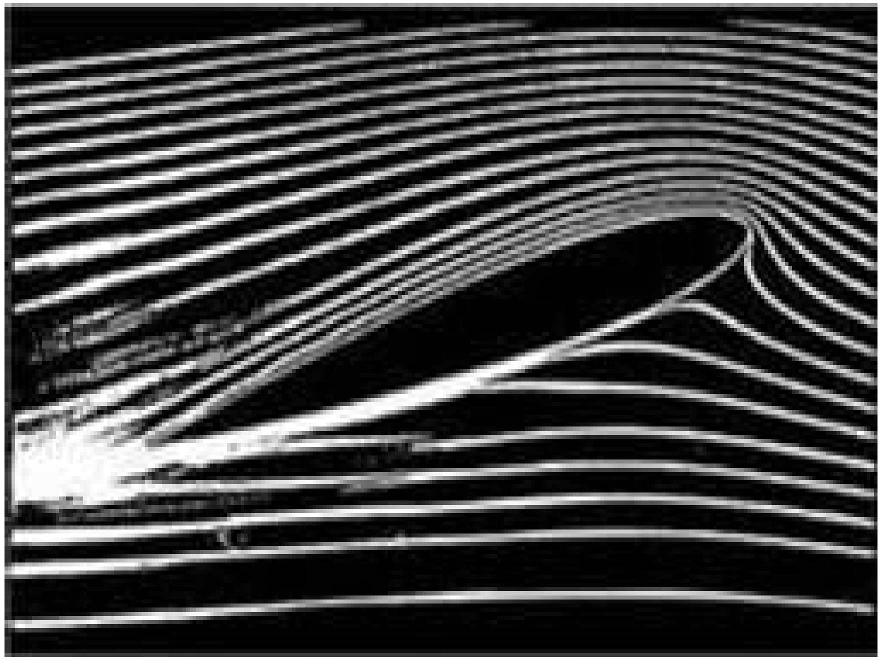
Also by simple geometric arguments using similar triangles, the slope of the streamline:

$$\frac{dy}{dx} = \frac{v}{u}$$

2.2.2 Streamlines



$$\mathbf{V} \times d\mathbf{r} = 0$$



Pathlines and streamlines are identical in steady flows

2.2.2 Streamlines

Characteristics of streamlines:

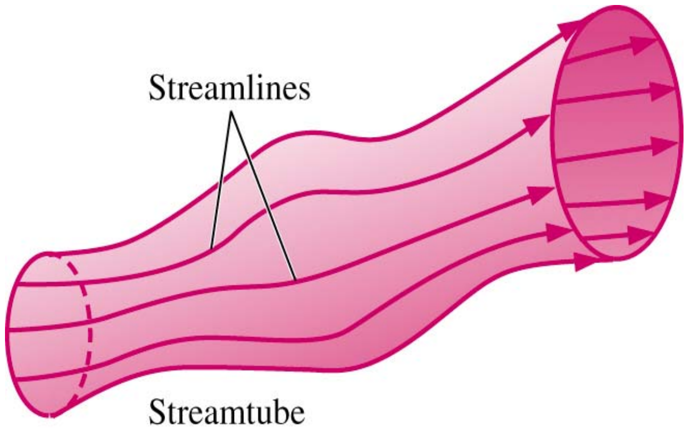
- Instantaneous quantities
- Tangential direction of the streamlines are identical to the velocity vectors. Streamline cluster density reflects the magnitude of velocity: streamlines **close** together ⇒ **high** velocity, streamline **far apart** ⇒ **low** velocity
- The streamlines **never intersect** each other except at a point of **zero** velocity, because at any point, there can be only one direction of the velocity
- The streamlines **never interrupt** in the fluid
- Since the velocity vector in the flow field is everywhere tangent to the streamline, the fluid **cannot cross** the streamlines and the streamlines can be regarded as fixed walls

2.2.2 Streamlines

Streamtube

Definition: a set of streamlines that intersect a closed loop in space.

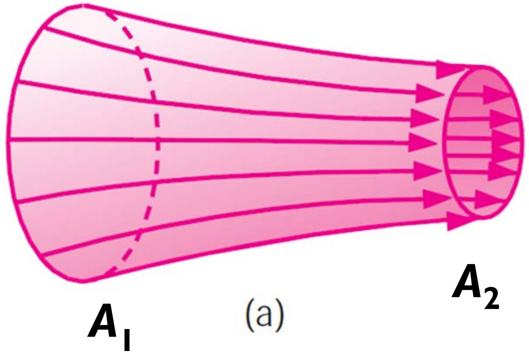
Streamtubes are instantaneous quantities like streamlines, defined at a particular instant in time according to the velocity field at that instant. When the flow is steady, the shape of the streamtube does not change with t, like a real pipe. **No** fluid crosses a stream tube's surface because the fluid velocity vector is everywhere tangent to it.

