复习

- **流函数：** 二维不可压流体的流动，均存在流函数。

$$\psi = \int -vdx + udy \quad \longleftrightarrow \quad \begin{cases} \dfrac{\partial \psi}{\partial x} = -v \\ \dfrac{\partial \psi}{\partial y} = u \end{cases}$$

- **动量方程(动量守恒)**

$$\iiint_{MV} \rho \frac{d\mathbf{V}}{dt} d\mathcal{V} = \iiint_{MV} (\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}) d\mathcal{V}$$

积分形式的动量方程

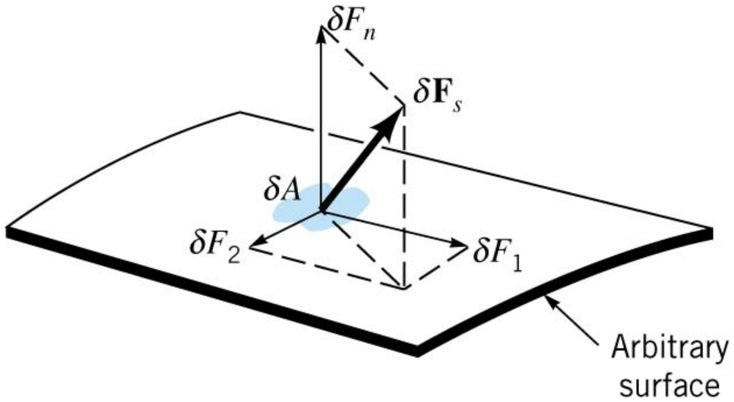
$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}$$

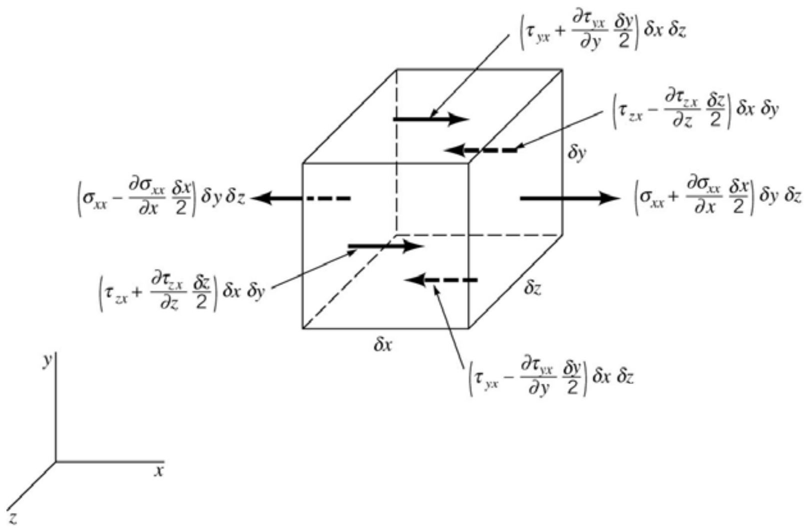
微分形式的动量方程

复习

对牛顿流体可以建立总表面力与应变率关系：

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} = \left(-p + \lambda\frac{\partial u_l}{\partial x_l}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$





复习

- 不可压缩流体表面力与应变率关系(本构方程)

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- 不可压缩流体运动的动量方程(动量守恒)

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p + \mu\nabla^2\mathbf{V} + \rho\mathbf{g}$$

- 不可压缩流体运动的基本控制方程(Navier-Stokes方程)

$$\begin{cases} \nabla\cdot\mathbf{V} = 0 \\ \dfrac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\dfrac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V} + \mathbf{g} \end{cases}$$

写成分量形式：

Continuity: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

x-component: $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x$$

y-component: $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+g_y$$

z-component: $$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+g_z$$

如果体积力为重力，则有 $g_x=g_y=0$ 和 $g_z=-g$。

或用Einstein指标法表示：

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

(I) (II) (III) (IV) (V)

或：

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

这里 $\nu = \mu/\rho$ **是运动粘性系数(kinematic viscosity)。**

3.5 流体运动基本控制方程

先看NS动量方程各项的物理意义：

(I) – 局部加速度 (local acceleration);

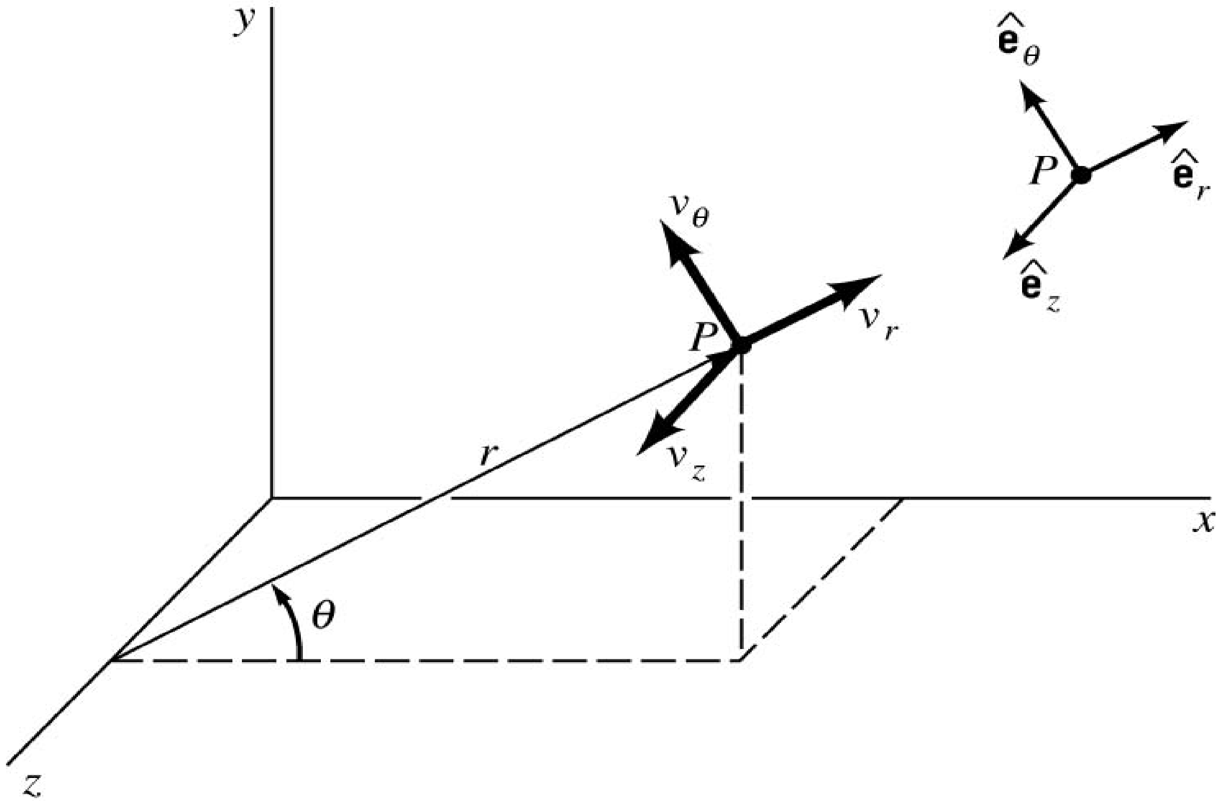
(II) – 变位加速度(convective acceleration)，惯性项(inertia)，对流项(convection)，非线性项(nonlinear term of the equation);

