Class-6

NA26018

Finite Element Analysis of Solids and Fluids

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Gauss-Legendre Quadrature

- **In the Newton-Cotes quadrature, the base point locations have been specified. If the x_l are not specified, then there will be $2r + 2$ undetermined parameters, $r + 1$ weights wl and $r + 1$ base points x_l, which define a polynomial of degree $2r + 1$**

- **The Gauss-Legendre quadrature is based on the idea that the base points xl and the weights w can be chosen so that the sum of the $r + 1$ appropriately weighted values of the function yields the integral exactly when $F(x)$ is a polynomial of degree $2r + 1$ or less**

The Gauss-Legendre quadrature formula is given by

$$\int_a^b F(x)dx = \int_{-1}^{1} \hat{F}(\xi)d\xi \approx \sum_{\mathrm{I}=1}^{r} \hat{F}(\xi_I)w_I$$

Numerical integration

$$\int_a^b F(x)dx = \int_{-1}^{1} \hat{F}(\xi)d\xi \approx \sum_{I=1}^{r} \hat{F}(\xi_I)w_I$$

where wI are the weight factors, $\xi_I$are the base points [roots of the Legendre polynomial $P_{r+1}(\xi)$], and F is the transformed integrand

$$\hat{F}(\xi) = F(x(\xi))J(\xi), dx = Jd\xi$$

where J is the Jacobian of the transformation between x anbd ξ. The weight factors and Gauss points for the Gauss-Legendre quadrature are given for $r = 1 \sim 6$ in Table

Numerical integration

Weights and Gauss points for the Gauss–Legendre quadrature.

$$\int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{r} F(\xi_i)w_i$$

Points, $\xi_i^\dagger$	r	Weights, w_i
0.0000000000	1	2.0000000000
±0.5773502692	2	1.0000000000
0.0000000000	3	0.8888888889
±0.7745966692		0.5555555555
±0.3399810435	4	0.6521451548
±0.8611363116		0.3478548451
0.0000000000	5	0.5688888889
±0.5384693101		0.4786286705
±0.9061798459		0.2369268850
±0.2386191861	6	0.4679139346
±0.6612093865		0.3607615730
±0.9324695142		0.1713244924

†Note that $0.57735\ldots = 1/\sqrt{3}$, $0.77459\ldots = \sqrt{3/5}$, and $0.888\ldots = 8/9$, and $0.555\ldots = 5/9$.

The Gauss-Legendre quadrature is more frequently used than the Newton-Cotes quadrature because it requires fewer base points（hence, a saving in computation） to achieve the same accuracy

- **A polynomial of degree p is integrated exactly by employing $r = 0.5(p+1)$ Gauss points. When p+1 is odd, one should pick the nearest larger integer**

$$r = \left\lceil \frac{1}{2}(p+1) \right\rceil$$

Numerical integration

$$\int_{x_a}^{x_b} F(x)dx = \int_{-1}^{1} \hat{F}\,(\xi)d\xi, \qquad \hat{F}(\xi)d\xi = F(x(\xi))dx$$

so that the Gauss-Legendre quadrature can be used to evaluate the integral over $[-1, 1]$. The differential element dx in the global coordinate system x is related to the differential element $d\xi$ in the natural coordinate system ξ by

$$dx = \frac{dx}{d\xi}d\xi = J_e d\xi$$

$$J_e = \frac{dx}{d\xi} = \frac{d}{d\xi}\left(\sum_{i=1}^{m} x_i^e \hat{\psi}_i^e\right) = \sum_{i=1}^{m} x_i^e \frac{d\hat{\psi}_i^e}{d\xi}$$

$$x = \sum_{i=1}^{m} x_i^e \hat{\psi}_i^e(\xi)$$

Numerical integration

For example, consider the integral

$$K_{ij}^e = \int_{x_a}^{x_b} a(x)\, d\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\, dx$$

Using the chain rule of differentiation we have

$$\frac{d\psi_i^e(x)}{dx} = \frac{d\psi_i^e(\xi)}{d\xi}\frac{d\xi}{dx} = J^{-1}\frac{d\psi_i^e(\xi)}{d\xi}$$

Since

$$x = \sum_{i=1}^{m} x_i^e \hat{\psi}_i^e \qquad dx = \frac{dx}{d\xi}d\xi = J_e d\xi$$

$$K_{ij}^e = \int_{-1}^{1} a(x(\xi))\frac{1}{J}\frac{d\psi_i^e}{d\xi}\frac{1}{J}\frac{d\psi_j^e}{d\xi}J d\xi \approx \sum_{I=1}^{r}\hat{F}_{ij}^e(\xi_I)w_I$$

where

$$\hat{F}_{ij}^e = a\frac{1}{J}\frac{d\psi_i^e}{d\xi}\frac{d\psi_j^e}{d\xi}, J = \sum_{i=1}^{m} x_i^e \frac{d\hat{\psi}_i^e}{d\xi}$$

Numerical integration

Determine the exact number of Gauss points required to evaluate the following element coefficients

$$K_{ij}^e = \int_{x_a}^{x_b} \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}dx = \int_{-1}^{+1} \frac{d\psi_i^e}{d\xi}\frac{d\psi_j^e}{d\xi}(J)^{-2}Jd\xi \equiv \int_{-1}^{+1} G_{ij}^K(\xi)d\xi$$

$$\approx \sum_{I=1}^{N^K} G_{ij}^K(\xi_I)W_I$$

$$M_{ij}^e = \int_{x_a}^{x_b} \psi_i^e\psi_j^e dx = \int_{-1}^{+1} \psi_i^e(\xi)\psi_j^e(\xi)Jd\xi \equiv \int_{-1}^{+1} G_{ij}^M(\xi)d\xi$$

$$\approx \sum_{I=1}^{N^M} G_{ij}^M(\xi_I)W_I$$

$$f_i^e = \int_{x_a}^{x_b} \psi_i^e dx = \int_{-1}^{+1} \psi_i^e(\xi)Jd\xi \equiv \int_{-1}^{+1} G_{ij}^F(\xi)d\xi$$

$$\approx \sum_{I=1}^{N^F} G_{ij}^F(\xi_I)W_I$$

A polynomial of degree p is integrated exactly by employing $r = 0.5(p+1)$ Gauss points.