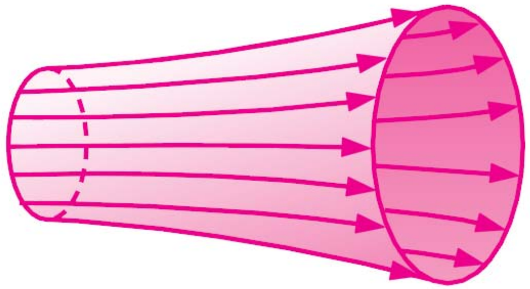


2.2.2 Streamlines

Streamtube

Let's take **two cross-sections** of a streamtube, with cross-sectional areas A_1 and A_2. The velocities **perpendicular** to the cross-sections are V_1 and V_2, respectively. The rate at which **mass** is entering the streamtube is $\rho_1 A_1 V_1$; the rate at which it is leaving is $\rho_2 A_2 V_2$. If the flow is **steady**, the mass is conserved and then $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$, and if flow is **incompressible**, then $A_1 V_1 = A_2 V_2$.





In an incompressible flow field, a streamtube (*a*) **decreases** in diameter as the flow **accelerates** or converges and (*b*) **increases** in diameter as the flow **decelerates** or diverges.

The fluid speed increases when the cross-sectional area of the streamtube decreases.

2.2.2 Streamlines

Flux

Definition: the flow rate of a property (volume, mass, weight) per unit area of a spatial curved surface per unit of time — volumetric flux, mass flux, weight flux.

Volumetric flux: the rate of volume flow across a unit area

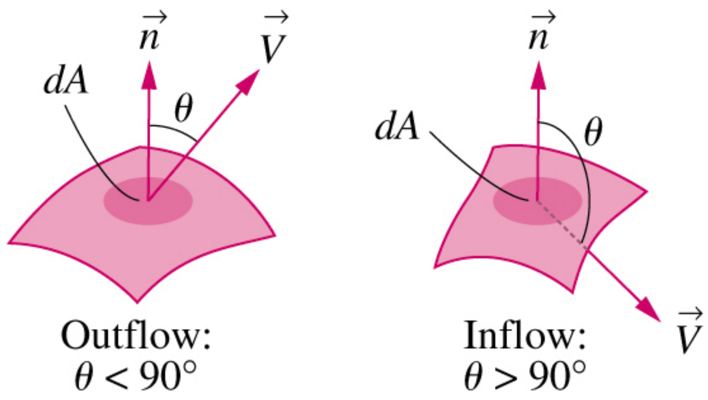
$$Q_1 = \iint_S v_n ds = \iint_S \mathbf{V} \cdot \mathbf{n} ds$$

Mass flux: the rate of mass flow per unit area

$$Q_2 = \iint_S \rho v_n ds$$

Weight flux: the rate of weight flow per unit area

$$Q_3 = \iint_S \rho g v_n ds$$



For a closed surface S, take the unit outer normal as positive, then the flux is:

$$\oiint_S (\rho \mathbf{V} \cdot \mathbf{n}) dS = \iiint_\Omega \nabla \cdot (\rho \mathbf{V}) d\Omega$$

$\vec{V} \cdot \vec{n} = |\vec{V}||\vec{n}| \cos\theta = V \cos\theta$

If $\theta < 90°$, then $\cos\theta > 0$ (outflow).

If $\theta > 90°$, then $\cos\theta < 0$ (inflow).

If $\theta = 90°$, then $\cos\theta = 0$ (no flow).

Gauss' theorem

Application1: Pathlines and Streamlines

A two-dimensional velocity field is given as: $u = \dfrac{x}{1+t}$, $v = y$

(1) Determine the pathline equation for $(x)_{y=1\,,\,t=0} = 1$

(2) Determine the streamline equation for $(x)_{t=0} = a, \quad (y)_{t=0} = b$

Solution: (1)

The pathline equation is: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{x}{1+t}, \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} = y$

Integrating to give: $\ln x = \ln(t+1) + \ln C_1, \qquad \ln y = t + \ln C_2$

i.e., $x = C_1(1+t), \;\; y = C_2 e^t$

From $(x)_{y=1\,,\,t=0} = 1$: $C_1 = 1, \quad C_2 = 1$

Thus, the pathline equation is: $\begin{cases} x = 1+t \\ y = e^t \end{cases}$

Application1: Pathlines and Streamlines

(2) The streamline equation is: $\frac{1+t}{x}\mathrm{d}x = \frac{1}{y}\mathrm{d}y$

Integrating, yield: $(1+t)\ln x = \ln y + \ln C$

i.e., $x^{1+t} = Cy$

From $(x)_{t=0} = a, \quad (y)_{t=0} = b:\ C = \frac{a}{b}$

Thus, the streamline equation at *t*=0 and (*a*, *b*) is: $y = \frac{b}{a}x$

Application2

Assume the pathline equation of a fluid particle is:

$$x = C_1 e^t - t - 1$$

$$y = C_2 e^t + t - 1 \qquad \text{where } C_1, C_2, C_3 \text{ are constant}$$

$$z = C_3$$

Determine: (1) The pathline equation of the fluid particle at $x=a$, $y=b$, $z=c$, and $t=0$;

(2) The velocity of any fluid particle;

(3) The expression of velocity field by Eularian description;

(4) Are the acceleration field by Eularian description and the acceleration field by converting Lagrangian description to Eularian description the same?

