Class-6

NA26018

Finite Element Analysis of Solids and Fluids

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2020年6月19日

Introduction

- **Interpolation functions for the basic elements**
- **Isoparametric element and coordinate transformation**
- **Numerical integration and modeling considerations**

Introduction

- In the previous courses, we studied the finite element analysis of second-order equation and its analogues in the fields of heat transfer, solid mechanics
- As part of this study, we developed the interpolation functions for the basic elements, namely, the linear triangular and rectangular elements
- These elements, which were developed in connection with the finite element analysis of a second-order partial differential equation in a single variable, are useful in all finite element models that admit Lagrange interpolation of the primary variables of the weak formulation

If a library of interpolation functions is available, then we can select admissible functions for the model from the library

Objectives

- The objective of this course is to develop a library of two-dimensional triangular and rectangular elements of the Lagrange family (i. e, elements over which only the function not its derivativesare interpolated) The Hermite cubic interpolation functions are also presented, without a derivation, for the sake of completeness and reference

- The regularly shaped elements, called master elements, for which interpolation functions are developed here can be used for numerical evaluation of integrals defined on irregular elements, this requires a transformation of the geometry from the actual element shape to an associated master element

Once we have elements of different shapes and order at our disposal, we can choose appropriate elements and associated interpolation functions for a given problem

Element types

Triangular Elements

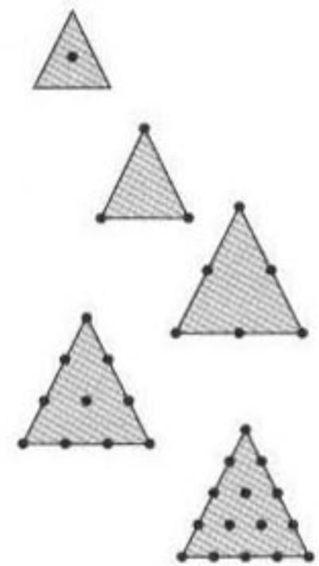
The linear (three-node) triangular element was developed in last course

Higher-order triangular elements (i.e, triangular elements with interpolation functions of higher degree) can be systematically developed with the help of the so-called Pascal's triangle, which contains the terms of polynomials of various degrees in the two coordinates x and y, as shown in Fig.

Element types

Here x and y denote some local coordinates; they do not, in general represent the global coordinates of the problem. We can view the position of the terms as the nodes of the triangle

Pascal's triangle	Degree of the complete polynomial	Number of terms in the polynomial	Element with nodes
1	0	1	
$x \quad y$	1	3	
$x^2 \quad xy \quad y^2$	2	6	
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3	10	
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4	15	
$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$	5	21	(Figure not shown)



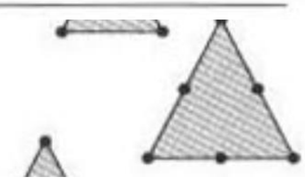
the constant term, the first and last terms of a given row being the vertices of the triangle

Element types

Of course, the shape of the triangle is arbitrary--not necessarily an equilateral triangle, as might appear from the position of the terms in Pascal's triangle

For example, a triangular element of order 2 (i.e, the degree of the polynomial is 2) contains six nodes, as can be seen from the third row of Pascal's triangle. The position of the six nodes in the triangle is at the three vertices and at the midpoints of the three sides. The polynomial involves six constants, which can be expressed in terms of the nodal values of the variable being interpolated as

$$\mathrm{u} = \sum_{i=1}^{6} u_i \psi_i(x, y)$$

Pascal's triangle	Degree of the complete polynomial	Number of terms in the polynomial	Element with nodes
x^2 xy y^2	2	6	

Element types

$$\mathrm{u} = \sum_{i=1}^{6} u_i \psi_i(x, y)$$

where ψ_i are the quadratic interpolation functions obtained following the same procedure as that used for the linear element. In general, a pth-order triangular element has a number of n nodes

$$\mathrm{n} = \frac{1}{2}(p+1)(p+2)$$

and a complete polynomial of pth degree is given by

$$\mathrm{u}(x, y) = \sum_{i=1}^{n} a_i x^r y^s = \sum_{j=1}^{n} u_j \psi_j, \qquad r + s \le p$$

Element types

- The location of the entries in Pascals triangle gives a symmetric location of nodal points in elements that will produce exactly the right number of nodes to define a Lagrange interpolation of any degree
- It should be noted that the Lagrange family of triangular elements (of order greater than zero) should be used for second-order problems that require only the dependent variables (not their derivatives) of the problem to be continuous at interelement boundaries
- It can be easily seen that the pth-degree polynomial associated with the pth-order Lagrange element, when evaluated on the boundary of the element, yields a pth-degree polynomial in the boundary coordinate

Element types

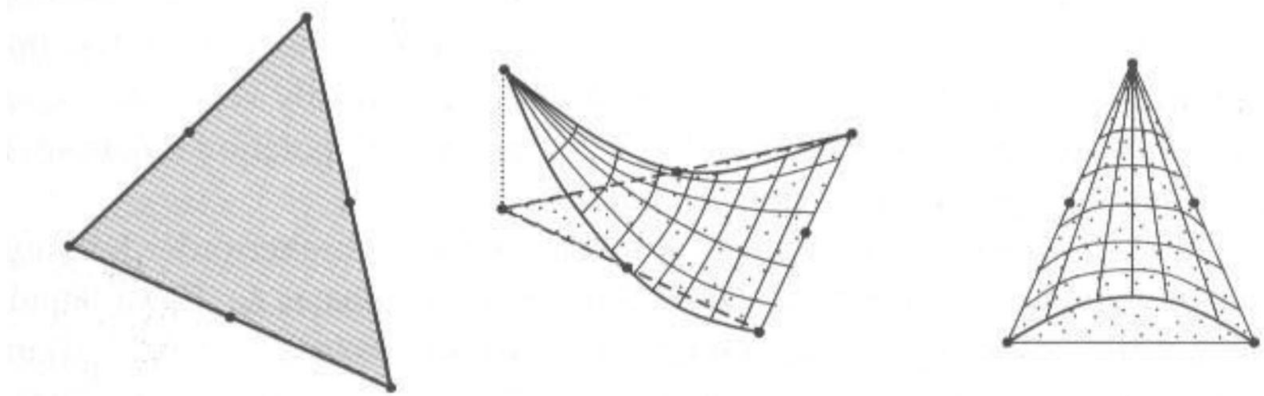
For example, the quadratic polynomial associated with the quadratic（six-node） triangular element shown in Fig. is given by

$$u^e(x,y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2$$

The derivatives of ue are

$$\frac{\partial u^e}{\partial x} = a_2 + a_4 y + 2a_5 x, \qquad \frac{\partial u^e}{\partial y} = a_3 + a_4 x + 2a_6 y$$

The element shown in Fig. is an arbitrary quadratic triangular element.