(III) – 压力梯度(pressure gradient);

(IV) – 体积力或重力(volume force or gravity);

(V) – 粘性扩散项(viscous diffusion of momentum owing to molecular viscosity of the fluid).

3.5 流体运动基本控制方程



直角坐标下(Rectangular Coordinates (x, y, z))

柱坐标下(Cylindrical Coordinates (r, θ, z))

NS方程在柱坐标下(Cylindrical Coordinates (r,θ, z))的表示：

Continuity: $$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}=0$$

r-component: $$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+g_r$$

θ-component: $$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}$$
$$+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+g_\theta$$

z-component: $$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+g_z$$

3.6 流体运动基本控制方程

NS方程是针对真实流体的运动，为了简化问题，我们首先考虑理想流体，即流体不存在粘性，粘性系数为0，$\mu = \nu = 0$，这样NS方程可以简化为：

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{V} = 0$$

上面的控制方程称为理想流体的Euler方程。

由于 $\mathbf{V} \cdot \nabla \mathbf{V} = \nabla \left(\frac{V^2}{2} \right) - \mathbf{V} \times (\nabla \times \mathbf{V}) = \nabla \left(\frac{V^2}{2} \right) - \mathbf{V} \times \mathbf{\Omega} = \nabla \left(\frac{V^2}{2} \right) - 2\mathbf{V} \times \boldsymbol{\omega}$

所以Euler方程可改写为：

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla \left(\frac{V^2}{2} \right) - 2\mathbf{V} \times \boldsymbol{\omega} = -\frac{1}{\rho} \nabla p + \mathbf{f},$$

这种形式的Euler方程称为Lamb方程。

3.6 流体运动基本控制方程

$$\mathbf{V}\times(\nabla\times\mathbf{V})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ u & v & w\\ \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} & \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} & \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\end{vmatrix}=\left[v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right]\vec{i}+\left[w\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)-u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]\vec{j}+\left[u\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-v\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\right]\vec{k}$$

$$=\left[\frac{1}{2}\left(\frac{\partial v^2}{\partial x}+\frac{\partial w^2}{\partial x}\right)-\left(v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]\vec{i}+\left[\frac{1}{2}\left(\frac{\partial w^2}{\partial y}+\frac{\partial u^2}{\partial y}\right)-\left(u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial z}\right)\right]\vec{j}+\left[\frac{1}{2}\left(\frac{\partial u^2}{\partial z}+\frac{\partial v^2}{\partial z}\right)-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}\right)\right]\vec{k}$$

$$=\left[\frac{1}{2}\frac{\partial\left(u^2+v^2+w^2\right)}{\partial x}-\left(\frac{1}{2}\frac{\partial u^2}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]\vec{i}+\left[\frac{1}{2}\frac{\partial\left(u^2+v^2+w^2\right)}{\partial y}-\left(u\frac{\partial v}{\partial x}+\frac{1}{2}\frac{\partial v^2}{\partial y}+w\frac{\partial v}{\partial z}\right)\right]\vec{j}+\left[\frac{1}{2}\frac{\partial\left(u^2+v^2+w^2\right)}{\partial z}-\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{1}{2}\frac{\partial w^2}{\partial z}\right)\right]\vec{k}$$

$$=\left[\frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}\right)\vec{i}+\frac{\partial}{\partial y}\left(\frac{u^2+v^2+w^2}{2}\right)\vec{j}+\frac{\partial}{\partial z}\left(\frac{u^2+v^2+w^2}{2}\right)\vec{k}\right]-\left[\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\vec{i}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\vec{j}+\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\vec{k}\right]$$

$$=\nabla\left(\frac{u^2+v^2+w^2}{2}\right)-\left[\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)u\vec{i}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)v\vec{j}+\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)w\vec{k}\right]$$

$$=\nabla\left(\frac{V^2}{2}\right)-(\mathbf{V}\cdot\nabla)\begin{pmatrix}u\vec{i}\\ v\vec{j}\\ w\vec{k}\end{pmatrix}=\nabla\left(\frac{V^2}{2}\right)-\mathbf{V}\cdot\nabla\mathbf{V}$$

$$\mathbf{V}\cdot\nabla\mathbf{V}=\nabla\left(\frac{V^2}{2}\right)-\mathbf{V}\times(\nabla\times\mathbf{V})$$

3.6 流体运动基本控制方程

梯度 $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_i} \quad (i = 1, 2, 3)$

散度 $\nabla \cdot \mathbf{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (u, v, w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u_i}{\partial x_i}$

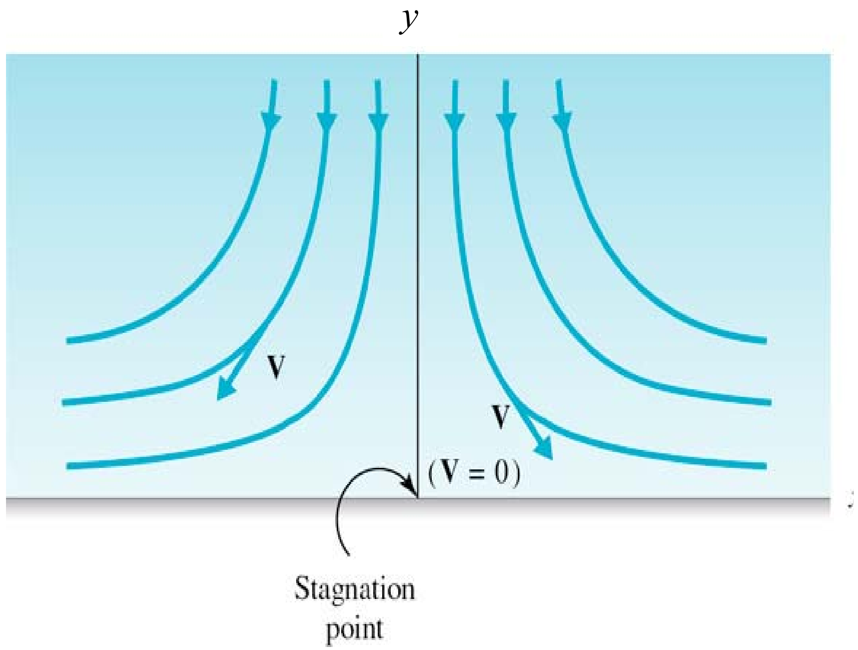
旋度 $\nabla \times \mathbf{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}\vec{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}\vec{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}\vec{k}$

