Class-5

NA26018

Finite Element Analysis of Solids and Fluids

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Introduction

Elasticity is the part of solid mechanics that deals with stress and deformation of solid continua

Linearized elasticity is concerned with

- small deformations

(i.e, strains and displacements that are very small compared to unity)

- linear elastic solids

(i.e, obey Hooke's law)

Plane elasticity problems:

There is a class of problems in elasticity whose solutions (i. e, displacements and stresses) are not dependent on one of the coordinates because of geometry, boundary conditions, and external applied loads

Such problems are called Plane elasticity problems

Plane elasticity

The plane elasticity problems considered here are grouped into plane strain and plane stress problems

- **Both classes of problems are described by a set of two coupled partial differential equations**
- **Those Eqs. are expressed in terms of two dependent variables that represent the two components of the displacement vector**

Plane stress ($x - y$): no stress in z, no external force in z-dir
Plane strain ($x - y$): no strain in z, z-displacement is constrained

The governing equations of plane strain problems differ from those of the plane stress problems only in the constitutive equations (coefficients) of the differential equations

Plane elasticity

The equations of motion, strain-displacement Relations constitutive relations are the same. It differ from each other only on account of the difference in the constitutive equations for the two cases

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & 0 \\ \bar{c}_{12} & \bar{c}_{22} & 0 \\ 0 & 0 & \bar{c}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

Elasticity stiffness for plane strain

$$\bar{c}_{11} = \frac{E_1(1-\nu_{12})}{(1+\nu_{12})(1-\nu_{12}-\nu_{21})}$$

$$\bar{c}_{22} = \frac{E_2(1-\nu_{21})}{(1+\nu_{21})(1-\nu_{12}-\nu_{21})}$$

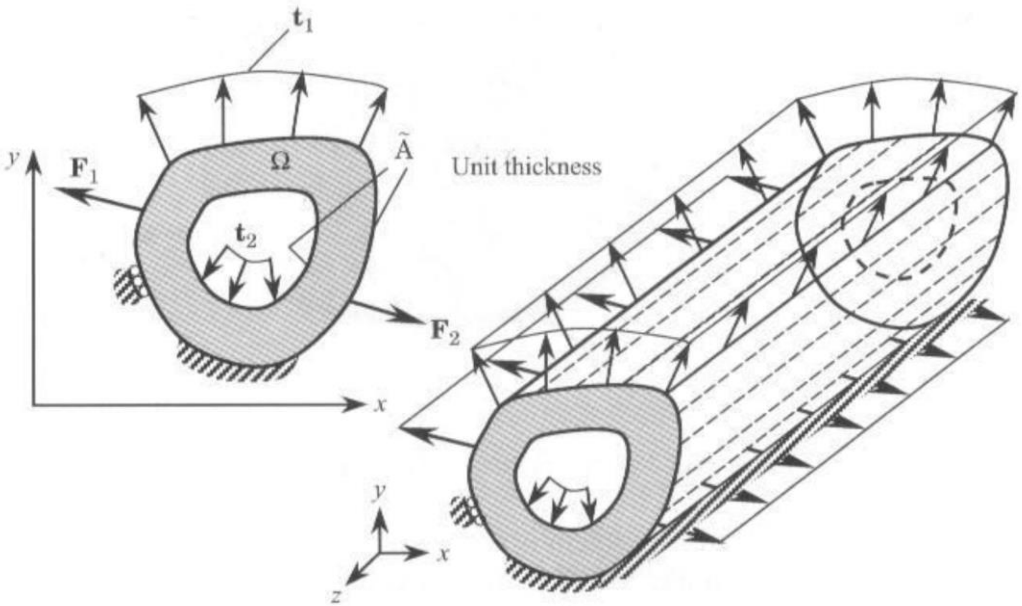
$$\bar{c}_{12} = \nu_{12}\bar{c}_{22}, \qquad \bar{c}_{66} = G_{12}$$

Elasticity stiffness for plane stress

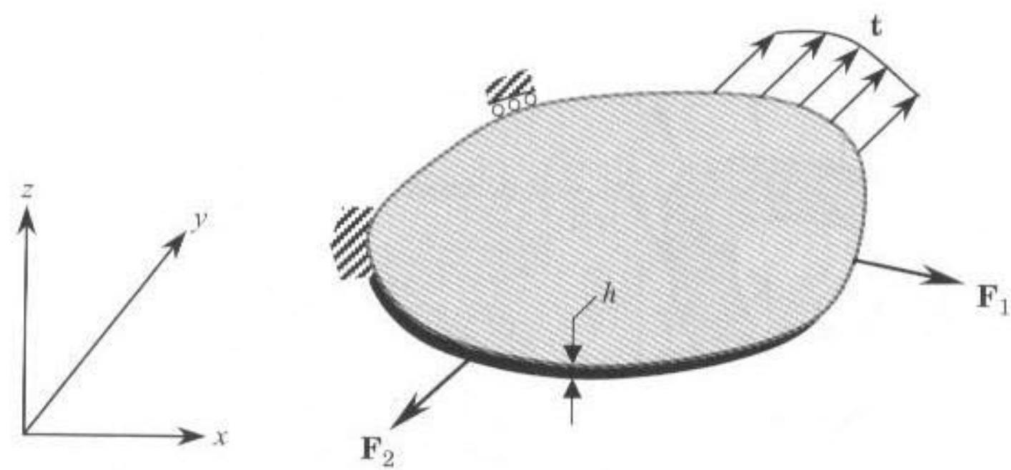
$$\hat{c}_{11} = \frac{E_1}{(1-\nu_{12}\nu_{21})} \qquad \hat{c}_{22} = \frac{E_2}{(1-\nu_{12}\nu_{21})}$$