Element types

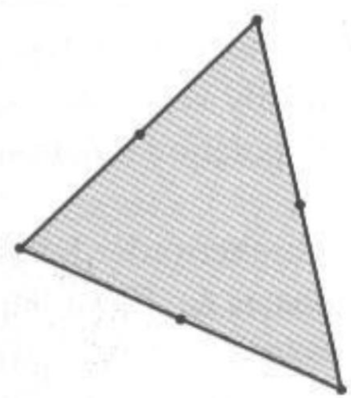
- By rotating and translating the (x, y) coordinate system, we obtain the (s, t) coordinate system
- Since the transformation from the (x, y) system to the (s, t) system involves only rotation (which is linear) and translation, a kth-degree polynomial in the (x, y) coordinate system is still a kth-degree polynomial in the (s, t) system

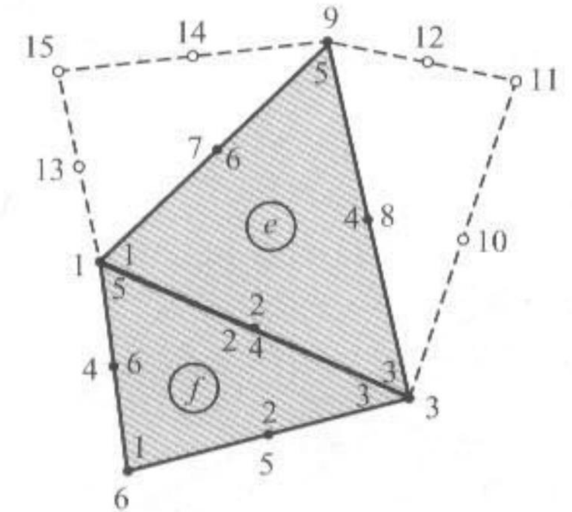
$$u^e(s,t) = \hat{a}_1 + \hat{a}_2 s + \hat{a}_3 t + \hat{a}_4 st + \hat{a}_5 s^2 + \hat{a}_6 t^2$$

where $\hat{a}_i$, $(i = 1, 2, \dots, 6)$ are constants that depend on a_i, and the angle of rotation α. Now by setting $t = 0$, we get the restriction of u to side 1-2-3 of element Ω^e

$$u^e(s,t) = \hat{a}_1 + \hat{a}_2 s + \hat{a}_5 s^2$$

which is a quadratic polynomial in s





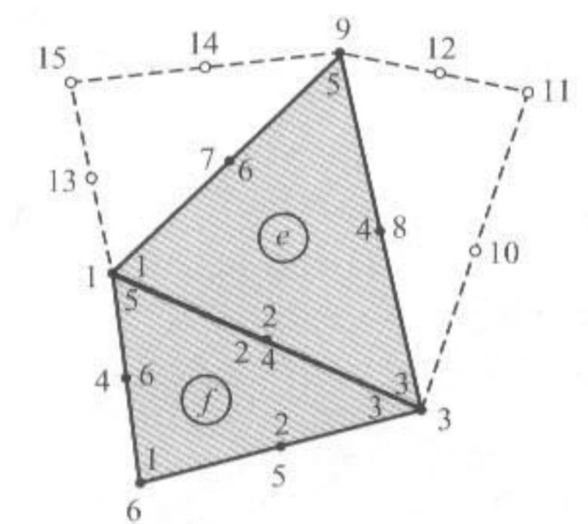
Element types

If a neighboring element Ω^f has its side 5-4-3 in common with side 1-2-3 of element Ω^e, then the function u on side 5-4-3 of element Ω^f also a quadratic polynomial

$$u^f(s,0) = \hat{b}_1 + \hat{b}_2 s + \hat{b}_5 s^2$$

Since the polynomials are uniquely defined by the same nodal values $U_1 = u_1^e = u_5^f$, $U_2 = u_2^e = u_4^f$, and $U_3 = u_3^e = u_3^f$, we have $u^e(s,0) = u^f(s,0)$ and hence the function u is uniquely defined on the interelement boundary of elements e and f

The ideas discussed above can be easily extended to three dimensions, in which case Pascal's triangle takes the form of a Christmas tree and the elements are of a pyramid shape, called tetrahedral elements



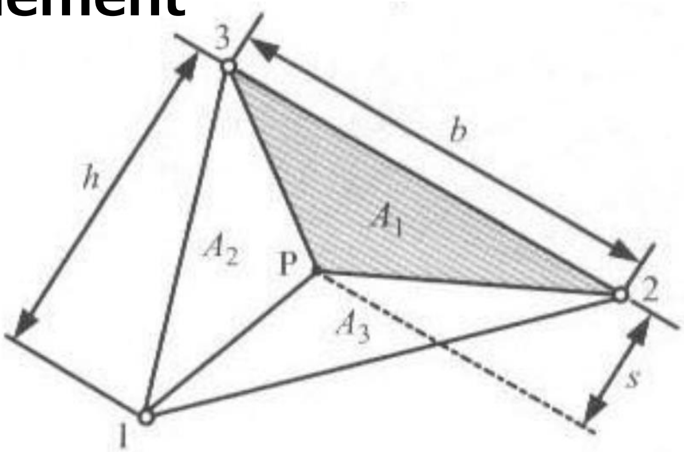
Element types

The alternative derivation of the interpolation functions for the higher-order Lagrange family of triangular elements is simplified by use of the area coordinates L_i

For triangular elements it is possible to construct three non-dimensionalized coordinates L_i ($i = 1, 2, 3$) that relate respectively to the sides directly opposite nodes 1, 2 and 3 such that

$$L_i = \frac{A_i}{A} \qquad A = \sum_{i=1}^{3} A_i$$

where A is the area of the triangle formed by nodes j and k and an arbitrary point P in the element, and A is the total area of the element



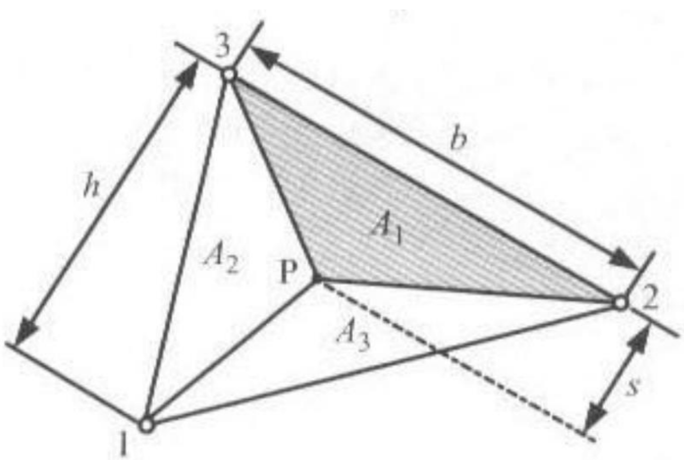
For example, A_1 is the area of the shaded triangle, which is formed by nodes 2 and 3 and point P. The point P is at a perpendicular distance of s from the side connecting nodes 2 and 3. We have $A_1 = 0.5bs$ and $A = 0.5bh$ Hence,

Element types

$$L_1 = \frac{A_1}{A} = \frac{s}{h}$$

Clearly, L_1 is zero on side 2-3 (hence, zero at nodes 2 and 3) and has a value of unity at node 1. Thus, L_1 is the interpolation function associated with node 1. Similarly, L_2 and L_3 are the interpolation functions associated with nodes 2 and 3, respectively. In summary, we have

$$\psi_i = L_i$$



for a linear triangular element. We shall use L_i to construct interpolation functions for higher-order triangular elements