Element type	N^K	N^M	N^F
Linear	1	2	1
Quadratic	2	3	2
Cubic	3	4	2

Integration over a Master Rectangular Element

Quadrature formulas for integrals defined over a rectangular master element $\hat{\Omega}_R$ can be derived from the one-dimensional quadrature formulae. We have

$$\int_{\hat{\Omega}_R} F(\xi,\eta)d\xi d\eta = \int_{-1}^{1}\left[\int_{-1}^{1}\mathrm{F}(\xi,\eta)d\eta\right]d\xi \approx \int_{-1}^{1}\left[\sum_{J=1}^{N}F(\xi,\eta_J)W_J\right]d\xi$$

$$\approx \sum_{I=1}^{M}\sum_{J=1}^{N}F(\xi_I,\eta_J)W_I W_J$$

where M and N denote the number of quadrature points in the ξ and η directions, (ξ_I,η_J) denote the Gauss points, and W_I and W_J, denote the corresponding Gauss weights

Numerical integration

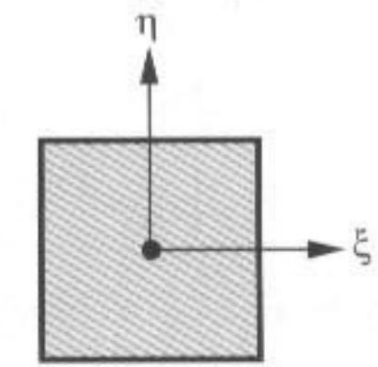
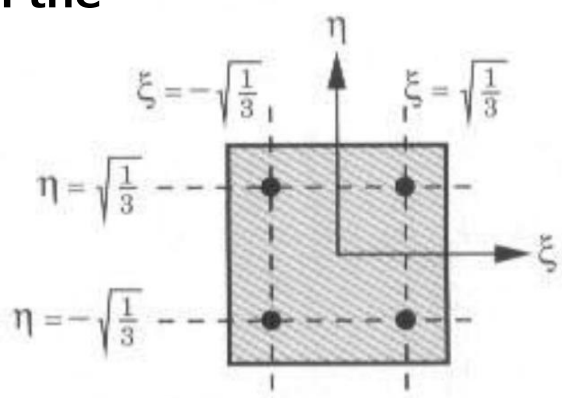
$$\int_{\hat{\Omega}_R} F(\xi,\eta)d\xi d\eta = \int_{-1}^{1}\left[\int_{-1}^{1}\mathrm{F}(\xi,\eta)d\eta\right]d\xi \approx \int_{-1}^{1}\left[\sum_{J=1}^{N} F(\xi,\eta_J)W_J\right]d\xi$$

$$\approx \sum_{I=1}^{M}\sum_{J=1}^{N} F(\xi_I,\eta_J)W_I W_J$$

The selection of the number of Gauss points is based on the same formula as that given in 1-D

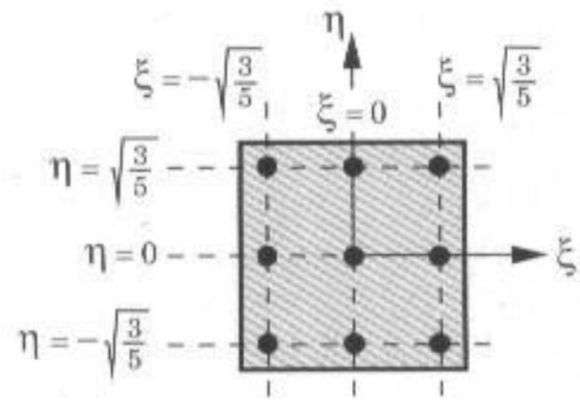
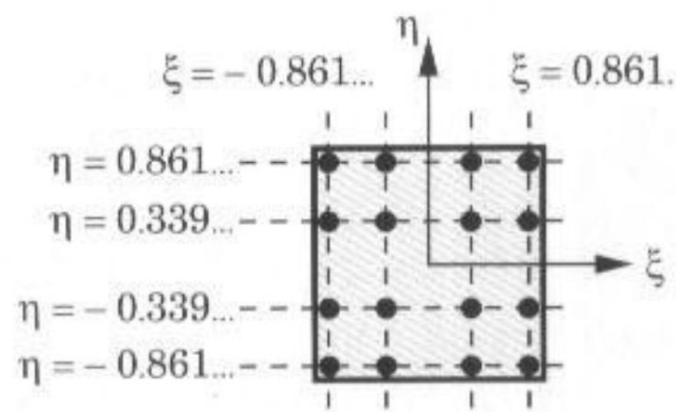
A polynomial of degree p is integrated exactly employing $N = int[0.5(p+1)]$**. In most cases, the interpolation functions are of the same degree in both** ξ **and** η**, and therefore** $M = N$**. When the integrand is of different degree in** ξ **and** η**, the number of Gauss points is selected on the basis of the largest-degree polynomial**

Numerical integration

Element type	Maximum polynomial degree	Order of integration $(r \times r)$	Order of the residual	Location of integration points* in master element
Constant $(r = 1)$	0	1×1	$O(h^2)$	
Linear $(r = 2)$	2	2×2	$O(h^4)$	

The maximum degree of the polynomial refers to the degree of the highest polynomial in ξ or η in the integrands of the element matrices

Numerical integration

Element type	Maximum polynomial degree	Order of integration ($r \times r$)	Order of the residual	Location of integration points* in master element
Quadratic ($r = 3$)	4	(3×3)	$O(h^6)$	
Cubic ($r = 4$)	6	(4×4)	$O(h^8)$	

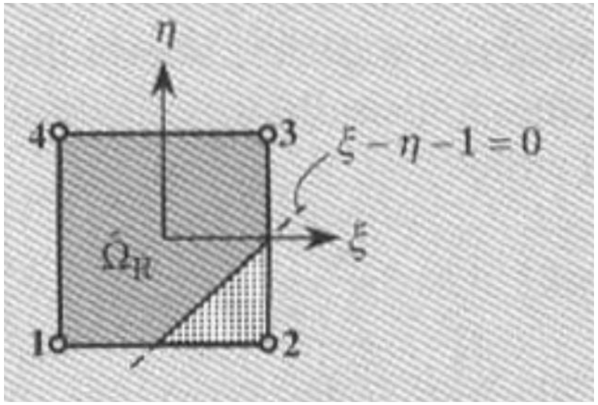
Numerical integration

The $N \times N$ gauss point locations are given by the tensor product of one-dimensional Gauss points ξ_i

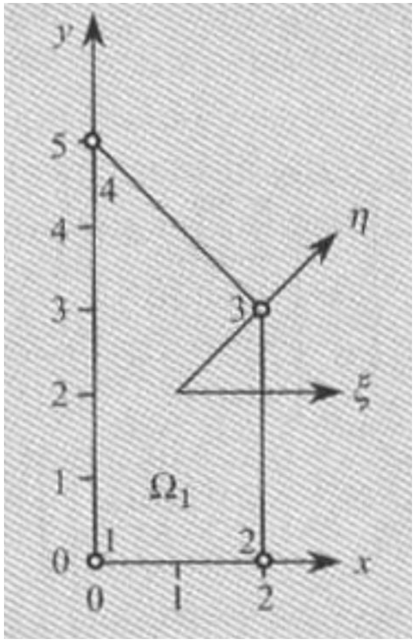
$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{Bmatrix} \{\xi_1, \xi_2, \dots, \xi_N\} \equiv \begin{bmatrix} (\xi_1, \xi_1) & (\xi_1, \xi_2) & \cdots & (\xi_1, \xi_N) \\ (\xi_2, \xi_1) & \ddots & & \vdots \\ \vdots & & & \\ (\xi_N, \xi_1) & \cdots & & (\xi_N, \xi_N) \end{bmatrix}$$

Example

Consider the quadrilateral element Ω_1. We wish to evaluate a $\partial\Psi_i/\partial x$ and $\partial\Psi_i/\partial y$ at (ξ,η)=(0,0) using the isoparametric formulation



$$\Psi_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$\Psi_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$\Psi_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$\Psi_4 = \frac{1}{4}(1-\xi)(1+\eta)$$