Application2

Solution: (1) The pathline equation

Substituting $t=0,\ x=a,\ y=b,\ z=c$ into the pathline equation, yields:

$$a=C_1-1,\quad b=C_2-1,\quad c=C_3$$

Thus, $$C_1=a+1,\quad C_2=b+1,\quad C_3=c$$

The pathline equation of the fluid particle at (a,b,c) is:

$$\begin{aligned} x&=(a+1)e^t-t-1 \\ y&=(b+1)e^t+t-1 \\ z&=c \end{aligned} \qquad (1)$$

Application2

(2) The velocity of any fluid particle:

$$
\begin{aligned}
u &= \frac{\partial x}{\partial t} = C_1 e^t - 1 = (a+1)e^t - 1 \\
v &= \frac{\partial y}{\partial t} = C_2 e^t + 1 = (b+1)e^t + 1 \qquad (2) \\
w &= \frac{\partial z}{\partial t} = 0
\end{aligned}
$$

Application2

(3) Expressing the velocity field by Eularian description

a, *b*, *c* are solved from Equation (1):

$$a = \frac{1}{e^t}(x+t+1)-1$$
$$b = \frac{1}{e^t}(y-t+1)-1 \qquad (3)$$
$$c = z$$

Substituting into Equation (2), yields:

$$u = \frac{\partial x}{\partial t} = (a+1)e^t - 1 = x+t$$
$$v = \frac{\partial y}{\partial t} = (b+1)e^t + 1 = y-t+2 \qquad (4)$$
$$w = \frac{\partial z}{\partial t} = 0$$

Application2

(4) Determining the acceleration field by Eularian description

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = x + t + 1$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = y - t + 1$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0$$

From Equation (1), the acceleration of the particle at (a, b, c) is:

$$a_x = \frac{\partial^2 x}{\partial t^2} = (a+1)e^t$$

$$a_y = \frac{\partial^2 y}{\partial t^2} = (b+1)e^t \qquad (5)$$

$$a_z = \frac{\partial^2 z}{\partial t^2} = 0$$

Substituting Equation (3) to (5), yields:

$$a_x = x + t + 1$$

$$a_y = y - t + 1$$

$$a_z = 0$$

By comparison, the two results are the same

2.3 Deformation and Rotation of Fluid Elements

In theoretical mechanics, the motion of a rigid body can be split into the translational motion and the rotational motion.

$$\boldsymbol{V} = \boldsymbol{V}_M + \boldsymbol{\omega} \times \boldsymbol{r}$$

$\boldsymbol{V}_M$	Velocity of a reference point M
$\boldsymbol{r}$	Radius vector between a moving point and the reference point
$\boldsymbol{\omega}$	Angular velocity

2.3 Deformation and Rotation of Fluid Elements

In fluid mechanics, to study the motion of the fluid, take a fluid element from the flow field. A point in this fluid element is M(*x*, *y*, *z*), an adjacent point is M' (*x*+dx, *y*+dy, *z*+dz). Let the velocity at M is **V**, then the velocity at M' is:

$$\mathbf{V}_{M'} = \mathbf{V}_M + \frac{\partial \mathbf{V}_M}{\partial x}\delta x + \frac{\partial \mathbf{V}_M}{\partial y}\delta y + \frac{\partial \mathbf{V}_M}{\partial z}\delta z + \cdots$$

$$= (u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz)\mathbf{i} + (\cdots)\mathbf{j} + (\cdots)\mathbf{k}$$

$$= [u + \underbrace{\frac{\partial u}{\partial x}dx}_{(1)} + \underbrace{\frac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dy}_{(2)} + \underbrace{\frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})dz}_{(3)}$$

$$+ \underbrace{\frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})dy}_{(4)} + \underbrace{\frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})dz}_{(5)}]\mathbf{i} + (\cdots)\mathbf{j} + (\cdots)\mathbf{k}$$

2.3 Deformation and Rotation of Fluid Elements

In the equation:

$$(1)\quad \varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} \quad \varepsilon_{zz}$$

$$(2)\quad \omega_z = -\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_x$$

$$(3)\quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$(4)\quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{zy}$$

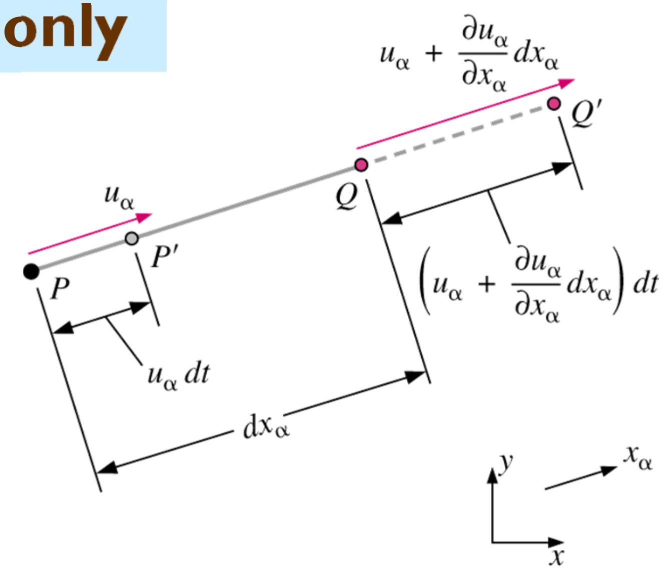
$$(5)\quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Physical meaning of each item in the equation