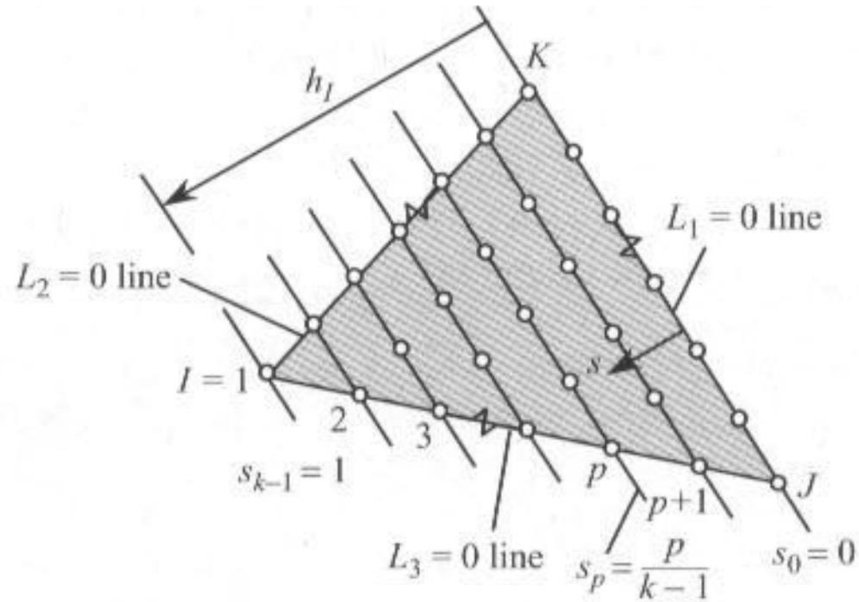
Element types

Consider a higher-order element with k nodes (equally spaced) per side. Then total number of nodes in the element is given by

$$\mathrm{n} = \sum_{i=0}^{k-1}(k-i) = k + (k-1) + \cdots + 1 = \frac{1}{2}k(k+1)$$

the degree of the interpolation functions is equal to $k-1$. For example, for the quadratic element we have $k - l = 2$ and $n = 6$

Let the corner (i. e, vertex) nodes be denoted by I, J and K, and let h_I be the perpendicular distance of the node from the side connecting J and K



Element types

Then the distance sp to the pth row parallel to side $J-K$ (under the assumption that the nodes are equally spaced along the sides and the rows) is given in nondimensional form by

$$s_p = \frac{p}{k-1}, \qquad s_0 = 0, \qquad s_{k-1} = 1$$

The interpolation function ψ_I should be zero at the nodes on the lines L_I=$0,1/(k-1),\dots,p/(k-1)$ $(p=0,1,\dots,k-2)$, and ψ_I should be equal to 1 at L_I=s_{k-1}. Thus, we have the necessary information for constructing the interpolation function ψ_I for vertex node $I(I=1,2,3)$

$$\psi_I = \frac{(L_I - s_0)(L_I - s_1)(L_I - s_2)\cdots(L_I - s_{k-2})}{(s_{k-1} - s_0)(s_{k-1} - s_1)\cdots(s_{k-1} - s_{k-2})} = \prod_{p=0}^{k-2} \frac{L_I - s_p}{s_{k-1} - s_p}$$

Element types

Similar expressions can be derived for nodes located at other than the vertices. In general ψ_i for node i is given by

$$\psi_i = \prod_{j=1}^{k-1} \frac{f_j}{f_j^i}$$

where f, are functions of L_1, L_2 and L_3, and f_j^i is the value of f_j at node i. The functions f_j are derived from the equations of $k-1$ lines that pass through all the nodes except node i

Example

First, consider the triangular element that has two nodes per side (i.e, $k=2$). This is the linear triangular element with the total number of nodes equal to three ($n=3$). For node 1, we have $k-2=0$ and

$$s_0 = 0, \qquad s_1 = 1, \qquad \psi_1 = \frac{L_1 - s_0}{s_1 - s_0} = L_1$$

Element types

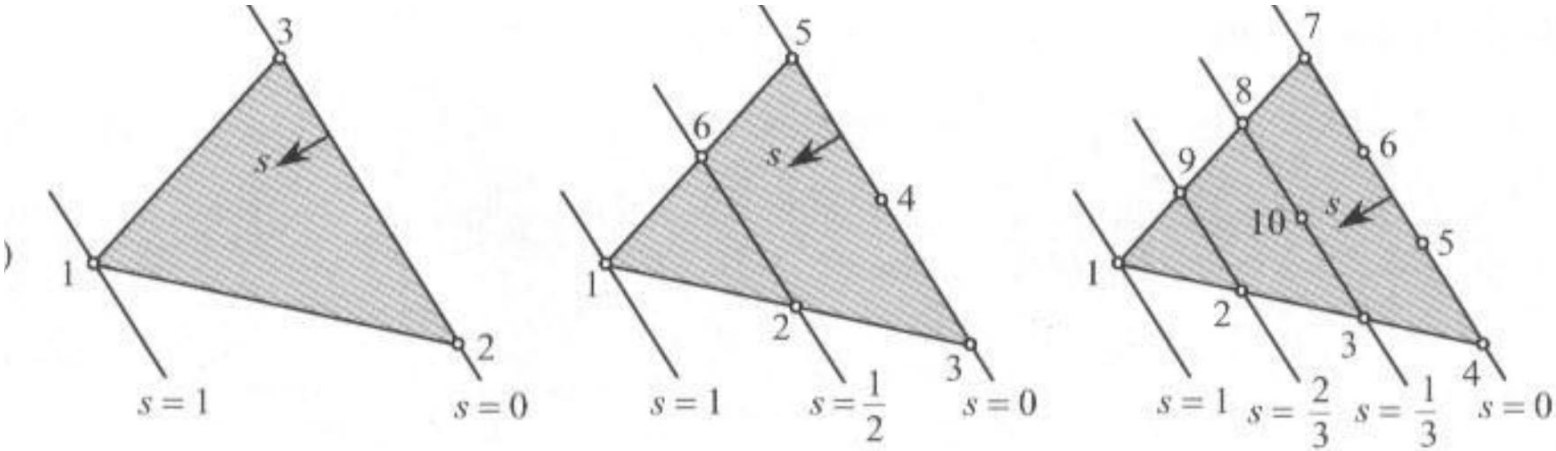
Similarly, for ψ_2 and ψ_3, we obtain

$$\psi_2 = L_2 , \qquad \psi_3 = L_3$$

Next, consider the triangular element with three nodes per side ($k = 3$). The total number of nodes is equal to six. For node 1, we have

$$s_0 = 0, \qquad s_1 = \frac{1}{2}, \qquad s_2 = 1$$

$$\psi_1 = \frac{L_1 - s_0}{s_2 - s_0}\frac{L_1 - s_1}{s_2 - s_1} = L_1(2L_1 - 1)$$