Numerical integration

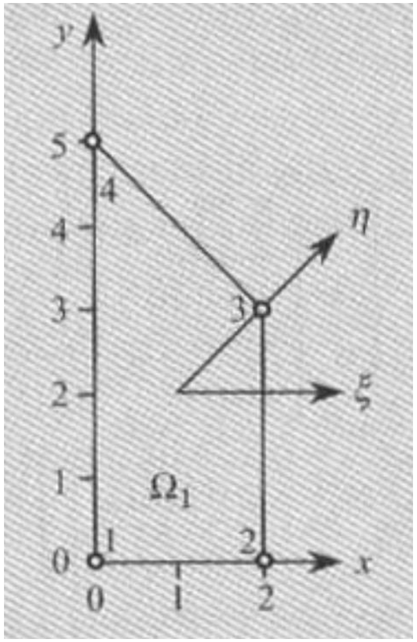
Recall that,

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^{m} x_j \frac{\partial \hat{\psi}_j^e}{\partial \xi}, \qquad \frac{\partial y}{\partial \xi} = \sum_{j=1}^{m} y_j \frac{\partial \hat{\psi}_j^e}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = \sum_{j=1}^{m} x_j \frac{\partial \hat{\psi}_j^e}{\partial \eta}, \qquad \frac{\partial y}{\partial \eta} = \sum_{j=1}^{m} y_j \frac{\partial \hat{\psi}_j^e}{\partial \eta}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \xi} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \xi} \\ \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \eta} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \hat{\psi}_1}{\partial \xi} & \frac{\partial \hat{\psi}_2}{\partial \xi} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \xi} \\ \frac{\partial \hat{\psi}_1}{\partial \eta} & \frac{\partial \hat{\psi}_2}{\partial \eta} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$



Numerical integration

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} 0.0 & 0.0 \\ 2.0 & 0.0 \\ 2.0 & 3.0 \\ 0.0 & 5.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2}(1+\eta) \\ 0 & \frac{1}{2}(4-\xi) \end{bmatrix}$$

The inverse of the Jacobian matrix is given by

$$[J]^{-1} = \begin{bmatrix} 1 & \frac{1+\eta}{4-\xi} \\ 0 & \frac{2}{4-\xi} \end{bmatrix}, \quad J_{11}^{*} = 1, \quad J_{21}^{*} = 0, \quad J_{12}^{*} = \frac{1+\eta}{4-\xi}, \quad J_{22}^{*} = \frac{2}{4-\xi}$$

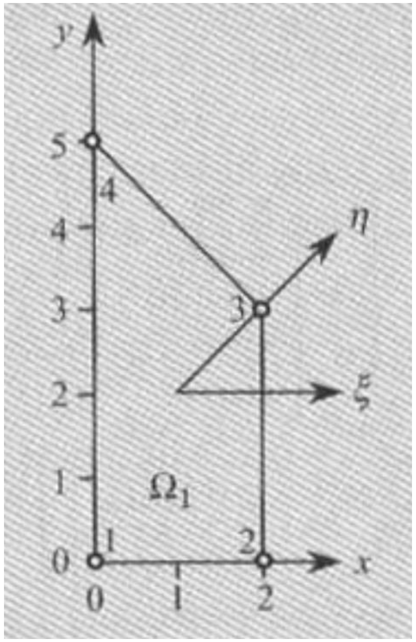
Numerical integration

Recall that,

$$\begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial \xi} \\ \dfrac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix}$$

$$\Psi_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$\Psi_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$\Psi_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$\Psi_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial \xi} + \frac{1+\eta}{4-\xi}\frac{\partial \psi_i}{\partial \eta}, \qquad \frac{\partial \psi_i}{\partial y} = \frac{2}{4-\xi}\frac{\partial \psi_i}{\partial \eta}$$



with

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i),$$
$$\frac{\partial \psi_i}{\partial \xi} = \frac{1}{4}\xi_i(1+\eta\eta_i), \qquad \frac{\partial \psi_i}{\partial \eta} = \frac{1}{4}\eta_i(1+\xi\xi_i)$$

Numerical integration

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{4}\xi_i(1+\eta\eta_i) + \frac{1}{4}\left(\frac{1+\eta}{4-\xi}\right)\eta_i(1+\xi\xi_i)$$

$$\frac{\partial \psi_i}{\partial y} = \frac{1}{4}\frac{2}{(4-\xi)}\eta_i(1+\xi\xi_i)$$

Thus, $\frac{\partial \psi_i}{\partial x}$ and $\frac{\partial \psi_i}{\partial y}$ at (ξ,η)=(0,0) is

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{4}\xi_i + \frac{1}{16}\eta_i\,, \qquad \frac{\partial \psi_i}{\partial y} = \frac{1}{8}\eta_i$$

Example

Consider the quadrilateral element in Fig. We wish to compute the following element matrices using the Gauss Legendre quadrature

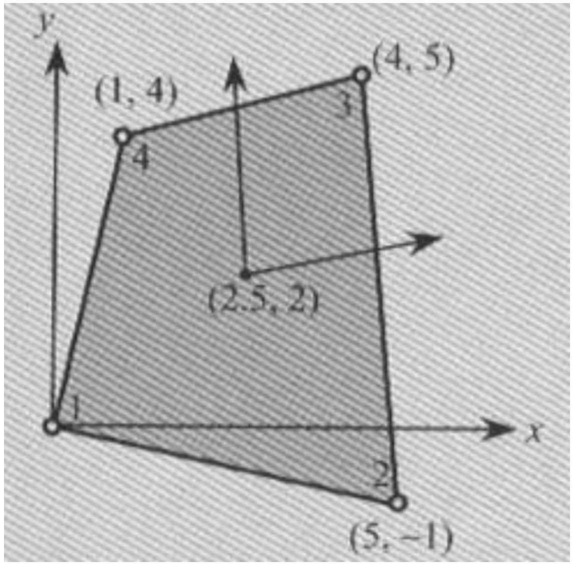
$$S_{ij}^{00} = \int_\Omega \psi_i \psi_j dxdy, \qquad S_{ij}^{11} = \int_\Omega \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} dxdy$$

$$S_{ij}^{22} = \int_\Omega \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} dxdy, \qquad S_{ij}^{12} = \int_\Omega \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial y} dxdy$$

The transformation equations are

$$x = 0 \cdot \hat{\psi}_1 + 5\hat{\psi}_1 + 4\hat{\psi}_3 + 1 \cdot \hat{\psi}_4 = \frac{1}{4}(10 + 8\xi - 2\xi\eta)$$

$$y = 0 \cdot \hat{\psi}_1 - 1 \cdot \hat{\psi}_2 + 5\hat{\psi}_3 + 4\hat{\psi}_4 = \frac{1}{4}(8 + 10\eta + 2\xi\eta)$$