(1) Physical meaning of $\varepsilon_{xx} = \dfrac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \dfrac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \dfrac{\partial w}{\partial z}$

2.3 Deformation and Rotation of Fluid Elements

Consider the linear strain in *x*-direction only



Change of length per unit length:

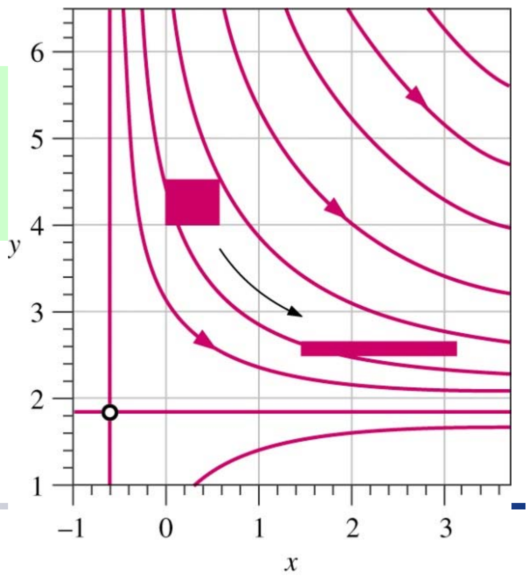
$$\varepsilon_x = \frac{\partial u}{\partial x} dx\, dt / dx = \frac{\partial u}{\partial x} dt$$

Rate of change of length per unit length:

$$\varepsilon_{xx} = \frac{\varepsilon_x}{dt} = \frac{\partial u}{\partial x}$$

It indicates the rate of increase or decrease in length of a fluid element in x-direction.

Similarly: $\varepsilon_{yy} = \dfrac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \dfrac{\partial w}{\partial z}$

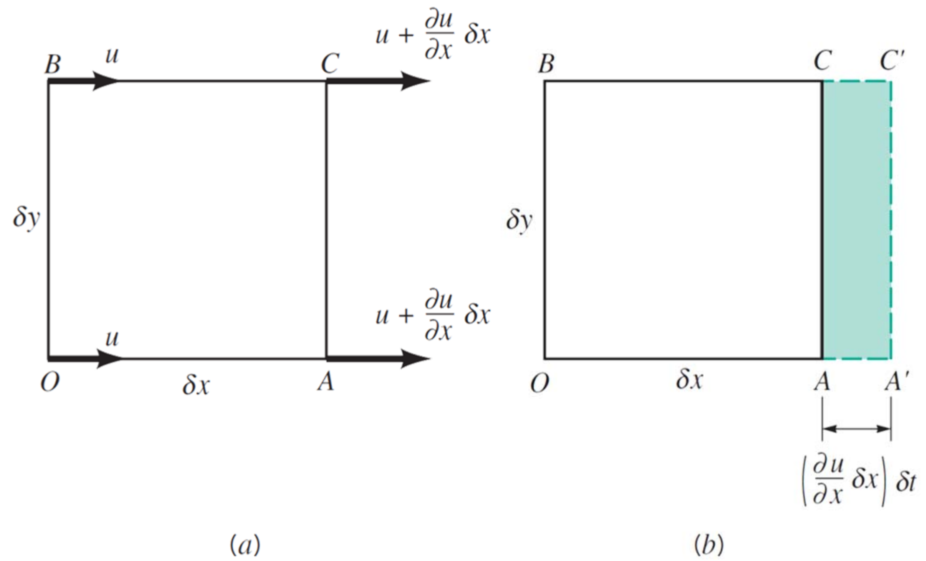


$\varepsilon_{xx}, \ \varepsilon_{yy}, \ \varepsilon_{zz}$ are called linear strain rate

2.3 Deformation and Rotation of Fluid Elements

Apparently: $$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V}$$

$\nabla \cdot \mathbf{V}$ is the **divergence of velocity**, denoting the rate of change of volume of a fluid element per unit volume (**volumetric strain/dilatation rate**).