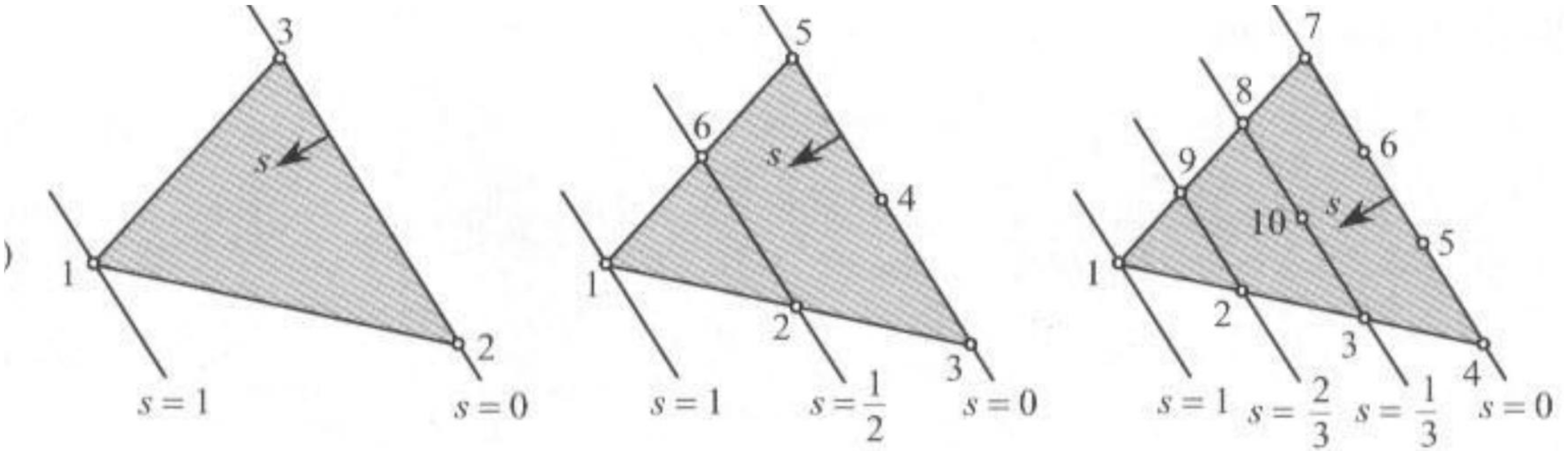
Element types

The function should vanish at nodes 1, 3, 4, 5 and 6, and should be equal to 1 at node 2. Equivalently, ψ_2 should vanish along the lines connecting nodes 1 and 5, and 3 and 5. These two lines are given in terms of L_1 and L_2 (note that the subscripts of L refer to the nodes in the three-node triangular element) as L_2=0 and L_1=0. Hence, we have

$$\psi_2 = \frac{L_2 - s_0}{s_1 - s_0}\frac{L_1 - s_0}{s_1 - s_0} = \frac{L_2 - 0}{\frac{1}{2}}\frac{L_1 - 0}{\frac{1}{2}} = 4L_1L_2$$

Similarly,

$$\psi_3 = L_2(2L_2 - 1), \qquad \psi_4 = 4L_2L_3, \qquad \psi_5 = L_3(2L_3 - 1), \qquad \psi_6 = 4L_1L_3$$



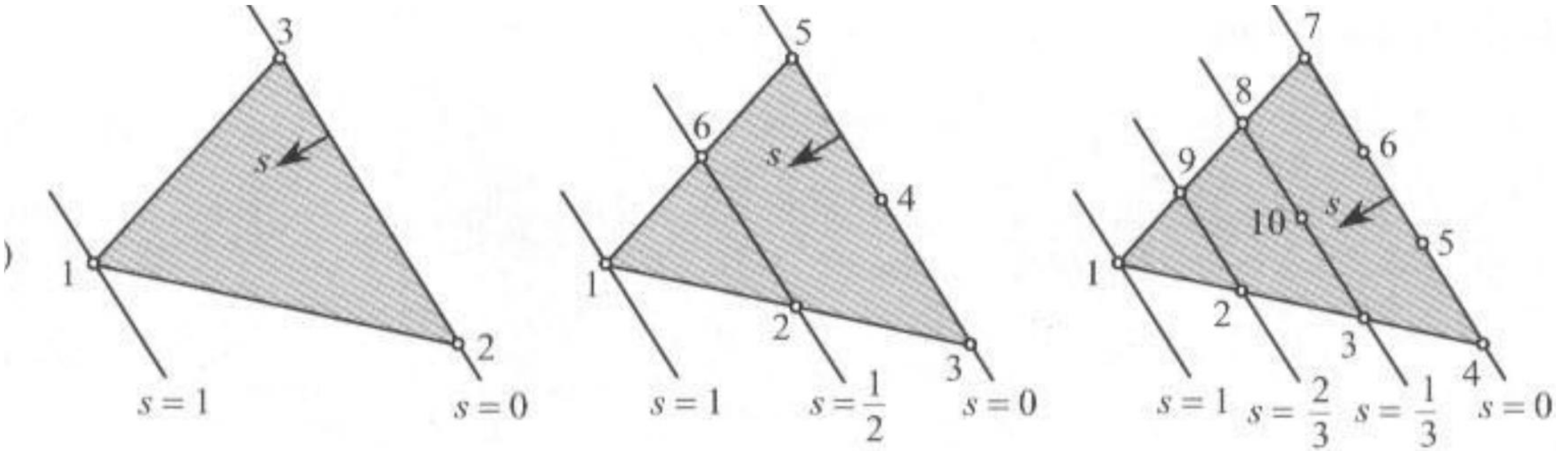
Element types

As a last example, consider the cubic element (i.e, $k-1=3$). For ψ_1 we note that it must vanish along lines L_1=0, L_1=1/3 and L_1=2/3. Therefore, we have

$$\psi_1 = \frac{L_1 - 0}{1 - 0} \frac{L_1 - \frac{1}{3}}{1 - \frac{1}{3}} \frac{L_1 - \frac{2}{3}}{1 - \frac{2}{3}} = \frac{1}{2} L_1 (3L_1 - 1)(3L_1 - 2)$$

The function ψ_2 must vanish along lines L_1=0, L_2=0, and L_1=1/3 (and node 2 is at a distance of 2/3 along L_i and a distance of 1/3 along L_2)

$$\psi_2 = \frac{L_1 - 0}{\frac{2}{3} - 0} \frac{L_2 - 0}{\frac{1}{3} - 0} \frac{L_1 - \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}} = \frac{9}{2} L_2 L_1 (3L_1 - 1)$$



Element types

Similarly, we can derive other functions, Thus we have

$$\psi_1 = \frac{1}{2}L_1(3L_1 - 1)(3L_1 - 2), \qquad \psi_2 = \frac{9}{2}L_2L_1(3L_1 - 1)$$

$$\psi_3 = \frac{9}{2}L_1L_2(3L_2 - 1), \qquad \psi_4 = \frac{1}{2}L_2(3L_2 - 1)(3L_2 - 2)$$

$$\psi_5 = \frac{9}{2}L_2L_3(3L_2 - 1), \qquad \psi_6 = \frac{9}{2}L_2L_3(3L_3 - 1)$$

$$\psi_7 = \frac{1}{2}L_3(3L_3 - 1)(3L_3 - 2), \qquad \psi_8 = \frac{9}{2}L_3L_1(3L_3 - 1)$$

$$\psi_9 = \frac{9}{2}L_1L_3(3L_1 - 1), \qquad \psi_{10} = 27L_1L_2L_3$$

Rectangular Elements

Analogous to the Lagrange family of triangular elements, the Lagrange family of rectangular elements can be developed from a rectangular array.

- **Since a linear rectangular element has four corners (hence, four nodes), the polynomial should have the first four terms** $1, x, y, xy$ **(which form a parallelogram in Pascal's triangle and a rectangle in the array given in Fig)**
- **The coordinates** (x, y) **are usually taken to be the element (i. e, local) coordinates**