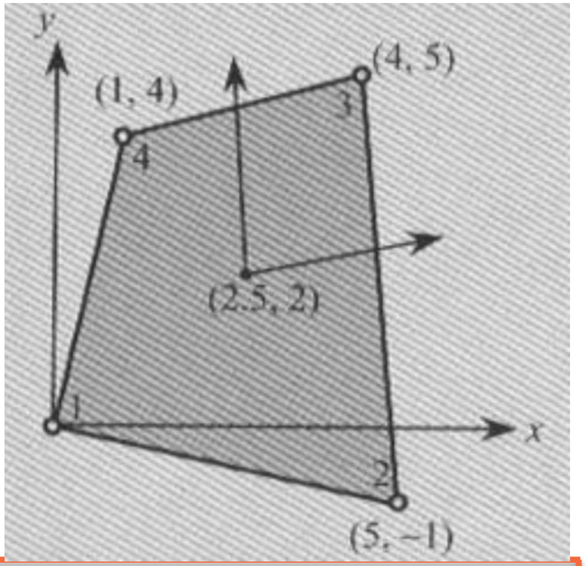
Numerical integration

The jacobian matrix and its inverse are

$$[J]=\begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}=\frac{1}{4}\begin{bmatrix}8-2\eta & 2\eta\\ -2\xi & 10+2\xi\end{bmatrix}, J=\frac{1}{4}[(4-\eta)(5+\xi)+\xi\eta]=\frac{1}{4}(20+4\xi-5\eta)$$

$$[J]^{-1}=\frac{1}{4J}\begin{bmatrix}10+2\xi & -2\eta\\ 2\xi & 8-2\eta\end{bmatrix},\quad J_{11}^{*}=\frac{10+2\xi}{20+4\xi-5\eta},\qquad J_{12}^{*}=\frac{2\eta}{20+4\xi-5\eta},$$

$$J_{21}^{*}=\frac{2\xi}{20+4\xi-5\eta},\qquad J_{22}^{*}=\frac{8-2\eta}{20+4\xi-5\eta}$$



Numerical integration

The matrix $[J]$ transforms base vectors $\hat{e}_x$=(1, 0) and $\hat{e}_x$=(0, 1) in the $x-y$ system to the base vectors $\hat{e}_\xi$ and $\hat{e}_\eta$ in the $\xi-\eta$ system,

$$\frac{1}{4}\begin{bmatrix} 8-2\eta & 2\eta \\ -2\xi & 10+2\xi \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} 8-2\eta \\ 2\eta \end{Bmatrix}, \qquad \frac{1}{4}\begin{bmatrix} 8-2\eta & 2\eta \\ -2\xi & 10+2\xi \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} -2\xi \\ 10+2\xi \end{Bmatrix}$$

$$\hat{e}_\xi = \frac{1}{4}\left[(8-2\eta)\hat{e}_x + 2\eta\hat{e}_y\right], \hat{e}_\eta = \frac{1}{4}\left[-2\xi\hat{e}_x + (10+2\xi)\hat{e}_y\right]$$

Hence, the area element dxdy in the $x-y$ system is related to the area element $d\xi\ d\eta$ in the $\xi-\eta$ system by

$$dxdy = \frac{1}{16}\begin{vmatrix} 8-2\eta & -2\xi \\ 2\eta & 10+2\xi \end{vmatrix} d\xi d\eta = J d\xi d\eta$$

Numerical integration

The coefficients S_{ij}^{00} and S_{ij}^{11}, can be expressed in natural coordinates (for numerical evaluation) as

$$S_{ij}^{00} = \int_{\Omega} \psi_i \psi_j dxdy = \int_{-1}^{1}\int_{-1}^{1} \psi_i \psi_j J d\xi d\eta$$

$$S_{ij}^{11} = \int_{\Omega} \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} dxdy = \int_{-1}^{1}\int_{-1}^{1} \left(J_{11}^{*}\frac{\partial \psi_i}{\partial \xi} + J_{12}^{*}\frac{\partial \psi_i}{\partial \eta}\right)\left(J_{11}^{*}\frac{\partial \psi_j}{\partial \xi} + J_{12}^{*}\frac{\partial \psi_j}{\partial \eta}\right) J d\xi d\eta$$

Recall in last Example,

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i), \frac{\partial \psi_i}{\partial \xi} = \frac{1}{4}(1+\eta\eta_i), \frac{\partial \psi_i}{\partial \eta} = \frac{1}{4}\eta_i(1+\xi\xi_i)$$

Note that the integrand of polynomial of the order $p = 3$ in each coordinate ξ and η. Hence. $N = M = 0.5(p+1) = 2$ will evaluate S_{ij} exactly

Numerical integration

$$S_{11}^{00} = \int_{\Omega} \psi_1 \psi_1 dxdy = \int_{-1}^{1}\int_{-1}^{1} \psi_1 \psi_1 J d\xi d\eta$$

$$= \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1} (1-\xi)^2(1-\eta)^2(20+4\xi-5\eta)d\xi d\eta$$

$$= \frac{1}{64}\sum_{i,j=1}^{2} (1-\xi_1)^2(1-\eta_1)^2(20+4\xi_1-5\eta_1)$$

where ξ_i and η_i are the Gauss points

$$(\xi_1,\eta_2) = \left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),(\xi_2,\eta_2) = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \qquad (\xi_1,\eta_1) = \left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),(\xi_2,\eta_1) = \left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

$$s_{11}^{00} = \frac{1}{64}\left[\left(1+\frac{1}{\sqrt{3}}\right)^4\left(20-\frac{4}{\sqrt{3}}+\frac{5}{\sqrt{3}}\right)+\left(1+\frac{1}{\sqrt{3}}\right)^2\left(1-\frac{1}{\sqrt{3}}\right)^2\left(20-\frac{4}{\sqrt{3}}-\frac{5}{\sqrt{3}}\right)\right.$$

$$= \frac{1}{64}\left[\frac{1120}{9}+\frac{160}{9}+\frac{32}{3\sqrt{3}}(-\frac{4}{\sqrt{3}}+\frac{5}{\sqrt{3}})\right] = \frac{1312}{576} = 2.27778$$

Numerical integration

$$S_{ij}^{11} = \int_{\Omega} \frac{\partial \psi_1}{\partial x}\frac{\partial \psi_2}{\partial x} dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left(J_{11}^{*}\frac{\partial \psi_1}{\partial \xi} + J_{12}^{*}\frac{\partial \psi_2}{\partial \eta}\right)\left(J_{11}^{*}\frac{\partial \psi_1}{\partial \xi} + J_{12}^{*}\frac{\partial \psi_2}{\partial \eta}\right) J d\xi d\eta$$

$$= \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}[-(10+2\xi)(1-\eta)+2\eta(1-\xi)][(10+2\xi)(1-\eta)+2\eta(1-\xi)]$$

$$\times \frac{1}{(20+4\xi-5\eta)} d\xi d\eta$$

$$= \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}[-(10+2\xi)^2(1-\eta)^2+4\eta^2(1-\xi)^2]\frac{1}{(20+4\xi-5\eta)} d\xi d\eta$$

By using 2 point Gauss integration,

$$S_{12}^{11} = \frac{1}{64}\int_{-1}^{1}\int_{-1}^{1}[-(10+2\xi)^2(1-\eta)^2+4\eta^2(1-\xi)^2]\frac{1}{(20+4\xi-5\eta)} d\xi d\eta$$

$$\approx \sum_{i,j=1}^{2}\left[-(10+2\xi_i)^2\left(1-\eta_j\right)^2+4{\eta_j}^2(1-\xi_i)^2\right]\frac{1}{64\left(20+4\xi_i-5\eta_j\right)}$$