Change of volume in *x*-direction:

$$\delta \forall_x = \left[\left(u + \frac{\partial u}{\partial x}\delta x\right)\delta y \cdot \delta z \cdot \delta t\right] - \left(u \cdot \delta y \cdot \delta z \cdot \delta t\right) = \frac{\partial u}{\partial x}\delta x \cdot \delta y \cdot \delta z \cdot \delta t$$

Change of volume in *x*-direction per unit volume per unit time:

$$\frac{\delta \forall_x}{\forall \cdot \delta t} = \frac{\frac{\partial u}{\partial x} \cdot \delta x \cdot \delta y \cdot \delta z \cdot \delta t}{\delta x \cdot \delta y \cdot \delta z \cdot \delta t} = \frac{\partial u}{\partial x} \quad \text{as } \delta x,\ \delta t \to 0$$

Similarly, in *y* and *z* directions:

$$\frac{\delta \mathcal{V}_y}{\mathcal{V} \cdot \delta t} = \frac{\partial v}{\partial y}, \qquad \frac{\delta \mathcal{V}_z}{\mathcal{V} \cdot \delta t} = \frac{\partial w}{\partial z} \quad \text{as } \delta y,\ \delta z,\ \delta t \rightarrow 0$$

Thus, the total change of volume per unit volume per unit time is:

$$\frac{\delta \mathcal{V}}{\mathcal{V} \cdot \delta t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathrm{V} \quad \text{as } \delta x,\ \delta y,\ \delta z,\ \delta t \rightarrow 0$$

The **divergence of velocity** $\nabla \cdot \mathbf{V}$ can be used to express the **volumetric strain/dilatation rate**

⇩

$\nabla \cdot \mathbf{V} = 0 \quad \Leftrightarrow$ Incompressible flow

2.3 Deformation and Rotation of Fluid Elements

$\nabla \cdot \mathbf{V} > 0$, indicates the fluid flows out of the element and is called a source flow

$\nabla \cdot \mathbf{V} < 0$, indicates the fluid flows into the element and is called a sink flow

$\nabla \cdot \mathbf{V} = 0$, indicates the velocity field of the incompressible fluid is without the source flow.

2.3 Deformation and Rotation of Fluid Elements

(2) Physical meaning of ε_{xy}, ε_{xz}, ε_{zy}

(3) Physical meaning of ω_x, ω_y, ω_z

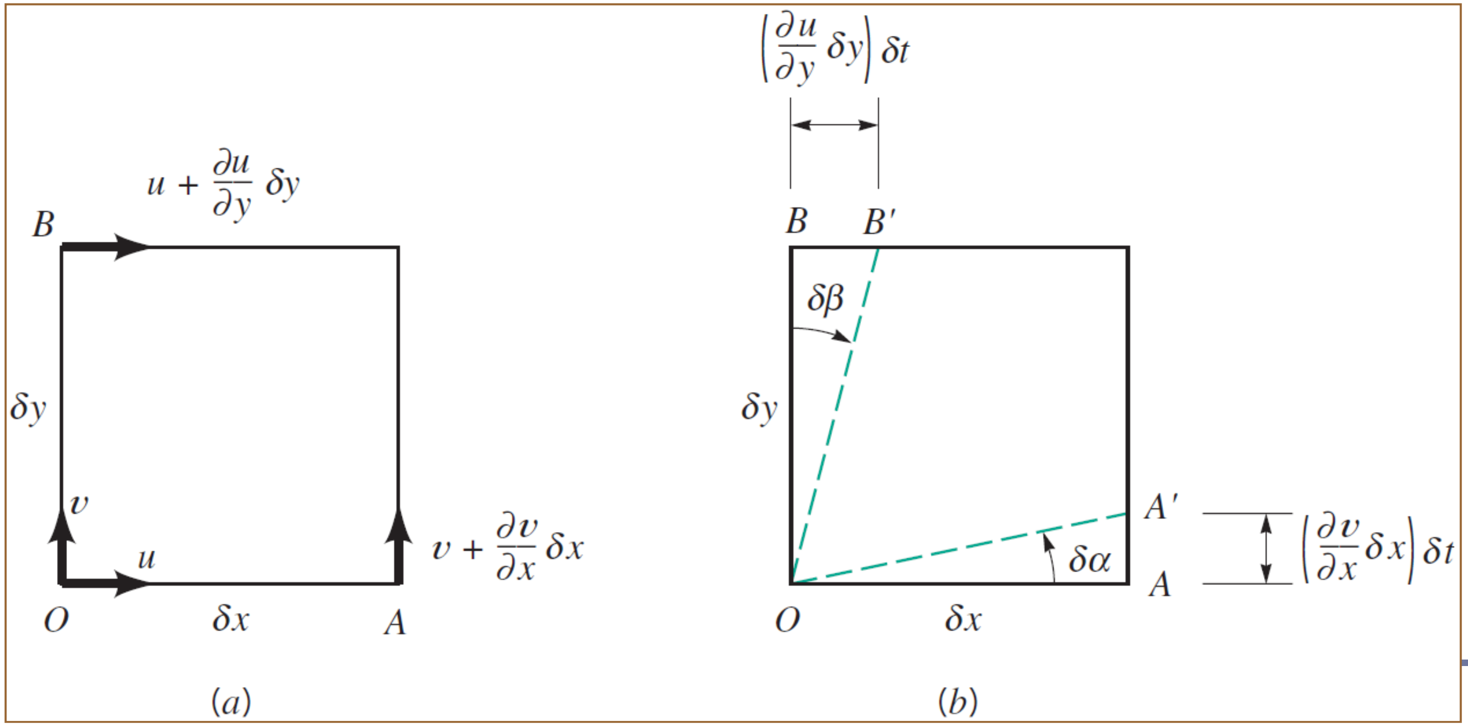
2.3 Deformation and Rotation of Fluid Elements