In general, a pth-order lagrange rectangular element has n nodes, with

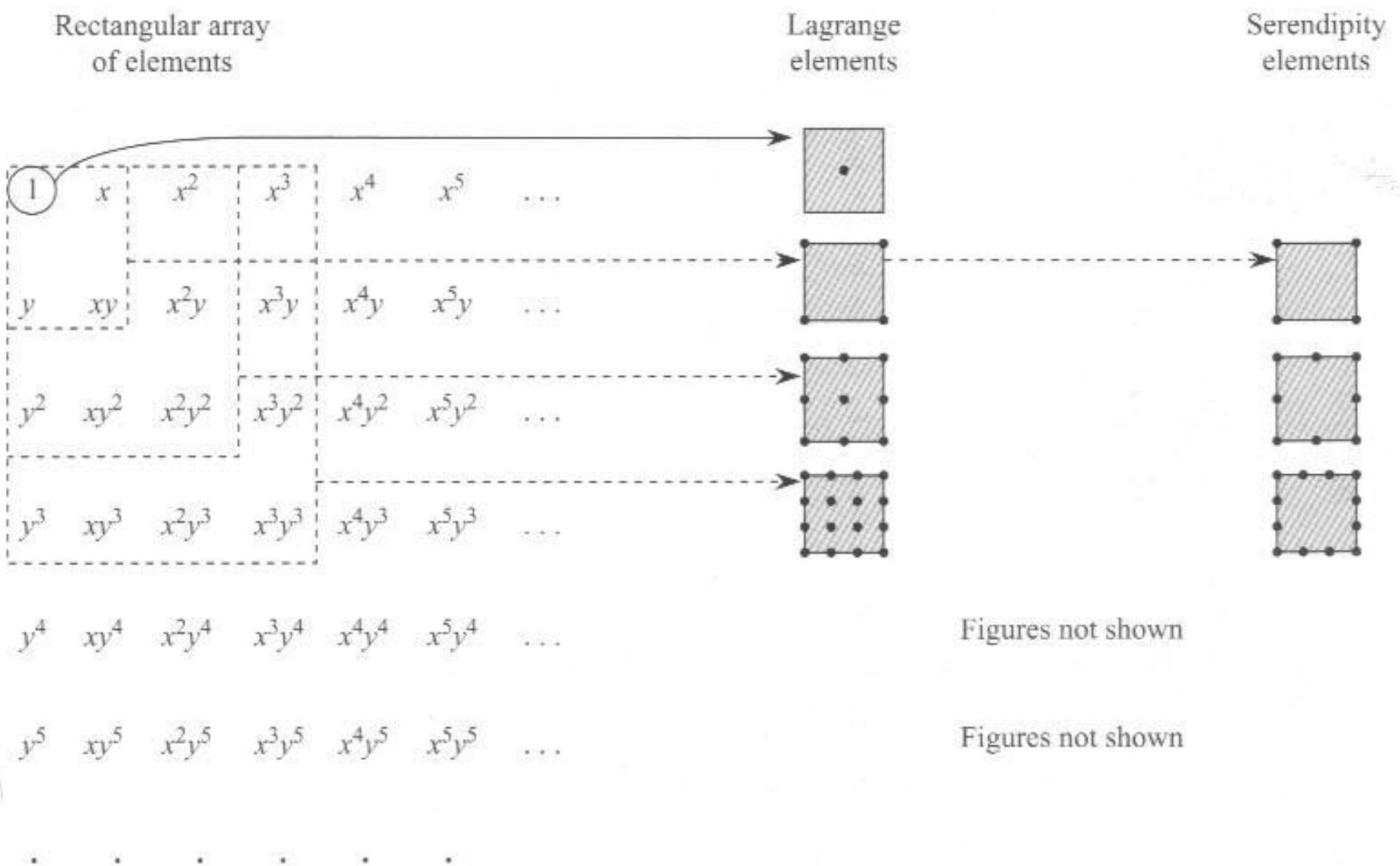
$$\mathrm{n} = (p+1)^2 \qquad (p = 0,1,\cdots)$$

Element types

- The associated polynomial contains the terms from the pth parallelogram or the pth rectangle in Fig.
- When $p = 0$, it is understood (as in triangular elements) that the node is at the center of the element (i.e, the variable is a constant on the entire element)
- The Lagrange quadratic rectangular element has nine nodes, and the associated polynomial is given by

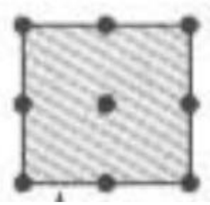
$$
\begin{aligned}
u(x, y) &= a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2 \\
&\quad + a_7 x^2 y + a_8 xy^2 + a_9 x^2 y^2 \\
\frac{\partial u}{\partial x} &= a_2 + a_4 y + 2a_5 x + 2a_7 xy + a_8 y^2 + 2a_9 xy^2 \\
\frac{\partial u}{\partial y} &= a_3 + a_4 x + 2a_6 y + a_7 x^2 + 2a_8 xy + 2a_9 x^2 y
\end{aligned}
$$

Element types



Element types

- The polynomial contains the complete polynomial of the second degree plus the third-degree terms x^2y and xy^2 and also the x^2y^2 term
- 4 of the nine nodes are placed at the four corners, 4 at the midpoints of the sides, and 1 at the center of the element
- The polynomial is uniquely determined by specifying its values at each of the 9 nodes



$$u(x,y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 + a_7x^2y + a_8xy^2 + a_9x^2y^2$$

$$\frac{\partial u}{\partial x} = a_2 + a_4y + 2a_5x + 2a_7xy + a_8y^2 + 2a_9xy^2$$

$$\frac{\partial u}{\partial y} = a_3 + a_4x + 2a_6y + a_7x^2 + 2a_8xy + 2a_9x^2y$$

Element types

- Moreover, along the sides of the element, the polynomial is quadratic (with three terms-as can be seen by setting $y = 0$) and is determined by its values at the three nodes on that side
- If two rectangular elements share a side and the polynomial is required to have the same values from both elements at the three nodes of the elements, then u is uniquely defined along the entire side (shared by the two elements)

$$u(x,y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 + a_7x^2y + a_8xy^2 + a_9x^2y^2$$

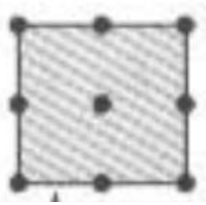
$$\frac{\partial u}{\partial x} = a_2 + a_4y + 2a_5x + 2a_7xy + a_8y^2 + 2a_9xy^2$$

$$\frac{\partial u}{\partial y} = a_3 + a_4x + 2a_6y + a_7x^2 + 2a_8xy + 2a_9x^2y$$

Element types

- **Note that the normal derivatives of u approximated by the quadratic Lagrange polynomials is quadratic in the tangential direction and linear in the normal direction (i.e, $\partial u/\partial x$ is quadratic in y and linear in x, and $\partial u/\partial y$ is quadratic in x and linear in y)**

$$u(x,y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2$$
$$+a_7x^2y + a_8xy^2 + a_9x^2y^2$$
$$\frac{\partial u}{\partial x} = a_2 + a_4y + 2a_5x + 2a_7xy + a_8y^2 + 2a_9xy^2$$
$$\frac{\partial u}{\partial y} = a_3 + a_4x + 2a_6y + a_7x^2 + 2a_8xy + 2a_9x^2y$$



Element types

The pth-order Lagrange rectangular element has the pth-degree polynomial

$$u(x,y)=\sum_{i=1}^{n} a_i x^j y^k \qquad (j+k\le p+1; j,k\le p)$$

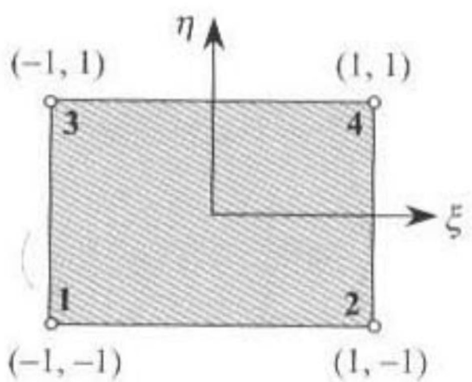
$$=\sum_{i=1}^{n} u_i \psi_i$$

ψ_i **are called the pth-order Lagrange interpolation functions**

- **The Lagrange interpolation functions associated with rectangular elements can be obtained from corresponding one-dimensional Lagrange interpolation functions by taking the tensor product of the x direction (one-dimensional) interpolation functions with the y direction (one-dimensional) interpolation functions**