变位加速度 $\mathbf{V} \cdot \nabla \mathbf{V} = \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times (\nabla \times \mathbf{V}) = \nabla\left(\frac{V^2}{2}\right) - \mathbf{V} \times \mathbf{\Omega}$

3.6 流体运动基本控制方程

例子：一个水柱冲向下面的平板，速度势 $\phi=-k\left(x^2-y^2\right)$ (k为常数)，ρ为流体密度，不考虑粘性。证明沿平板的压力分布为：

$$\frac{\partial p}{\partial x}=-4\rho k^2 x$$



证明：由速度势可以得到速度分布：

$$u=\frac{\partial \phi}{\partial x}=-2kx,\quad v=\frac{\partial \phi}{\partial y}=2ky \tag{1}$$

可以看出，这个是二维定常理想流体的流动，由Euler方程可以得到：

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x} \tag{2}$$

沿平板表面，有 $y=0, v=0$，代入上式得：

$$\frac{\partial p}{\partial x} = -\rho u \frac{\partial u}{\partial x} \qquad (3)$$

由方程(1)，可以得到： $$\frac{\partial u}{\partial x} = -2k$$

把上式代入(3)式，得到：

$$\frac{\partial p}{\partial x} = -\rho(-2kx)(-2k) = -4\rho k^2 x$$

3.7 Bernoulli方程

如果我们考虑：

- 理想流体 (ideal fluid, Inviscid flow)；
- 不可压流体 (constant density, incompressible flow)；
- 定常流动 (steady flow)；
- 体积力为重力 (gravity)；

这样Lamb方程可以改写为：

$$\frac{\partial \mathbf{V}}{\partial t}+\nabla\left(\frac{V^2}{2}\right)-2\mathbf{V}\times\boldsymbol{\omega}=-\frac{1}{\rho}\nabla p+\mathbf{f}$$

$$\nabla\left(\frac{V^2}{2}\right)+\nabla\left(\frac{p}{\rho}\right)+\nabla(gz)=2\mathbf{V}\times\boldsymbol{\omega}$$

3.7 Bernoulli方程

$$\nabla\left(\frac{V^2}{2}\right)+\nabla\left(\frac{p}{\rho}\right)+\nabla(gz)=2\mathbf{V}\times\boldsymbol{\omega}$$

⇓

$$\nabla\left(\frac{V^2}{2}+\frac{p}{\rho}+gz\right)=2\mathbf{V}\times\boldsymbol{\omega}$$

⇓

$$\nabla H=2\mathbf{V}\times\boldsymbol{\omega},\quad H=\frac{V^2}{2}+\frac{p}{\rho}+gz$$

这里H称为Bernoulli函数。

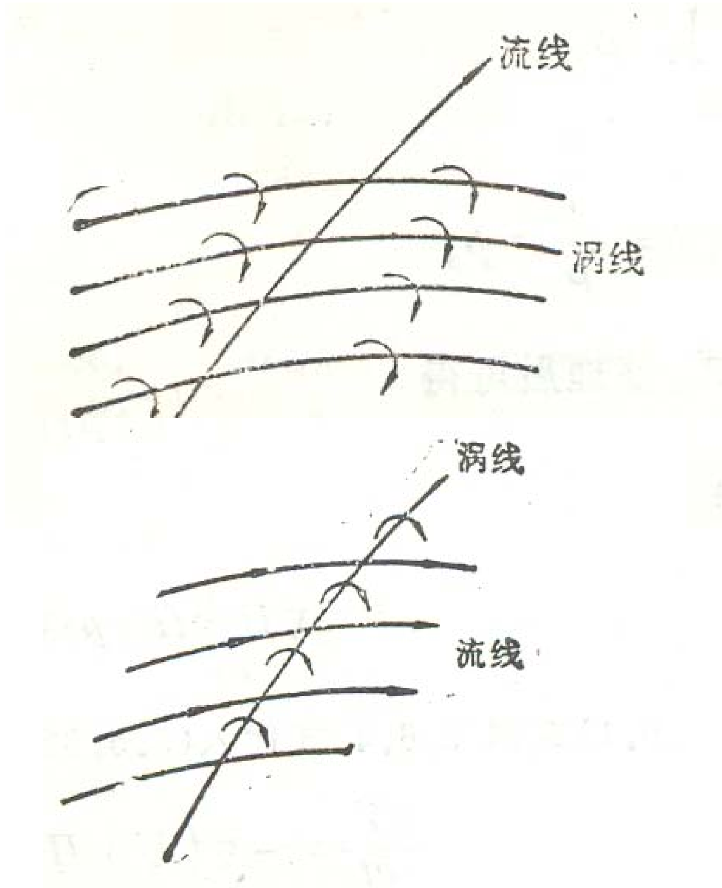
3.7 Bernoulli方程

在流线(V)或涡线(ω)上取一个微元段d*l*，对上式做点乘，即：

$$\nabla H \cdot dl = (2\mathbf{V} \times \boldsymbol{\omega}) \cdot dl$$

$$\left.\begin{aligned} \nabla H \cdot dl &= dH \\ (2\mathbf{V} \times \boldsymbol{\omega}) \cdot dl &= 0 \end{aligned}\right\} \Rightarrow dH = 0$$