For a fluid element (in the *x*–*y* plane), the change of line **OA** in a short time interval δt is:

$$\delta\alpha = tg\left(\delta\alpha\right) = \frac{\partial v}{\partial x}\delta x\delta t / \delta x = \frac{\partial v}{\partial x}\delta t$$

Similarly,

$$\delta\beta = tg\left(\delta\beta\right) = \frac{\partial u}{\partial y}\delta y\delta t / \delta y = \frac{\partial u}{\partial y}\delta t$$



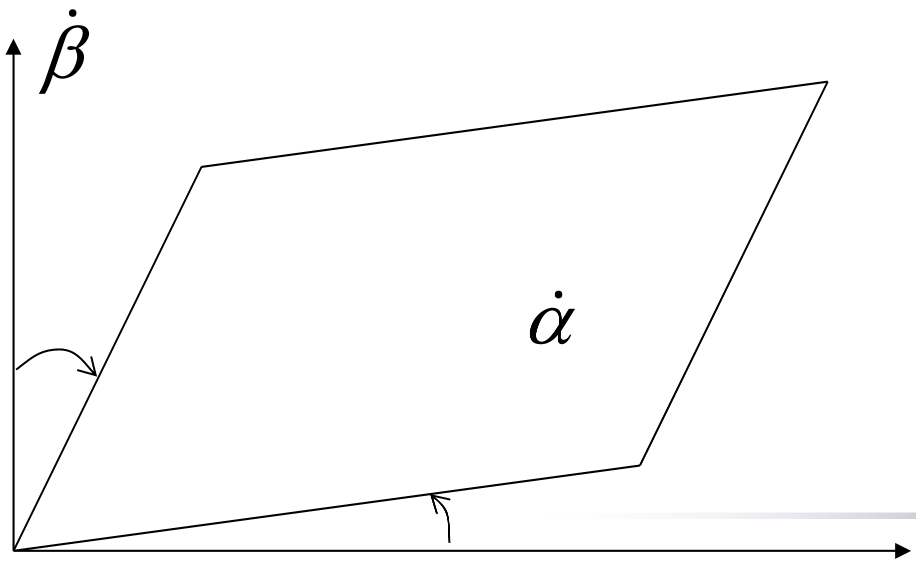
2.3 Deformation and Rotation of Fluid Elements

The angular velocity of line OA is:

$$\dot{\alpha} = \frac{d\alpha}{dt} = \frac{\partial v}{\partial x} \quad \text{as} \quad \delta x,\ \delta t \to 0$$

Similarly, the angular velocity of line OB is:

$$\dot{\beta} = \frac{d\beta}{dt} = \frac{\partial u}{\partial y} \quad \text{as} \quad \delta y,\ \delta t \to 0$$



2.3 Deformation and Rotation of Fluid Elements

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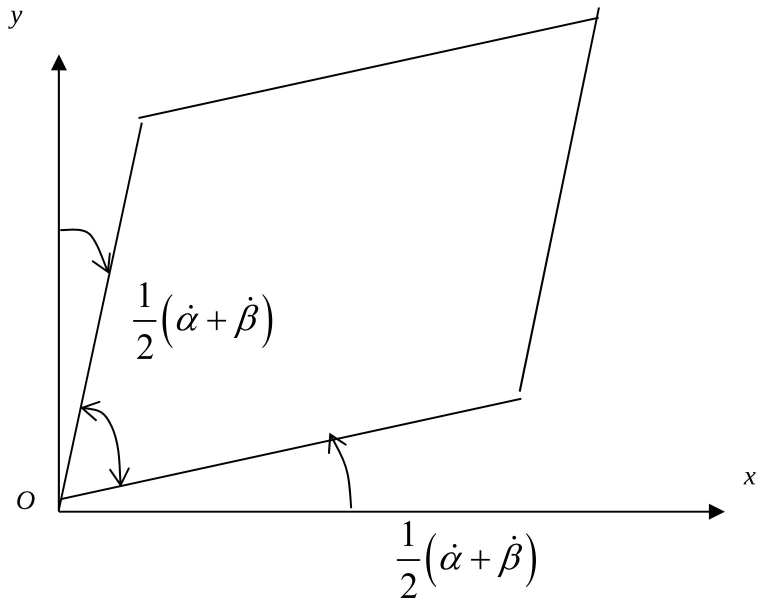
$$\therefore\ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(\dot{\alpha} + \dot{\beta})$$