Integration over a Master Triangular Element

In the preceding section we discussed numerical integration on quadrilateral elements that can be used to represent very general geometries as well as field variables in a variety of problems

Here we discuss numerical integration on triangular elements

- **Master triangular elements can be obtained in a natural way from associated master rectangular elements**

Since quadrilateral elements can be geometrically distorted, it is possible to distort a quadrilateral element to obtain a required triangular element by moving the position of the corner nodes to one of the neighboring nodes. In actual computation, this is achieved by assigning the same global node number to two corner nodes of the quadrilateral element

Numerical integration

We choose the unit right isosceles triangle as the master element

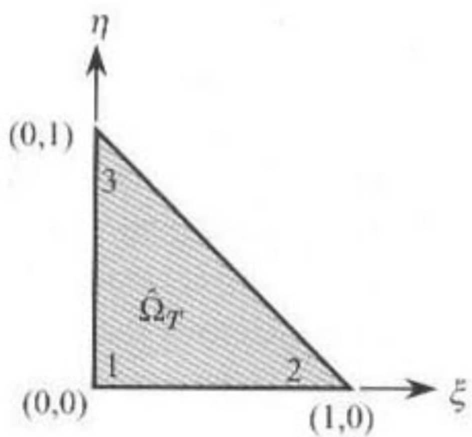
An arbitrary triangular element Ω^e can be generated from the master triangular element $\hat{\Omega}^T$ by transformation

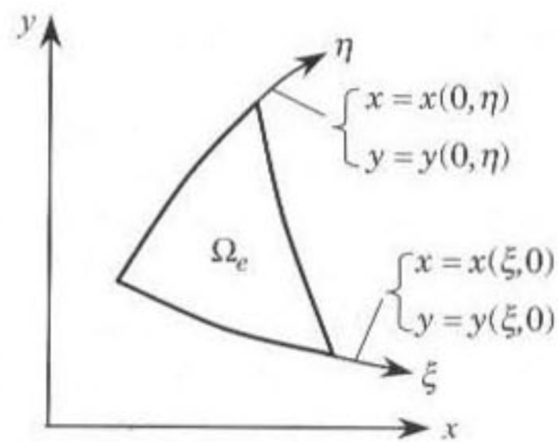
- **The coordinate lines $\xi = 0$ and $\eta = 0$ in $\hat{\Omega}^T$ correspond to the skew curvilinear coordinate lines 1-3 and 1-2 in Ω^e**

For the 3-node triangular element, the transformation is taken to be

$$x = \sum_{i=1}^{3} x_i \hat{\psi}_i(\xi,\eta), \qquad y = \sum_{i=1}^{3} y_i \hat{\psi}_i(\xi,\eta)$$

$$\hat{\psi}_1 = 1 - \xi - \eta, \qquad \hat{\psi}_2 = \xi, \qquad \hat{\psi}_3 = \eta$$





Numerical integration

The inverse transformation from element Ω^e to $\widehat{\Omega}^T$ is given by

$$\xi = \frac{1}{2A}[(x-x_1)(y_3-y_1)-(y-y_1)(x_3-x_1)]$$

$$\eta = \frac{1}{2A}[(x-x_1)(y_1-y_2)+(y-y_1)(x_2-x_1)]$$

where A is the area of Ω^e

The Jacobian matrix for the linear triangular element is given by

$$[J]^{-1} = \begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix} = \begin{bmatrix} \gamma_3 & -\beta_3 \\ -\gamma_2 & \beta_2 \end{bmatrix}$$

$$\left.\begin{aligned} \alpha_i &= x_j y_k - x_k y_j \\ \beta_i &= y_j - y_k \\ \gamma_i &= -(x_j - x_k) \end{aligned}\right\} (i \neq j \neq k; i, j, \text{ and } k \text{ permute in a natural order})$$

$$[J]^{-1} = \frac{1}{J}\begin{bmatrix} \beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{bmatrix}, J = \beta_2\gamma_3 - \gamma_2\beta_3 = 2A$$

Numerical integration

Recall that,

$$\hat{\psi}_1 = 1 - \xi - \eta, \qquad \hat{\psi}_2 = \xi, \qquad \hat{\psi}_3 = \eta$$

$$\begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial \xi} \\ \dfrac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix}$$

$$\frac{\partial \psi_1}{\partial x} = -\frac{\beta_2 + \beta_3}{2A} = \frac{\beta_1}{2A}, \qquad \frac{\partial \psi_1}{\partial y} = -\frac{\gamma_2 + \gamma_3}{2A} = \frac{\gamma_1}{2A}$$

$$\frac{\partial \psi_2}{\partial x} = \frac{\beta_2}{2A}, \qquad \frac{\partial \psi_2}{\partial y} = \frac{\gamma_2}{2A}, \qquad \frac{\partial \psi_3}{\partial x} = \frac{\beta_3}{2A}, \qquad \frac{\partial \psi_3}{\partial y} = \frac{\gamma_3}{2A}$$

Numerical integration

In a general case, the derivatives of ψ_i with respect to the global coordinates can be computed from the area coordinate (L_1, L_2) form

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial \psi_i}{\partial L_2}\frac{\partial L_2}{\partial x}$$
$$\frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_i}{\partial L_1}\frac{\partial L_1}{\partial y} + \frac{\partial \psi_i}{\partial L_2}\frac{\partial L_2}{\partial y}$$

$$\begin{Bmatrix} \dfrac{\partial \psi_i}{\partial x} \\ \dfrac{\partial \psi_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \psi_i}{\partial L_1} \\ \dfrac{\partial \psi_i}{\partial L_2} \end{Bmatrix}, [J] = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix}$$

Note that only L_1 and L_2 are treated as linearly independent coordinates, because L_3=1-L_1-L_2

Numerical integration

After transformation, integrals on $\hat{\Omega}_T$ have the form

$$\int_{\hat{\Omega}_T} G(\xi,\eta)d\xi d\eta = \int_{\hat{\Omega}_T} \hat{G}\,(L_1,L_2,L_3)dL_1 dL_2$$

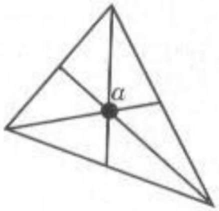
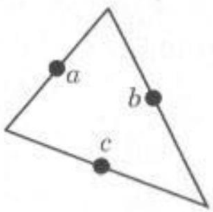
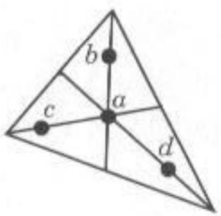
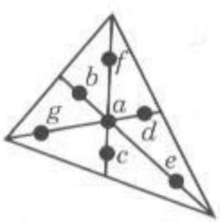
which can be approximated by the quadrature formula

$$\int_{\hat{\Omega}_T} \hat{G}\,(L_1,L_2,L_3)dL_1 dL_2 \approx \frac{1}{2}\sum_{l=1}^{N} W_I \hat{G}(S_l)$$

where W_I, and S_l denote the weights and integration points of the quadrature rule.