$$H = \frac{V^2}{2} + \frac{p}{\rho} + gz = C_l = \text{const}$$



因此，Bernoulli函数在同一条流线或涡线上为常数。上式称为Bernoulli方程(Bernoulli equation)。

Bernoulli方程各项的物理意义：

$$\frac{V^2}{2}+\frac{p}{\rho}+gz=C_l=\text{const}$$

可以换成以长度单位度量

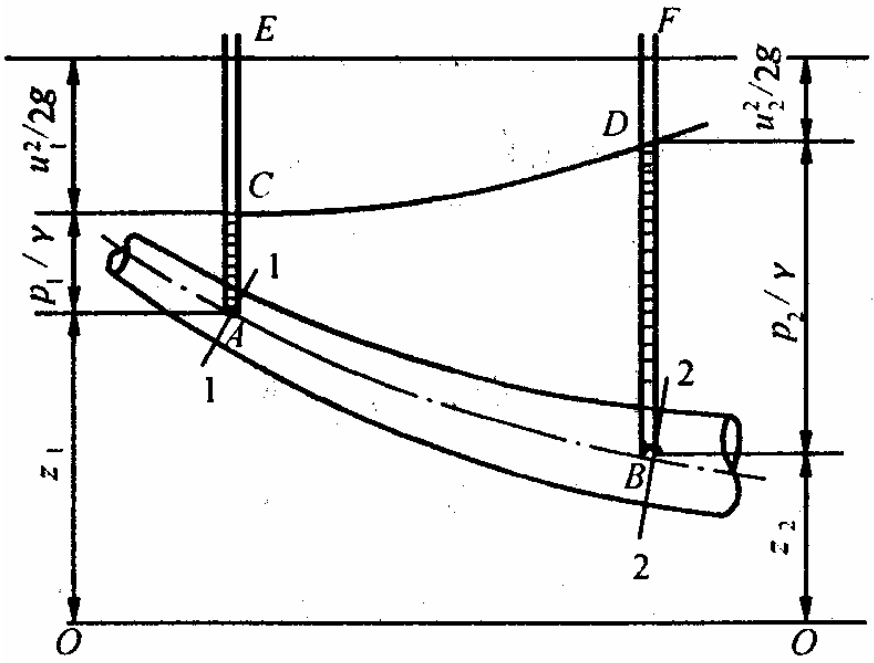
$$\frac{V^2}{2g}+\frac{p}{\rho g}+z=C_l=\text{const}$$

- 速度水头(velocity head)，代表动能(kinetic energy)
- 压力水头(pressure head)，表示压能(pressure energy)
- 位置水头(elevation head)，代表势能(potential energy)

速度水头 ＋ 压力水头 ＋ 位置水头 ＝ 常数

3.7 Bernoulli方程

Bernoulli方程，说明理想流体在流管中作定常流动时，单位体积的动能、重力势能以及该点的压强之和为一常量。

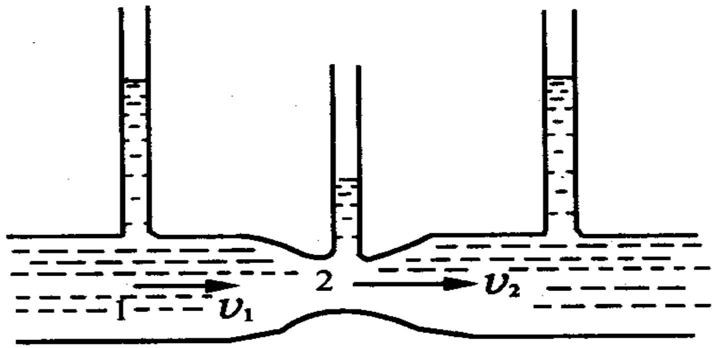


Bernoulli方程的适用范围:

适用于理想不可压流体作定常流动。

3.7 Bernoulli方程和动量方程例子

例子1：如图一个水平管内流动，问点1和点2处，哪点的压力大？



解：

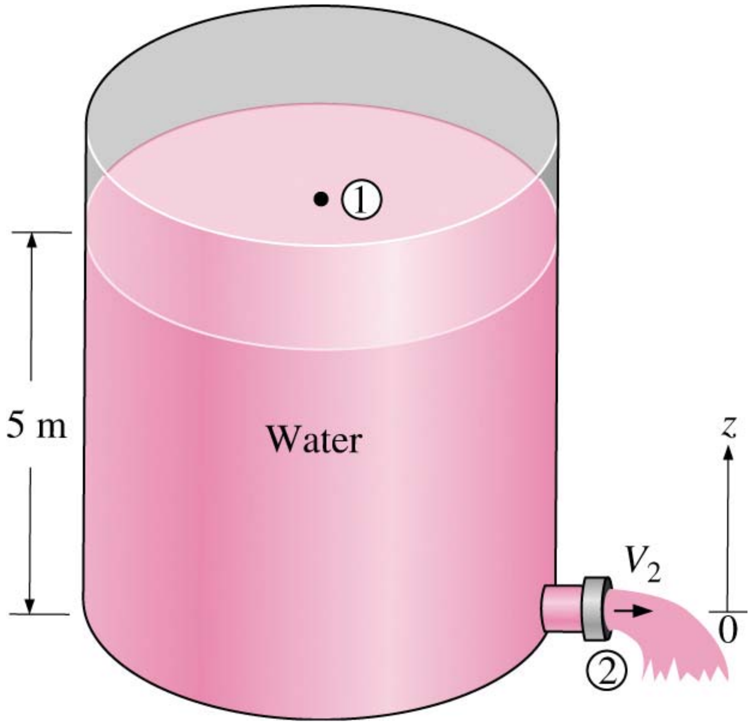
如图所示，1处横截面积远大于2处的横截面积，根据连续性方程可知，横截面小处流速大，2处的流速远大于1处。又由于管处于水平，根据伯努利方程有：

$$\frac{1}{2}\rho\upsilon_1^{\ 2}+P_1=\frac{1}{2}\rho\upsilon_2^{\ 2}+P_2$$

结论：在水平管中流动的流体，流速小的地方压强较大，流速大的地方压强较小，即点1处的压力比点2处的压力大。

3.7 Bernoulli方程和动量方程例子

例子2：如图一个水桶，装了5m高的水，水桶底部有一个阀门，问如果打开阀门，出口流速是多少?