Element types

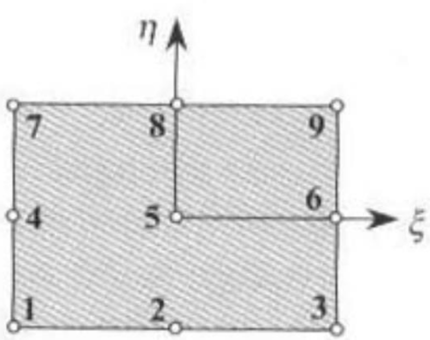
- **Let the x and y coordinates be taken along element sides with the origin of the coordinate system at the lower left corner of the rectangle.**
- **For an element with dimensions a and b along the x and y directions, the interpolation functions are given as follows**



Linear(p=1)

$$\begin{bmatrix}\psi_1 & \psi_3\\ \psi_2 & \psi_4\end{bmatrix} = \begin{Bmatrix}1-\dfrac{x}{a}\\ \dfrac{x}{a}\end{Bmatrix}\left\{1-\frac{y}{b} \quad \frac{y}{b}\right\}$$

$$= \begin{bmatrix}\left(1-\dfrac{x}{a}\right)\left(1-\dfrac{y}{b}\right) & \left(1-\dfrac{x}{a}\right)\dfrac{y}{b}\\ \dfrac{x}{a}\left(1-\dfrac{y}{b}\right) & \dfrac{x}{a}\dfrac{y}{b}\end{bmatrix}$$



Quadratic(p=2)

$$\begin{bmatrix}\psi_1 & \psi_4 & \psi_7\\ \psi_2 & \psi_5 & \psi_8\\ \psi_3 & \psi_6 & \psi_9\end{bmatrix} = \begin{Bmatrix}\dfrac{\left(x-\frac{1}{2}a\right)(x-a)}{\left(-\frac{1}{2}a\right)(-a)}\\ \dfrac{x(x-a)}{\frac{1}{2}a\left(\frac{1}{2}a-a\right)}\\ \dfrac{x\left(x-\frac{1}{2}a\right)}{a\left(\frac{1}{2}a\right)}\end{Bmatrix}\begin{Bmatrix}\dfrac{\left(y-\frac{1}{2}b\right)(y-b)}{\frac{1}{2}b^2}\\ \dfrac{y(y-b)}{-\frac{1}{4}b^2}\\ \dfrac{y(y-b/2)}{\frac{1}{2}b^2}\end{Bmatrix}^T$$

Element types

The two vectors are the one-dimensional interpolation functions along the x and directions, respectively. We obtain

$$\psi_1 = \left(1-\frac{2x}{a}\right)\left(1-\frac{x}{a}\right)\left(1-\frac{2y}{b}\right)\left(1-\frac{y}{b}\right),\qquad \psi_2 = \frac{4x}{a}(1-\frac{x}{a})(1-\frac{2y}{b})\left(1-\frac{y}{b}\right)$$

$$\psi_3 = \frac{x}{a}\left(\frac{2x}{a}-1\right)\left(1-\frac{2y}{b}\right)\left(1-\frac{y}{b}\right),\qquad \psi_4 = (1-\frac{2x}{a})(1-\frac{x}{a})\frac{4y}{b}\left(1-\frac{y}{b}\right)$$

$$\psi_5 = \frac{4x}{a}\left(1-\frac{x}{a}\right)\frac{4y}{b}\left(1-\frac{y}{b}\right),\qquad \psi_6 = \frac{x}{a}(\frac{2x}{a}-1)\frac{4y}{b}\left(1-\frac{y}{b}\right)$$

$$\psi_7 = \left(1-\frac{2x}{a}\right)\left(1-\frac{x}{a}\right)\frac{y}{b}\left(\frac{2y}{b}-1\right),\qquad \psi_8 = \frac{4x}{a}(1-\frac{x}{a})\frac{y}{b}\left(\frac{2y}{b}-1\right)$$

$$\psi_9 = \frac{x}{a}\left(\frac{2x}{a}-1\right)\frac{y}{b}\left(\frac{2y}{b}-1\right), \qquad (9.2.24)$$

Element types

Linear(p=1)

$$\begin{bmatrix}\psi_1 & \psi_3\\ \psi_2 & \psi_4\end{bmatrix} = \begin{Bmatrix}1-\dfrac{x}{a}\\ \dfrac{x}{a}\end{Bmatrix}\begin{Bmatrix}1-\dfrac{y}{b} & \dfrac{y}{b}\end{Bmatrix}$$

Quadratic(p=2)

$$\begin{bmatrix}\psi_1 & \psi_4 & \psi_7\\ \psi_2 & \psi_5 & \psi_8\\ \psi_3 & \psi_6 & \psi_9\end{bmatrix} = \begin{Bmatrix}\dfrac{\left(x-\frac{1}{2}a\right)(x-a)}{\left(-\frac{1}{2}a\right)(-a)}\\ \dfrac{x(x-a)}{\frac{1}{2}a\left(\frac{1}{2}a-a\right)}\\ \dfrac{x\left(x-\frac{1}{2}a\right)}{a\left(\frac{1}{2}a\right)}\end{Bmatrix}\begin{Bmatrix}\dfrac{\left(y-\frac{1}{2}b\right)(y-b)}{\frac{1}{2}b^2}\\ \dfrac{y(y-b)}{-\frac{1}{4}b^2}\\ \dfrac{y(y-b/2)}{\frac{1}{2}b^2}\end{Bmatrix}^T$$

***p*th Order**

$$\begin{bmatrix}\psi_1 & \psi_{p+2} & \cdots & \psi_k\\ \psi_2 & & & \\ \vdots & \ddots & & \vdots\\ \psi_p & & \ddots & \\ \psi_{p+1} & \psi_{2p+2} & \cdots & \psi_n\end{bmatrix} = \begin{Bmatrix}f_1\\ f_2\\ \vdots\\ f_{p+1}\end{Bmatrix}\begin{Bmatrix}g_1\\ g_2\\ \vdots\\ g_{p+1}\end{Bmatrix}^T$$

$$k=(p+1)p+1, n=(p+1)^2$$

Element types

***p*th Order**

$$\begin{bmatrix} \psi_1 & \psi_{p+2} & \cdots & \psi_k \\ \psi_2 & & & \\ \vdots & \ddots & & \vdots \\ \psi_p & & \ddots & \\ \psi_{p+1} & \psi_{2p+2} & \cdots & \psi_n \end{bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{p+1} \end{Bmatrix} \begin{Bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{p+1} \end{Bmatrix}^T$$

$$k = (p+1)p + 1, n = (p+1)^2$$

where $f_i\,(x)$ and $g_i\,(y)$ are the *p*th-order interpolants in x and y, respectively. For example the polynomial

$$f_i(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)\cdots(\xi - \xi_{i-1})(\xi - \xi_{i+1})\cdots(\xi - \xi_{p+1})}{(\xi_i - \xi_1)(\xi_i - \xi_2)\cdots(\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1})\cdots(\xi_i - \xi_{p+1})}$$

(where ξ_i is the ξ coordinate of node i) is the pth-degree interpolation polynomial in ξ that vanishes at points ξ_1, ξ_2,⋯, ξ_{i-1}, ξ_{i+1},⋯ ξ_{p+1}. We recall that (x, y) are the element coordinates