NEXT Table contains the location of integration points and weights for one-, three-, and seven-point quadrature rules over triangular elements

Numerical integration

Number of integration points	Degree of polynomial and order of the residual	Integration points and weights L_1	L_2	L_3	W	Nodes	Geometric locations
1	1; $O(h^2)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	a	
3	2; $O(h^3)$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$	a	
		$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	b	
		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	c	
4	3; $O(h^4)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{27}{48}$	a	
		0.6	0.2	0.2	$\frac{25}{48}$	b	
		0.2	0.6	0.2	$\frac{25}{48}$	c	
		0.2	0.2	0.6	$\frac{25}{48}$	d	
7	5; $O(h^6)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0.225	a	
		$\alpha_1^{\dagger}$	β_1	β_1		b	
		β_1	α_1	β_1	W_2	c	
		β_1	β_1	α_1		d	
		α_2	β_2	β_2		e	
		β_2	α_2	β_2	W_3	f	
		β_2	β_2	α_2		g	

Modeling considerations

Finite element analysis is a numerical simulation of a physical process. Therefore, finite element modeling involves assumptions concerning the representation of the system and/or its behavior.

- Valid assumptions can be made only if we have a qualitative understanding of how the process or system works
- A good knowledge of the basic principles governing the process and the finite element theory enable the development of a good numerical model of the actual process

Here we discuss several aspects of development of finite element models. Guidelines concerning element geometries, mesh refinements, and load representations are given

Element Geometries

Recall that the numerical evaluation of integrals over actual elements involves a coordinate transformation from the actual element to a master element

- The transformation is acceptable if and only if every point in the actual element is mapped uniquely into a point in the master element, and vice versa. Such mappings are termed one-to-one. This requirement can be expressed as

$$J^e \equiv \det[J^e] > 0 \qquad \text{everywhere in the element } \boldsymbol{\Omega_e}$$

where $[J^e]$ is the Jacobian matrix. Geometrically, the Jacobian represents the ratio of an area element in the real element to the corresponding area element in the master element

$$dA \equiv dxdy = J^e d\xi d\eta$$

NOTE: If J^e is zero, then a nonzero area element in the real element is mapped into zero area in the master element, which is unacceptable. Also, if $J^e < 0$, a right-handed coordinate system is mapped into a left-handed coordinate system

Modeling considerations

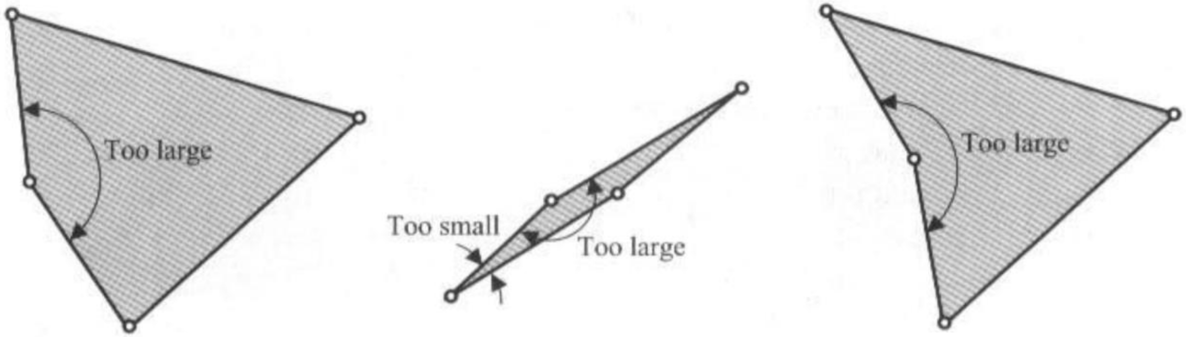
To ensure $J^e > 0$ and keep within the extreme limits of acceptable distortion, certain geometric shapes of real elements must be avoided. For example,

- **The interior angle at each vertex of a triangular element should not be equal to either 0° or 180°**

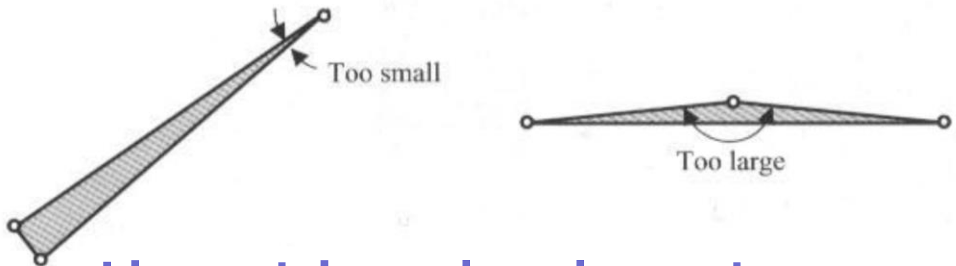
Indeed, in practice the angle should reasonably be larger than 0° and smaller than 180° to avoid numerical ill conditioning of element matrices. Although the acceptable range depends on the problem, the range 15°-165° can be used as a guide

Modeling considerations

e.g. Finite elements with unacceptable vertex angles



Linear quadrilateral elements



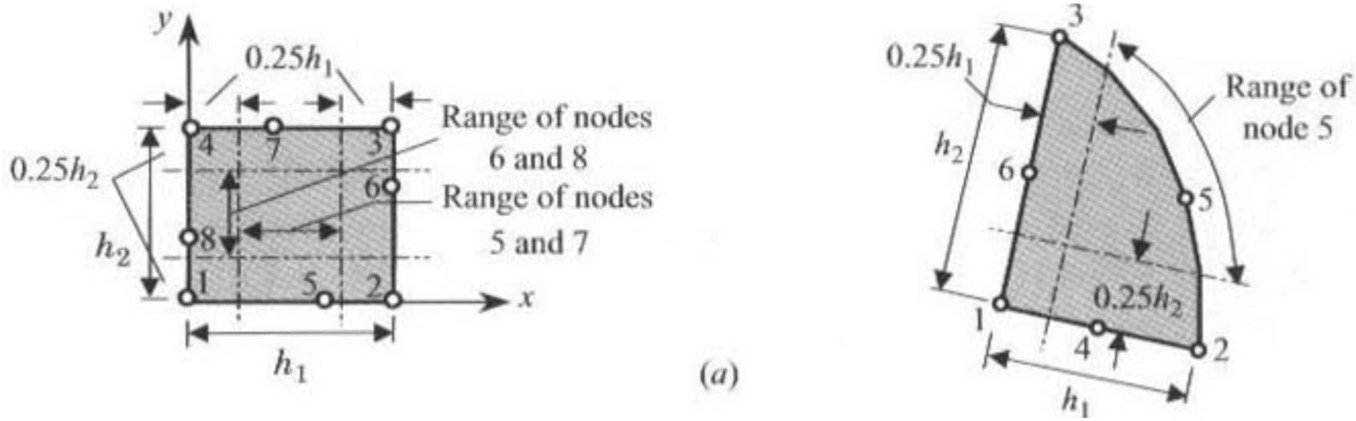
Linear triangular elements

Quadratic triangular elements

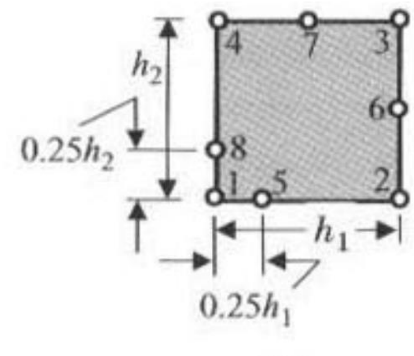
Modeling considerations

For higher-order Lagrange elements, the locations of the interior nodes contribute to the element distortion, and therefore they are constrained to lie within certain distance from the vertex nodes.

e.g., in the case of a quadratic element, the midside node should be at a distance not less than one-fourth of the length of the side from the vertex nodes



(a)



(b)

Eight node quadratic element and six-node quadratic triangular element

The quarter point quadrilateral element

Range of acceptable locations of the midside nodes for quadratic elements

Mesh Generation

Generation of a finite element mesh for a given problem should follow the guidelines listed below

1. The mesh should represent the geometry of the computational domain and load representation accurately
2. The mesh should be such that large gradients in the solution are adequately represented
3. The mesh should not contain elements with unacceptable geometries, especially in regions of large gradients

Within the above guidelines, the mesh used can be coarse or refined, and may consist of one or more orders and types of elements(e. g, linear and quadratic, triangular and quadrilateral)

Modeling considerations

A judicious choice of element order and type could save computational cost while giving accurate results.