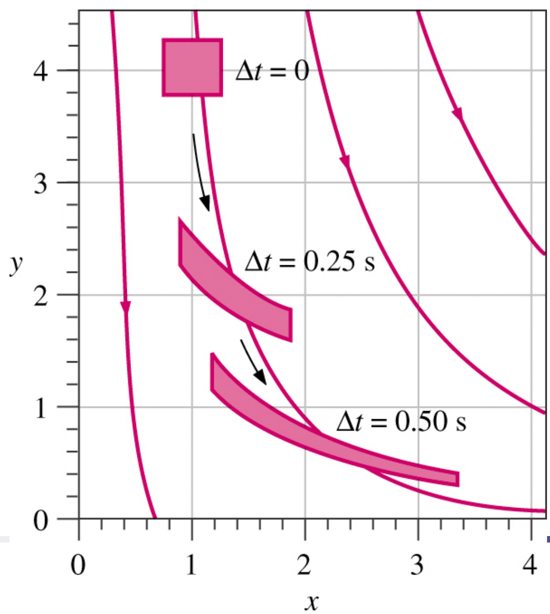
It denotes the rate of angular deformation of a right angle in the fluid element, it is called shear strain rate. When $\varepsilon_{xy} > 0$,the angle decreases; In contrast , the angle increases.



Similarly:

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

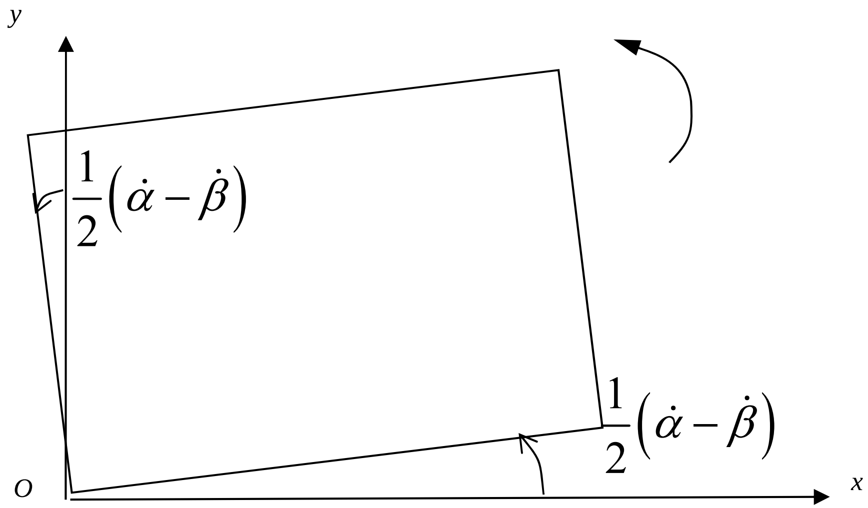
$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$



2.3 Deformation and Rotation of Fluid Elements

$$\omega_z = \frac{1}{2}\left(\dot{\alpha} - \dot{\beta}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

It denotes the rate of rotation, angular velocity of the fluid element. When $\omega_z > 0$, the fluid element rotates in counterclockwise direction; in contrast, it rotates in clockwise direction.



Similarly:

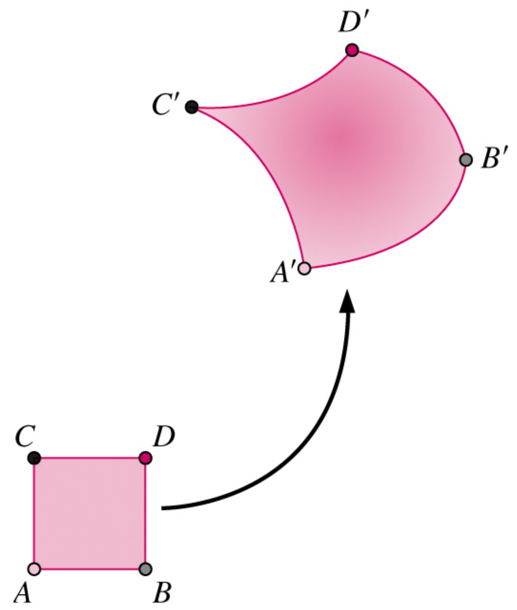
$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