解：

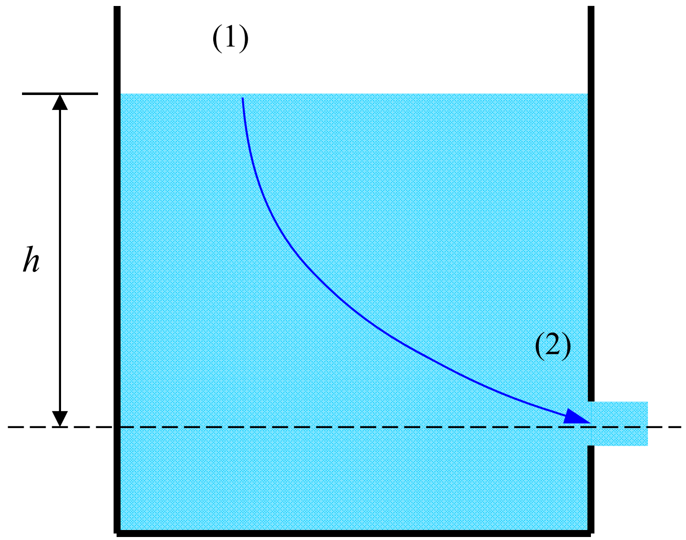
在水桶自由面和底部出口分别取两个点1和2，可以认为是一条流线上的两个点，参考位置选在水桶底部。根据Bernoulli方程，有：

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2$$

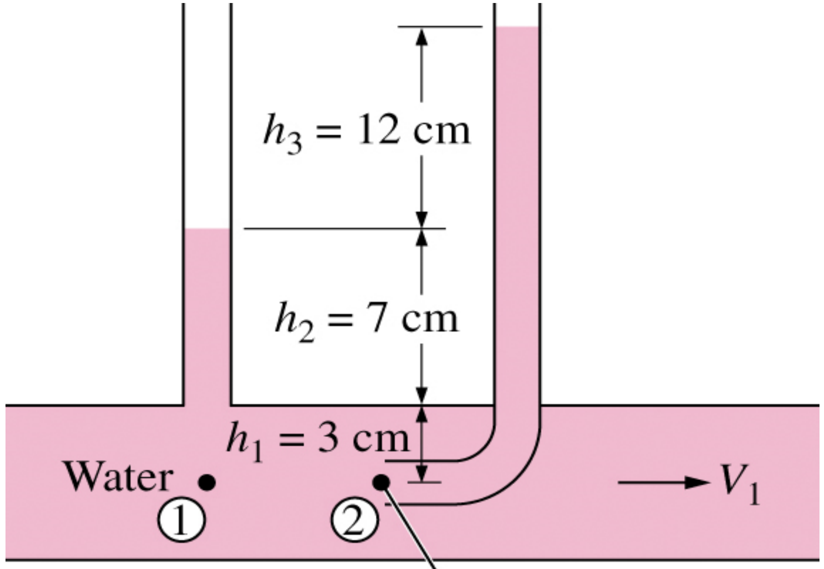
3.7 Bernoulli方程和动量方程例子

p_1和p_2都是大气压；由于水桶直径远远大于出水口的直径，由连续性方程可知，$V_1 << V_2$；点2位于水桶底部，即$z_2 = 0$，所以得到：

$$V_2 = \sqrt{2gz_1} = \sqrt{2\left(9.8 m/s^2\right)\left(5m\right)} = 9.9\, m/s$$



3.7 Bernoulli方程和动量方程例子



例子3： 用**Pitot**管(**Pitot tube**)测流速。如图所示，两个管子液面高度可以读出，试求点**1**处的流速。

解： 取**Pitot**管的前端点为**2**点，这样**1**点和**2**点处的压力为：

$$p_1 = \rho g(h_1 + h_2),\quad p_2 = \rho g(h_1 + h_2 + h_3)$$

选点**1**和点**2**的水平线为参考位置，所以$z_1=z_2=0$，点**2**处为驻点，速度为**0**，即$V_2=0$。

根据**Bernoulli**方程，有：

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

得到：

$$\frac{V_1^2}{2g} = \frac{p_2 - p_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

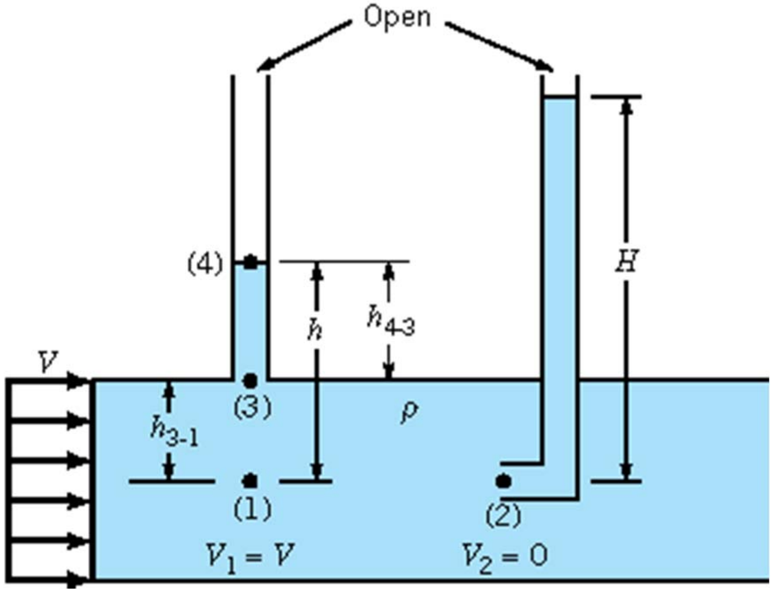
$$V_1 = \sqrt{2gh_3} = \sqrt{2(9.81\,\mathrm{m/s^2})(0.12\,\mathrm{m})} = 1.53\,\mathrm{m/s}$$

3.7 Bernoulli方程和动量方程例子

用**Pitot管(Pitot tube)**测流速。

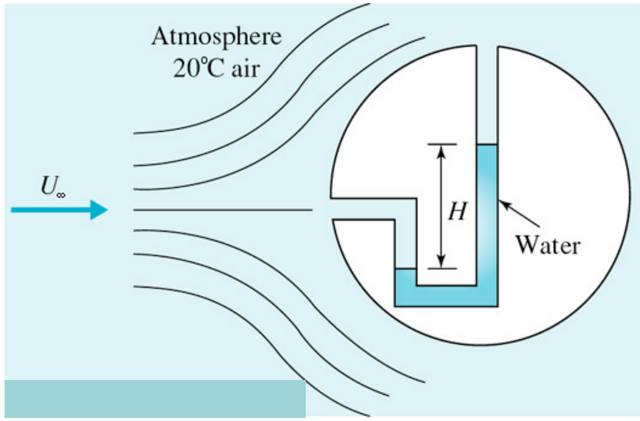
$$p_2 = p_1 + \frac{1}{2}\rho V_1^2 \quad \Rightarrow$$