- **It should be noted that the choice of elements and mesh is problem-dependent what works well for one problem may not work well for another problem**
- **An analyst with physical insight into the process being simulated can make a better choice of elements and mesh for the problem at hand**
- **We should start with a coarse mesh that meets the three requirements listed above exploit symmetries available in the problem, and evaluate the results thus obtained in light of physical understanding and approximate analytical and/or experimental information (these results can be used to guide subsequent mesh refinements and analyses)**

Modeling considerations

Generation of meshes of single element type is easy because elements of the same degree are compatible with each other

- **Mesh refinements involve several options. Refine the mesh by subdividing existing elements into two or more elements of the same type(h-version mesh refinement)**
- **Alternatively, existing elements can be replaced by elements of higher order (p-version mesh refinement)**

Generally, local mesh refinements should be such that very small elements are not placed adjacent to very large ones

h-version

p-version

Modeling considerations

Combining elements of different kinds naturally arises in solid and structural mechanics problems

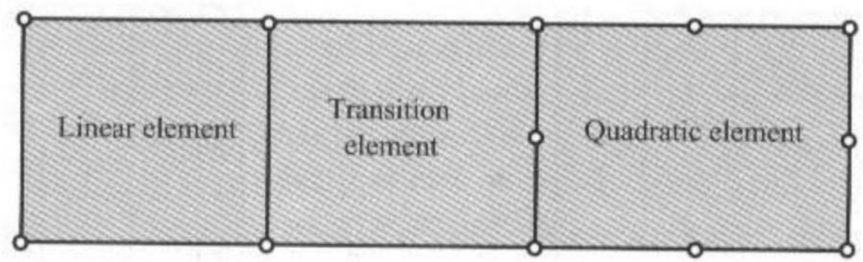
E.g., plate bending elements （2-D） can be connected to a beam element（1-D）

If the plate element is based on the classical plate theory, the beam element should be one based on the Euler-Bernoulli beam theory so that they have the same degrees of freedom at the connecting node. When a plane elasticity element is connected to a beam element, which are not compatible with the former in terms of the degrees of freedom at the nodes, we must construct a special element that makes the transition from the 2-D plane elasticity element to the 1-D beam element. Such elements are called transition elements

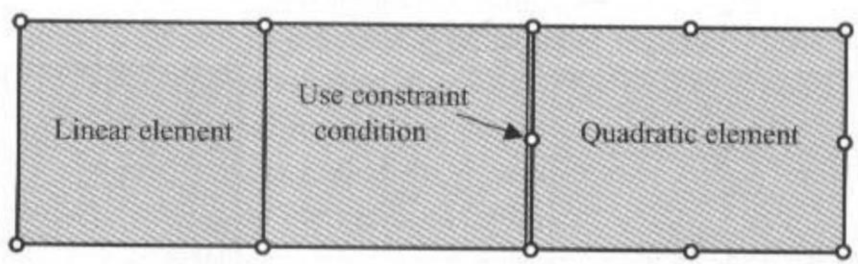
Modeling considerations

Combining elements of different order, say linear to quadratic elements, may be necessary to accomplish local mesh refinements. There are two ways to do this.

- **One way is to use a transition element, which has different number of nodes on different sides**
- **The other way is to impose a condition that constrains the midside node to have the same value as that experienced at the node by the lower-order element**



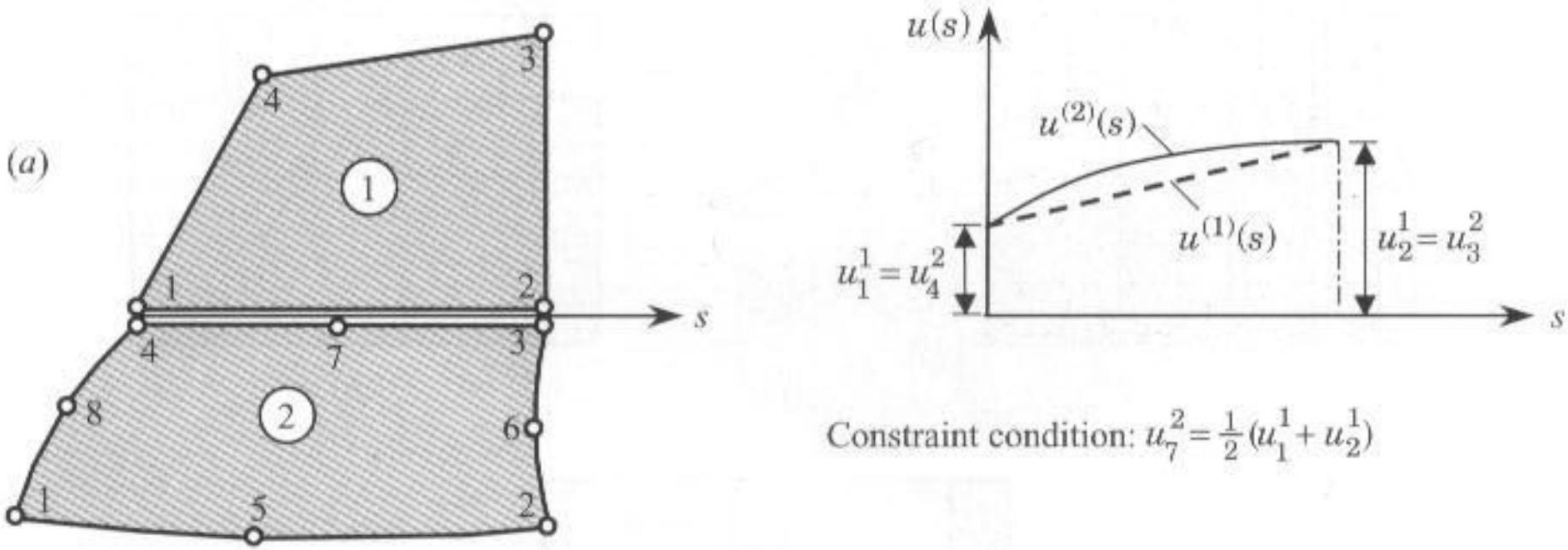
use of a transition element that has three sides linear and one side quadratic