Element types

It is convenient (for numerical integration purposes) to express the interpolation functions in terms of the natural coordinates ξ and η

$$\xi = \frac{2(x - x_1) - a}{a}, \qquad \eta = \frac{2(y - y_1) - b}{b}$$

where x_1 and y_1 are the global coordinates of node 1 in the local x and y coordinates. For a coordinate system with origin fixed at node 1 and coordinates parallel to the sides of the element, we have $x_1 = y_1 = 0$. In this case, the quadratic interpolation functions can be written in terms of the natural coordinates ξ and η as

$$\psi_7 = -\frac{1}{4}(\xi - \xi^2)(\eta + \eta^2)$$

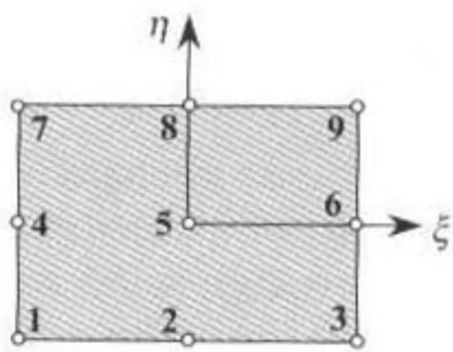
$$\psi_8 = \frac{1}{2}(1 - \xi^2)(\eta + \eta^2)$$

$$\psi_1 = \frac{1}{4}(\xi - \xi^2)(\eta - \eta^2) \qquad \psi_3 = -\frac{1}{4}(\xi + \xi^2)(\eta - \eta^2),$$

$$\psi_2 = -\frac{1}{2}(1 - \xi^2)(\eta - \eta^2) \qquad \psi_4 = -\frac{1}{2}(\xi - \xi^2)(1 - \eta^2),$$

$$\psi_5 = (1 - \xi^2)(1 - \eta^2) \qquad \psi_9 = \frac{1}{4}(\xi + \xi^2)(\eta + \eta^2)$$

$$\psi_6 = \frac{1}{2}(\xi + \xi^2)(1 - \eta^2)$$



The Serendipity Elements

- Since the internal nodes of the higher-order elements of the Lagrange family do not contribute to the interelement connectivity, they can be condensed out at the element level so that the size of the element matrices is reduced
- Alternatively, we can use the so-called serendipity elements to avoid the internal nodes present in the Lagrange elements. The serendipity elements are those rectangular elements which have no interior nodes. In other words, all the node points are on the boundary of the element. The interpolation functions for serendipity elements cannot be obtained using tensor products of one-dimensional interpolation functions. Instead, an alternative procedure that employs the interpolation properties is used

Here we illustrate how to construct the interpolation functions for the eight-node（quadratic）element using the natural coordinates

Element types

The interpolation function for node 1 should take on a value of zero at nodes 2, 3,⋯,8 and a value of unity at node 1. Equivalently, ψ_1 should vanish on the sides defined by the equations $1-\xi=0$, $1-\eta=0$, and $1+\xi+\eta=0$. Therefore, ψ_1 is of the form

$$\psi_1(\xi,\eta)=c(1-\xi)(1-\eta)(1+\xi+\eta)$$

where c is a constant that should be determined so as to yield $\psi_1(-1,-1)=1$. We obtain $c=-1/4$, and therefore

$$\psi_1(\xi,\eta)=-\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

We can construct other interpolation functions in a similar manner. We have

Element types

$$\psi_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), \qquad \psi_2 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\psi_3 = \frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta), \qquad \psi_4 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

$$\psi_5 = \frac{1}{2}(1+\xi)(1-\eta^2), \qquad \psi_6 = \frac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta)$$

$$\psi_7 = \frac{1}{2}(1-\xi^2)(1+\eta), \qquad \psi_8 = \frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta)$$

Note that all the ψ_i for the eight-node element have the form

$$\psi_i = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\xi^2 + c_6\eta^2 + c_7\xi^2\eta + c_8\xi\eta^2$$

The derivatives of ψ_i with respect to ξ and η are of the form

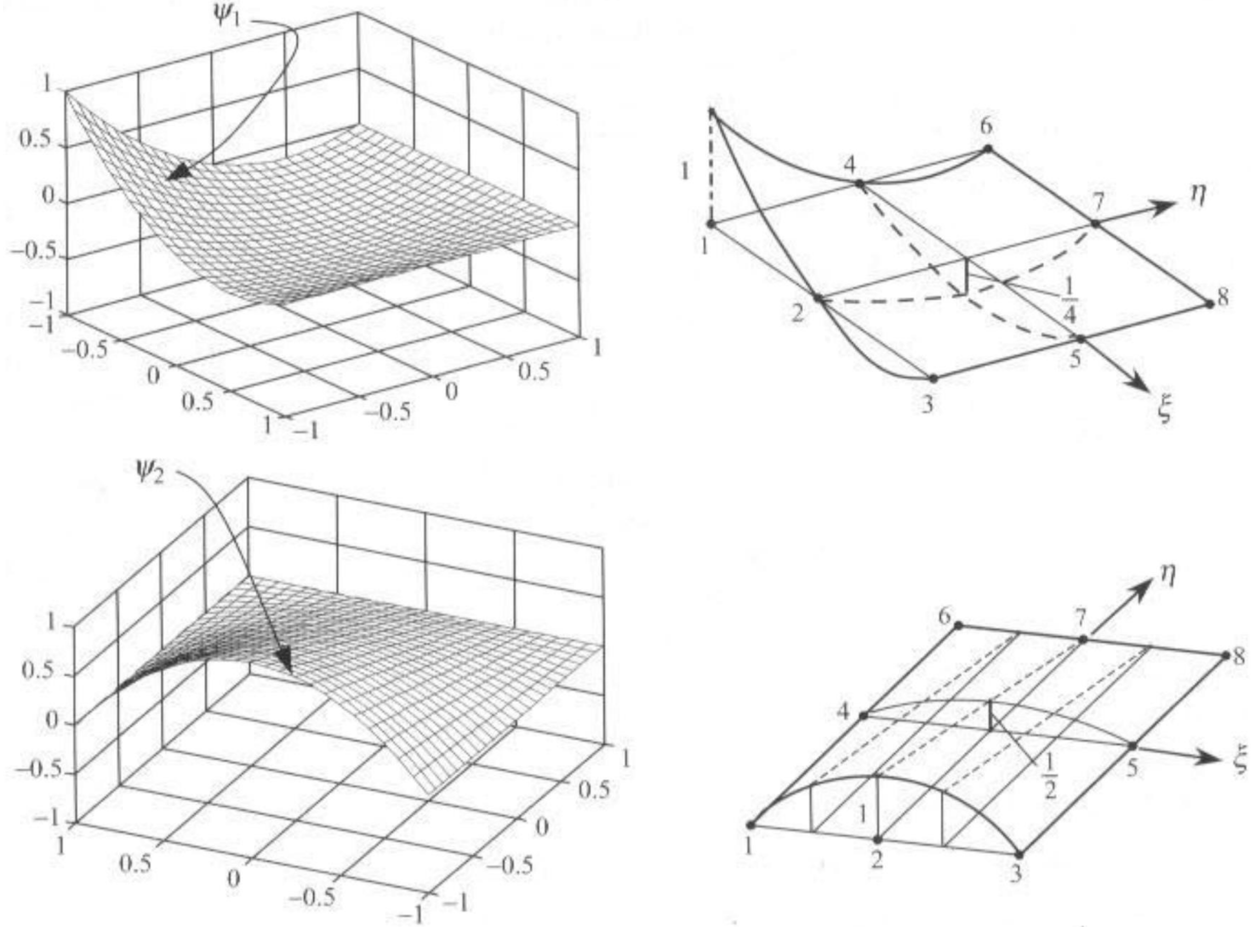
$$\frac{\partial\psi_i}{\partial\xi} = c_2 + c_4\eta + 2c_5\xi + 2c_7\xi\eta + c_8\eta^2$$

$$\frac{\partial\xi_i}{\partial\eta} = c_3 + c_4\xi + 2c_6\eta + c_7\xi^2 + 2c_8\xi\eta$$