2.3 Deformation and Rotation of Fluid Elements

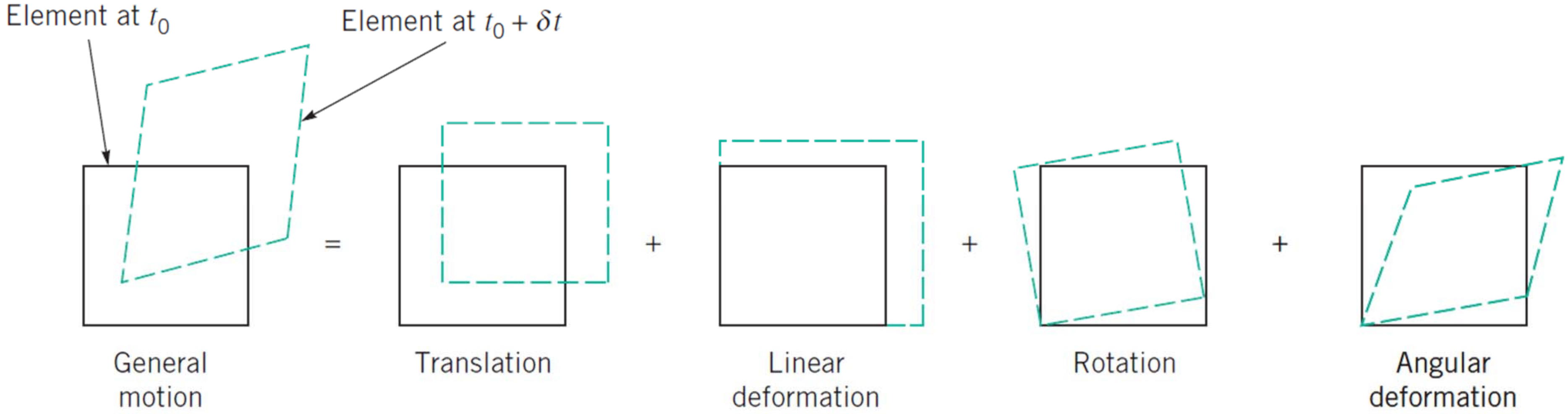
$$\mathbf{V}_{M'} = [\underbrace{u}_{(1)} + \underbrace{\omega_z dy + \omega_y dz}_{(2)} + \underbrace{\varepsilon_{xx} dx}_{(3)} + \underbrace{\varepsilon_{xy} dy + \varepsilon_{xz} dz}_{(4)}]\mathbf{i}$$

$$+ (\cdots)\mathbf{j} + (\cdots)\mathbf{k}$$

General motion = Translation (1) + Rotation (2) (rigid body motion) + Dilatation (3) (change in volume) + Angular deformation (4) (change in shape)

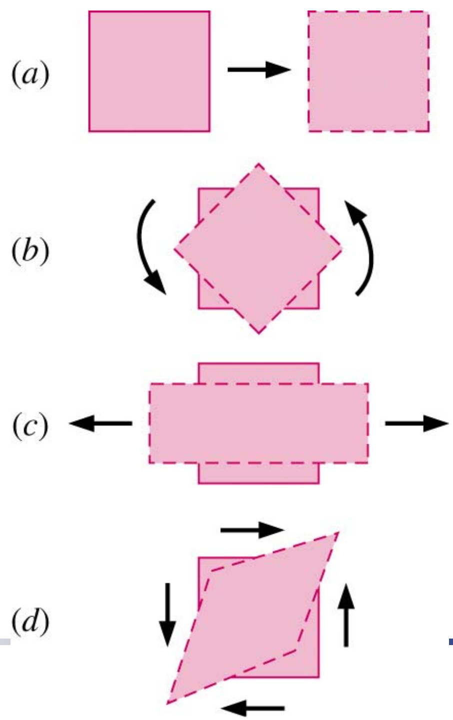


2.3 Deformation and Rotation of Fluid Elements



Helmholtz velocity decomposing theorem:

An fluid element may undergo **four** fundamental types of motion or deformation: (a) translation, (b) rotation, (c) linear strain, and (d) shear strain



2.3 Deformation and Rotation of Fluid Elements

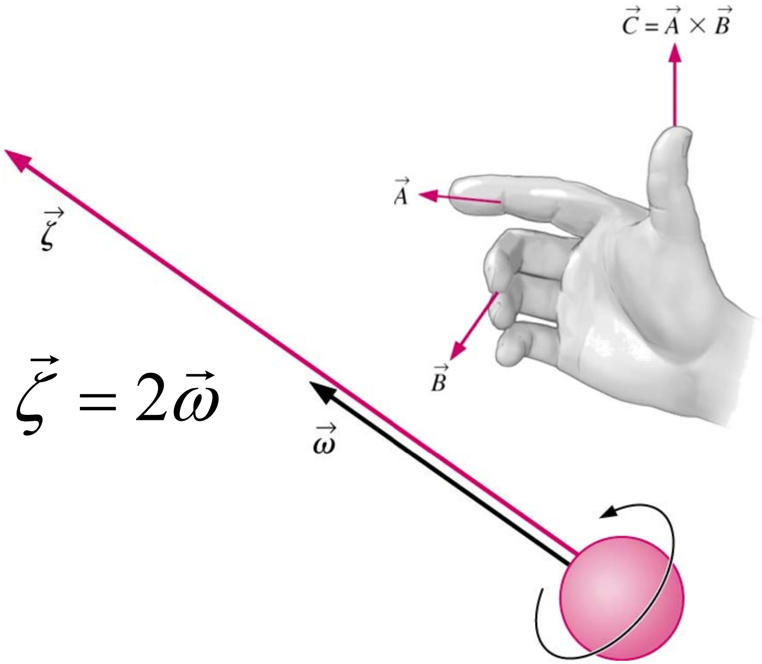
shear rate tensor $\mathbf{E} = e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

rate of rotation $\boldsymbol{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} = \frac{1}{2}\zeta$ (vorticity)

vorticity $\zeta = \mathbf{\Omega} = \nabla \times \mathbf{V}$ (curl of velocity)

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{2\omega_x}\vec{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{2\omega_y}\vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{2\omega_z}\vec{k}$$



$$\vec{\zeta} = 2\vec{\omega}$$