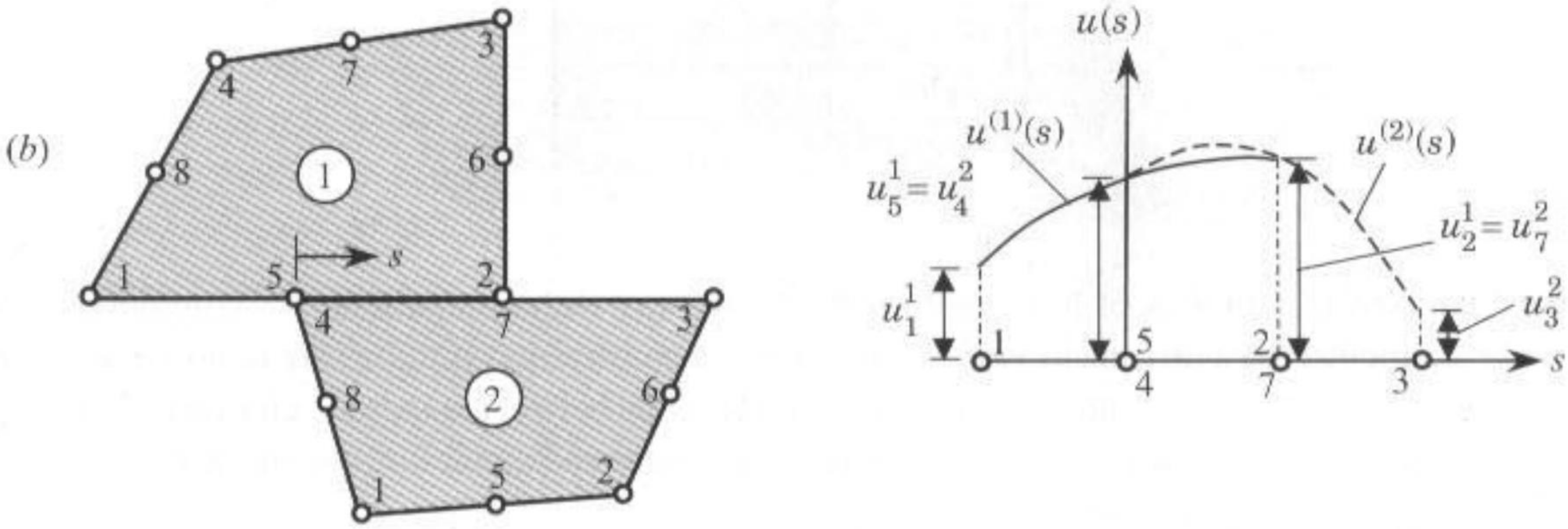
use of a linear constraint equation to connect a linear side to a quadratic side

Modeling considerations

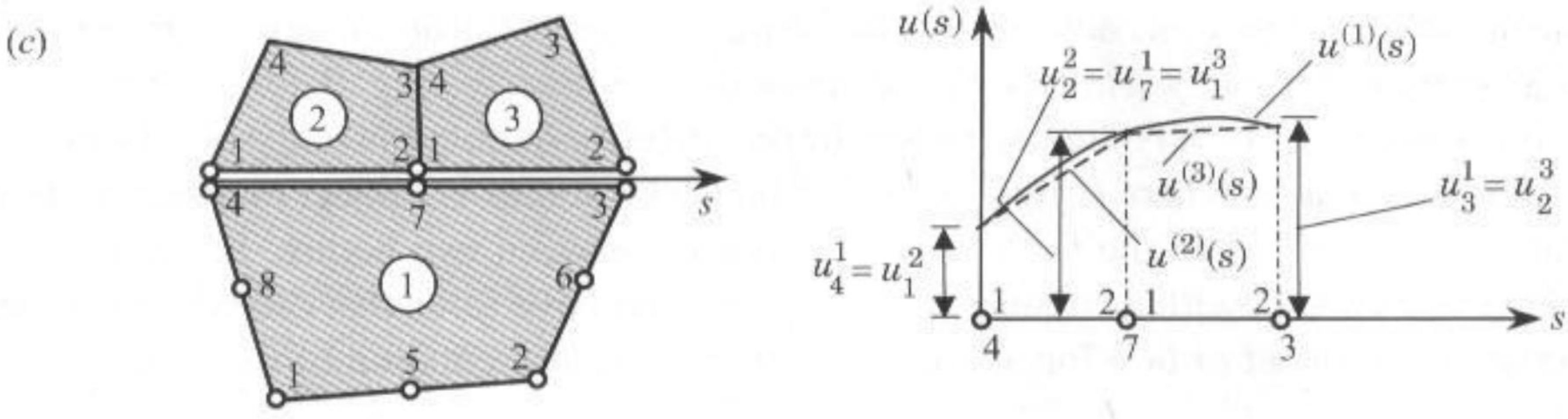
However, such combinations do **not enforce interelement continuity** of the solution along the entire interface. Fig. contains element connections that do not satisfy the C continuity along the connecting sides.



Modeling considerations



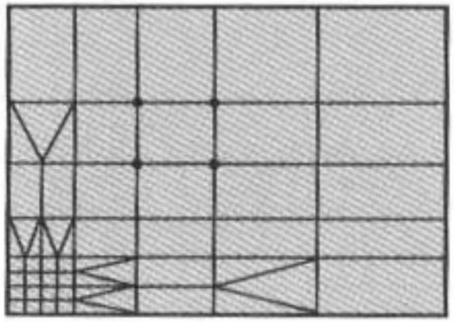
Modeling considerations



Modeling considerations

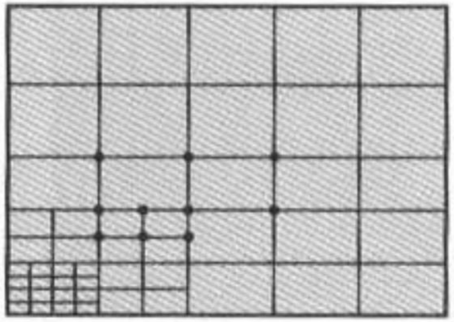
Examples of local mesh refinements

with compatible（Co-continuous）elements

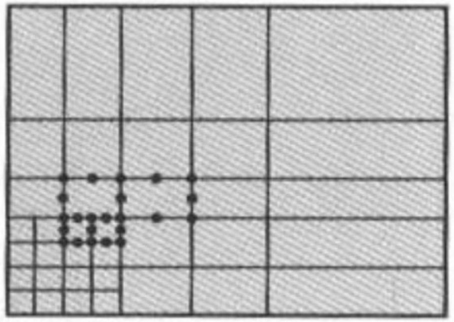


(a)

with transition elements（or when constraint conditions are imposed） between linear elements



(b)



(c)

with transition elements between quadratic elements

谢　谢！

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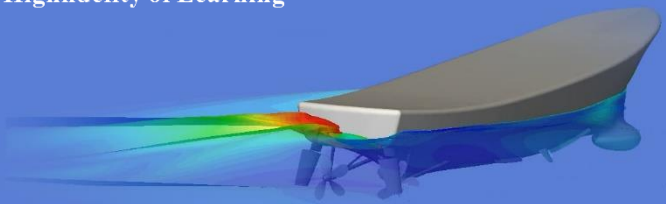

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