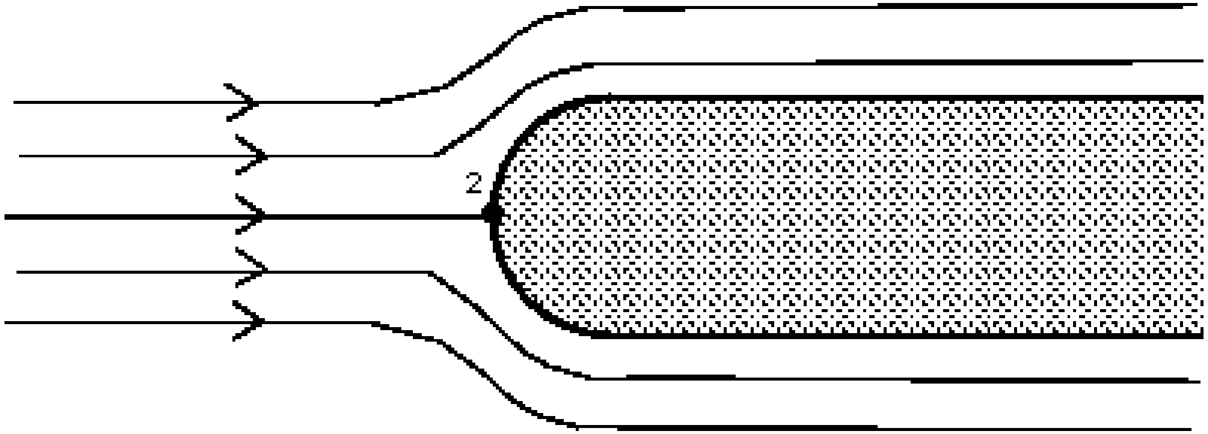
$$\rho g H = \rho g h + \frac{1}{2}\rho V^2 \quad \Rightarrow \quad V = \sqrt{2g(H-h)}$$

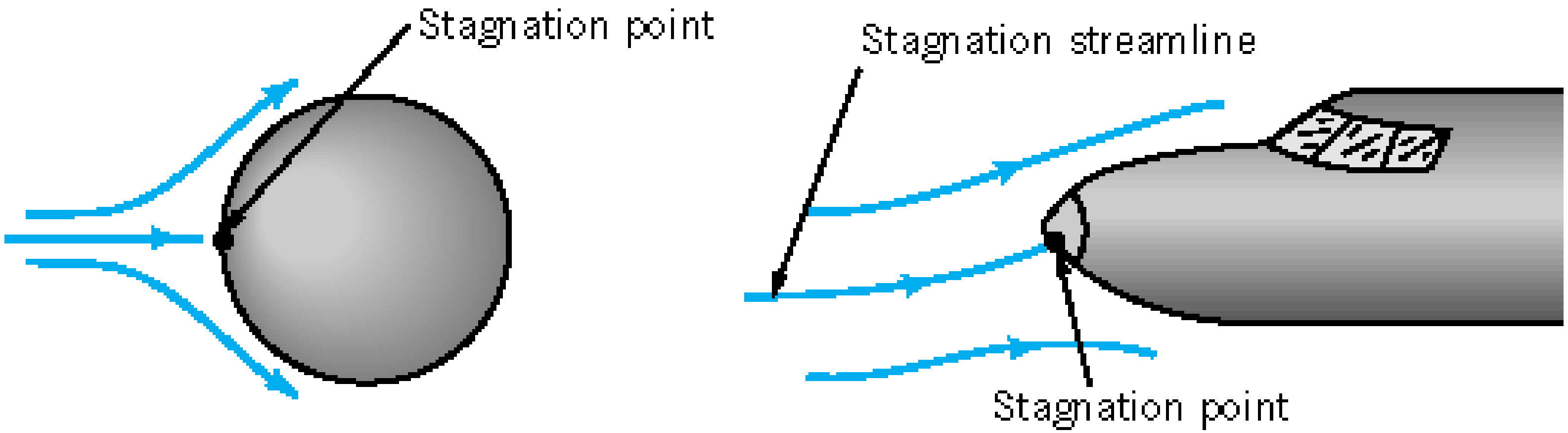


3.7 Bernoulli方程和动量方程例子

驻点(stagnation point)：流体流过钝体(blunt body)时，其前沿点处的流速为0，此点就称为驻点。



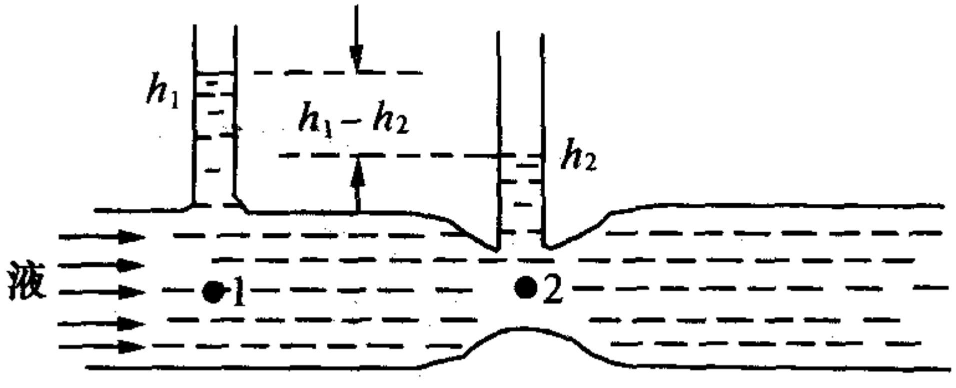




(a)　　　　(b)

3.7 Bernoulli方程和动量方程例子

例子4：用Venturi管(Venturi tube)测流量。如图所示，两个管子液面高度可以读出，试求管内的流量。



解：

如图所示，设管子粗、细两处的截面积、压强、流速分别为 S_1、P_1、υ_1 和 S_2、P_2、υ_2，管子粗细两处竖直管内的液面高度差为 $h = h_1 - h_2$，根据水平管伯努利方程有：

$$P_1 + \frac{1}{2}\rho\upsilon_1^2 = P_2 + \frac{1}{2}\rho\upsilon_2^2 \qquad P_1 - P_2 = \rho g(h_1 - h_2) = \rho g h$$