Element types

Plots of ψ_1 and ψ_2 for the eight-node serendipity element are shown

- **Noted that ψ_2 of the nine-node element is zero at the element center, whereas ψ_2 of the eight-node element is nonzero there**

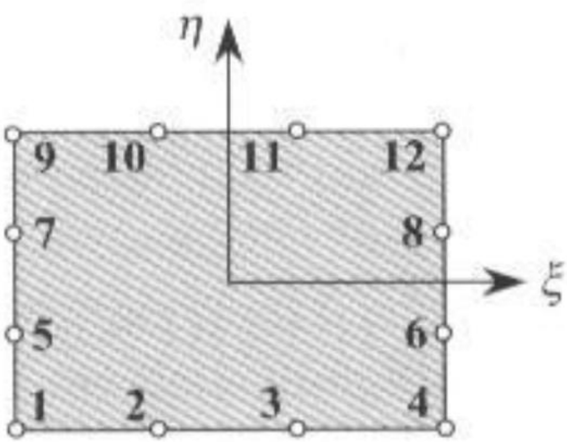


Element types

The interpolation functions ψ_i for the twelve-node element are of the form

$$\psi_i = \text{8-nodes element} + c_9\xi^3 + c_{10}\eta^3 + c_{11}\xi^3\eta + c_{12}\xi\eta^3$$

$$\psi_i = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\xi^2 + c_6\eta^2 + c_7\xi^2\eta + c_8\xi\eta^2$$



$$
\begin{aligned}
\psi_1 &= \frac{1}{32}(1-\xi)(1-\eta)\left[-10+9\left(\xi^2+\eta^2\right)\right], \quad \psi_2 = \frac{9}{32}(1-\eta)(1-\xi^2)(1-3\xi)\\
\psi_3 &= \frac{9}{32}(1-\eta)\left(1-\xi^2\right)(1+3\xi),\\
\psi_4 &= \frac{1}{32}(1+\xi)(1-\eta)[-10+9(\xi^2+\eta^2)]\\
\psi_5 &= \frac{9}{32}(1-\xi)\left(1-\eta^2\right)(1-3\eta), \quad \psi_6 = \frac{9}{32}(1+\xi)(1-\eta^2)(1-3\eta)\\
\psi_7 &= \frac{9}{32}(1-\xi)\left(1-\eta^2\right)(1+3\eta), \quad \psi_8 = \frac{9}{32}(1+\xi)(1-\eta^2)(1+3\eta)\\
\psi_9 &= \frac{1}{32}(1-\xi)(1+\eta)\left[-10+9\left(\xi^2+\eta^2\right)\right], \quad \psi_{10} = \frac{9}{32}(1+\eta)(1-\xi^2)(1-3\xi)\\
\psi_{11} &= \frac{9}{32}(1+\eta)\left(1-\xi^2\right)(1+3\xi), \qquad (9.2)\\
\psi_{12} &= \frac{1}{32}(1+\xi)(1+\eta)[-10+9(\xi^2+\eta^2)]
\end{aligned}
$$

Hermite Cubic Interpolation Functions

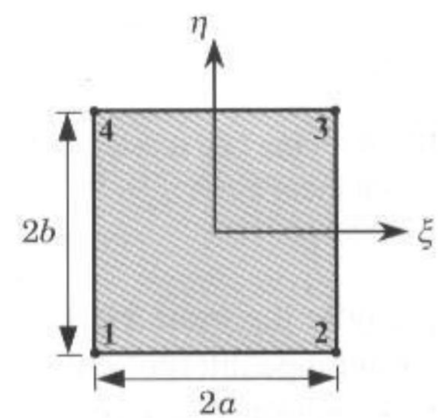
In the above discussion, we developed only the Lagrange interpolation functions for trangular and rectangular elements.

- **The Hermite family of interpolation functions (which interpolate the function and its derivatives) were not discussed**
- **Recall that such functions are required in the finite element formulation of fourth-order (or higher-order) differential equations (e.g, the Euler-Bernoulli beam theory).**

For the sake of completeness, while not presenting the details of the derivation, the Hermite cubic interpolation functions for two rectangular elements are summarized in Table. The first one is based on the interpolation of ($u, \partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x \partial y$) at each node, and the second one is based on the interpolation of ($u, \partial u/\partial x, \partial u/\partial y$) at each node

Element types

Element type	Interpolation functions	Remarks
Lagrange element		
Linear	$\frac{1}{4}(1+\xi_0)(1+\eta_0)$	Node $i = 1, \ldots, 4$
Quadratic	$\psi_i = \frac{1}{4}\xi_0(1+\xi_0)\,\eta_0(1+\eta_0)$	Corner node i
	$\psi_i = \frac{1}{2}\eta_0(1+\eta_0)(1-\xi^2)$	Side node i, $\xi_i = 0$
	$\psi_i = \frac{1}{2}\xi_0(1+\xi_0)(1-\eta^2)$	Side node i, $\eta_i = 0$
	$\psi_i = (1-\xi^2)(1-\eta^2)$	Interior node i
Serendipity element		
Quadratic	$\psi_i = \frac{1}{4}(1+\xi_0)(1+\eta_0)(\xi_0+\eta_0-1)$	Corner node i
	$\psi_i = \frac{1}{2}(1-\xi^2)(1+\eta_0)$	Side node i, $\xi_i = 0$
	$\psi_i = \frac{1}{2}(1+\xi_0)(1-\eta^2)$	Side node i, $\eta_i = 0$
Hermite cubic element		
Nonconforming element	$[I = 4(i-1)+1,\ i = 1, \ldots, 4]$	
Variable u	$\varphi_I = \frac{1}{16}(\xi+\xi_i)^2(\xi_0-2)(\eta+\eta_i)^2(\eta_0-2)$	
Derivative $(\partial u/\partial x)$	$\varphi_{I+1} = -\frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi_0-1)(\eta+\eta_i)^2(\eta_0-2)$	
Derivative $(\partial u/\partial y)$	$\varphi_{I+2} = -\frac{1}{16}(\xi+\xi_i)^2(\xi_0-2)\eta_i(\eta+\eta_i)^2(\eta_0-1)$	
Derivative $(\partial^2 u/\partial x\partial y)$	$\varphi_{I+3} = \frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi_0-1)\eta_i(\eta+\eta_i)^2(\eta_0-1)$	
Conforming element	$[I = 3(i-1)+1,\ i = 1, \ldots, 4]$	
Variable u	$\varphi_I = \frac{1}{8}(\xi_0+1)(\eta_0+1)(2+\xi_0+\eta_0-\xi^2-\eta^2)$	
Derivative $(\partial u/\partial x)$	$\varphi_{I+1} = \frac{1}{8}\xi_i(\xi_0+1)^2(\xi_0-1)(\eta_0+1)$	
Derivative $(\partial u/\partial y)$	$\varphi_{I+2} = \frac{1}{8}\eta_i(\xi_0+1)(\eta_0+1)^2(\eta_0-1)$	
	$\xi = (x-x_c)/a,\ \eta = (y-y_c)/b,\ \xi_0 = \xi\xi_i,\ \eta_0 = \eta\eta_i$	



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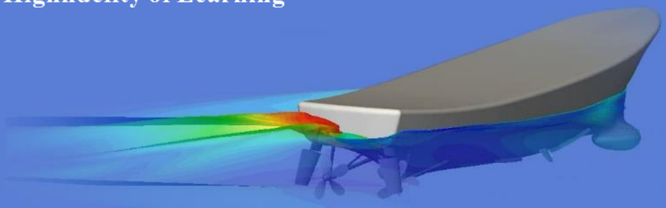

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