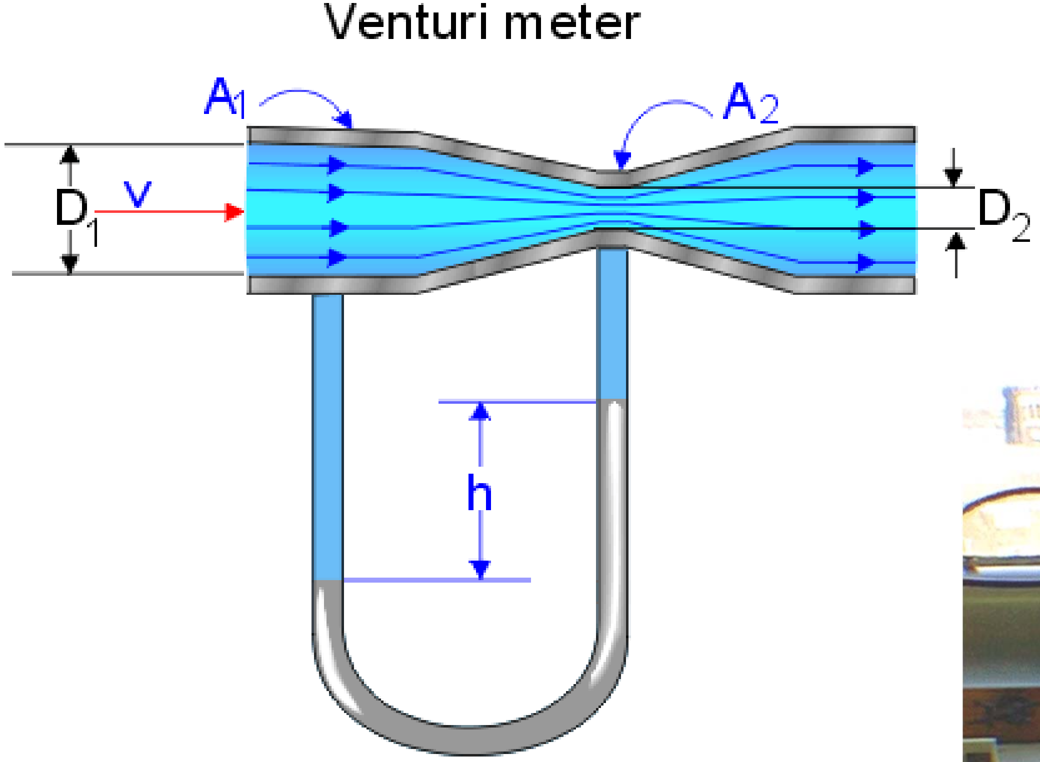
根据连续性方程 $S_1\upsilon_1 = S_2\upsilon_2$ 得：

$$\upsilon_1 = S_2\sqrt{\frac{2(P_1 - P_2)}{\rho(S_1^2 - S_2^2)}}$$

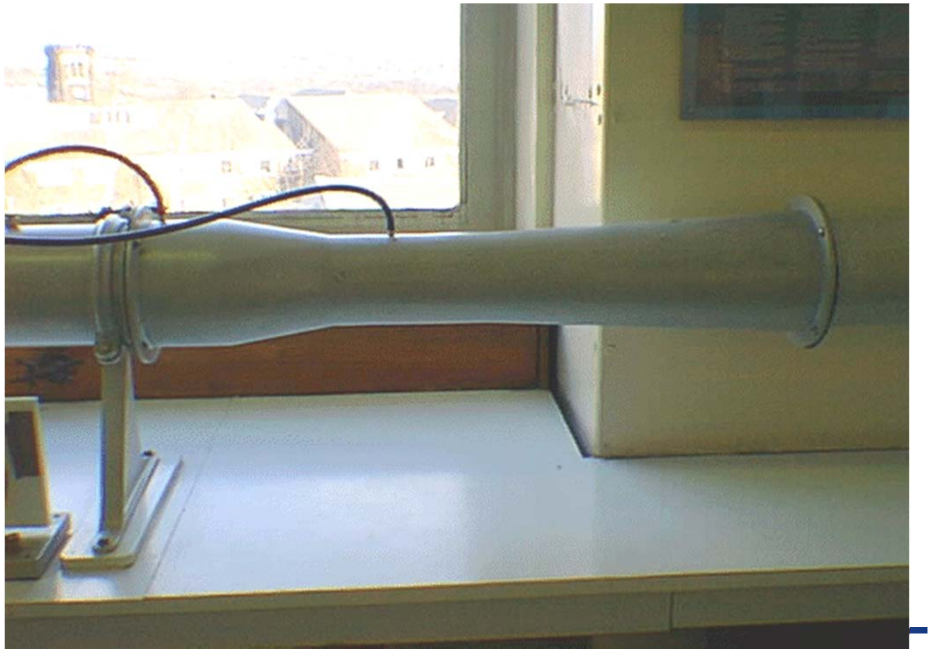
流量：$$Q = S_1\upsilon_1 = S_1 S_2\sqrt{\frac{2gh}{S_1^2 - S_2^2}}$$

3.7 Bernoulli方程和动量方程例子

用**Venturi管(Venturi tube)**测流量。

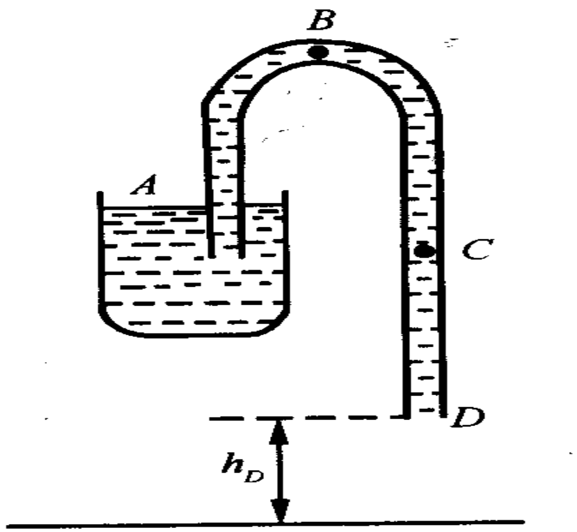


$$Q = A_1 A_2 \sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$



3.7 Bernoulli方程和动量方程例子

例子5： 虹吸管(siphoning tube)是用来从不能倾斜的容器中排出液体的装置。问什么条件下，虹吸管能够正常工作？



解：

（1）流体流速

视液体为理想流体，且排水管均匀，对容器内液面**A**和管口**D**，应用伯努利方程得：

$$\frac{1}{2}\rho\upsilon_A^2 + \rho g h_A + P_0 = \frac{1}{2}\rho\upsilon_D^2 + \rho g h_D + P_0$$

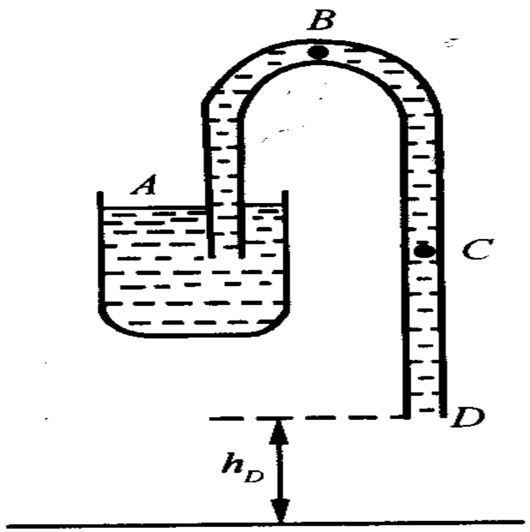
因为$S_A >> S_D$，由连续性方程可知：

$$\upsilon_A^2 << \upsilon_D^2$$

3.7 Bernoulli方程和动量方程例子

由上式得出管口处的流速为：

$$\upsilon_D = \sqrt{2g(h_A - h_D)} = \sqrt{2gh_{AD}}$$



（2）压强和高度的关系

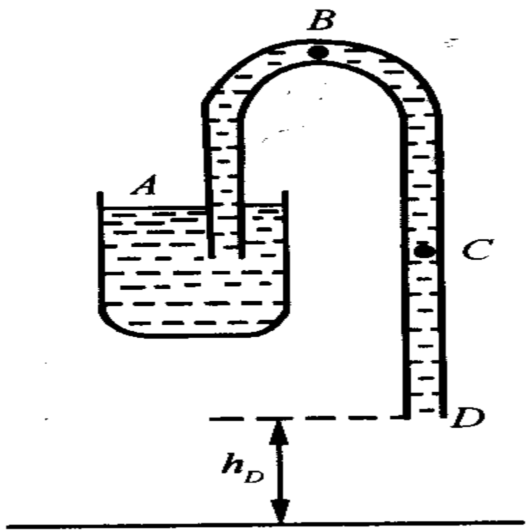
由于管子粗细均匀，由连续性方程知，$\upsilon_B = \upsilon_C = \upsilon_D$，对于B、C两点，应用伯努利方程有：

$$\rho g h_B + P_B = \rho g h_c + P_c \quad \Rightarrow \quad \rho g h + P = C$$

结论：粗细均匀的虹吸管中，处于较高处液面的压强小于较低处液面的压强。

3.7 Bernoulli方程和动量方程例子

对于A、B两处，应用伯努利方程, 由于 $\upsilon_A^2 << \upsilon_B^2$ ，有：



$$\rho g h_B + P_B + \frac{1}{2}\rho\upsilon_B^2 = \rho g h_A + P_0$$

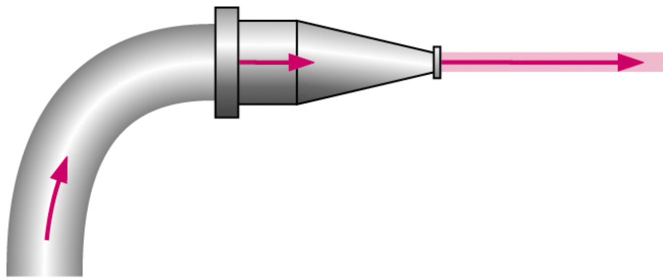
$$h_A - h_B = \frac{1}{\rho g}(P_B - P_0) + \frac{1}{2g}\upsilon_B^2$$

当$P_B=0$时，(h_B-h_A)有最大值，这是虹吸管能够正常工作的条件，即排水管的最高点与容器中液面之间的高度只能小于：

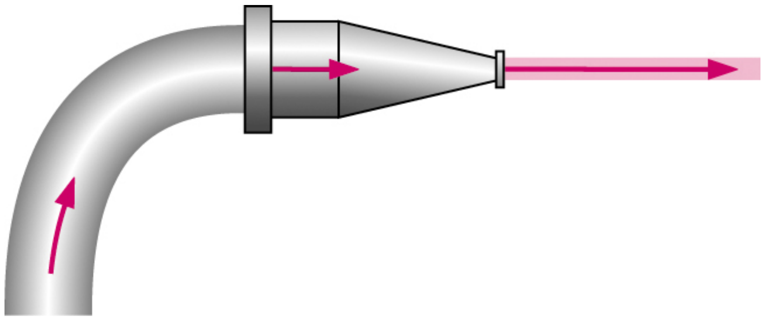
$$h_B - h_A = \frac{P_0}{\rho g} - \frac{\upsilon_B^2}{2g} < \frac{P_0}{\rho g}$$

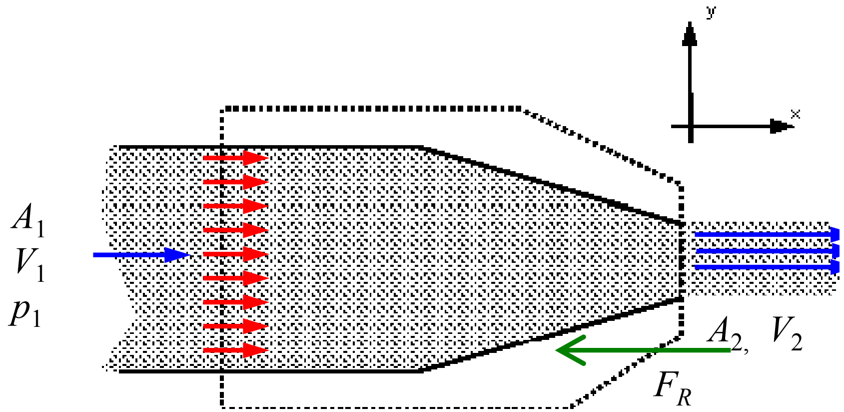
3.7 Bernoulli方程和动量方程例子

例子6：如图一个消防喷水咀，假定喷水流量为Q，喷水管横截面积为A_1，喷水咀口处面积为A_2，问消防员需要多大的握力，才能握住这个消防喷水咀？



3.7 Bernoulli方程和动量方程例子





解：如图取控制体，根据动量守恒，这个控制体受到的总作用力为：

$$\sum F = \rho Q (V_2 - V_1)$$

由质量守恒有： $Q = A_1 V_1 = A_2 V_2$ ⇒ $\sum F = \rho Q^2 \left(\frac{1}{A_2} - \frac{1}{A_1} \right)$

对喷水咀和水管内截面，用**Bernoulli**方程，可得：

$$\frac{p_1}{\rho g} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{V_2^2}{2g}$$

3.7 Bernoulli方程和动量方程例子

由于喷水咀水平，喷水咀口处压力为大气压，即p_2=0，所以

$$p_1 = \frac{\rho}{2}\left(V_2^2 - V_1^2\right) = \frac{\rho Q^2}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)$$

消防员的握力为F_R，因此：

$$\sum F = -F_R + p_1 A_1 = \rho Q^2 \left(\frac{1}{A_2} - \frac{1}{A_1}\right)$$

$$\Rightarrow F_R = \frac{\rho Q^2 A_1}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right) - \rho Q^2 \left(\frac{1}{A_2} - \frac{1}{A_1}\right)$$