$$\hat{c}_{12} = \nu_{12}\bar{c}_{22} = \nu_{21}\bar{c}_{11}, \hat{c}_{66} = G_{12}$$

Plane elasticity



plane strain problem



plane stress problem

Summary of Equations

1. Equations of Motion:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

or $$\mathbf{D}^*\boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

where f_x and f_y denote the components of the body force vector (measured per unit volume) along the x and y directions, respectively, ρ is the density of the material, and

$$\mathbf{D}^* = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}, \qquad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}, \qquad \mathbf{f} = \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}, \qquad \mathbf{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix}$$

Plane elasticity

2. Strain-Displacement Relations:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

or $$\varepsilon = \mathbf{D}\mathbf{u}, \varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xx} \\ 2\varepsilon_{xy} \end{Bmatrix}, \mathbf{D} = (\mathbf{D}^*)^{\mathrm{T}}$$

3. Stress-Strain (Constitutive) Relations:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{C}\varepsilon, \qquad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix}$$

Plane elasticity

Boundary Conditions:

- **Natural boundary conditions are**

$$\left.\begin{aligned} t_x \equiv \sigma_{xx} n_x + \sigma_{xy} n_y = \hat{t}_x \\ t_y \equiv \sigma_{xy} n_x + \sigma_{yy} n_y = \hat{t}_y \end{aligned}\right\} \textbf{ on } \Gamma_\sigma$$

or $\mathbf{t} \equiv \bar{\boldsymbol{\sigma}} \mathbf{n} = \hat{\mathbf{t}}$ on Γ_σ , $\mathbf{n} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$, $\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$

- **Essential boundary conditions are**

$$u_x = \hat{u}_x \,, \quad u_y = \hat{u}_y \quad \text{on } \Gamma_u$$

or $\mathbf{u} = \hat{\mathbf{u}}$ on Γ_u

(n_x, n_y) **the components of the unit normal vector on boundary** Γ

Γ_σ **and** Γ_u **:portions of the boundary**

t_x, t_y**: the components of the specified traction vector**

u_x, u_y **: the components of specified displacement vector**

Plane elasticity

Equations of motion can be expressed in terms of only the displacements u_x and u_y by substituting stress-strain and strain-displacement relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

$$-\frac{\partial}{\partial x}\left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y}\right) - \frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right] = f_x - \rho\frac{\partial^2 u_x}{\partial t^2}$$

$$-\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right] - \frac{\partial}{\partial y}\left(c_{12}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y}\right) = f_y - \rho\frac{\partial^2 u_y}{\partial t^2} \quad \textbf{(O-1)}$$

Plane elasticity

or $$-\mathbf{D}^*\mathbf{C}\mathbf{D}\mathbf{u} = \mathbf{f} + \rho\ddot{\mathbf{u}}$$

The boundary stress components (or tractions) can also be expressed in terms of the displacements

$$t_x = \left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y}\right)n_x + c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)n_y \qquad \textbf{(O-2)}$$

$$t_y = c_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)n_x + \left(c_{12}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y}\right)n_y$$

or $$\mathbf{t} = \bar{\mathbf{n}}\mathbf{C}\mathbf{D}\mathbf{u}, \qquad \bar{\mathbf{n}} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}$$

Weak Formulations

Preliminary Comments:
There are two different ways of constructing the weak forms and associated finite element model of the plane elasticity equations

1. **Uses the principle of virtual displacements (or the principle of minimum total potential energy)**

expressed in terms of matrices relating displacements to strains, strains to stresses, and the equations of motion. This approach is often used in most finite element texts on solid mechanics

2. **Follows a procedure consistent with the previous sections and employs the weak formulation of Eqs. (O-1, O-2) to construct the finite element model**

Both methods give mathematically the same finite element model, but differ in their algebraic forms

Plane elasticity

Weak Form of the Governing Differential Equations

The second approach, which has been used throughout the course, does not require knowledge of the principles of virtual displacements or the total minimum potential energy but only needs the governing differential equations of the problem

- **Use the 3-step procedure for each of the two differential equations: multiply the equation with a weight function wi and integrate by parts to trade the differentiation equally between the weight function and the dependent variables (u_x, u_y)**

$$0 = \int_{\Omega_e} h_e \left(\frac{\partial w_1}{\partial x} \sigma_{xx} + \frac{\partial w_1}{\partial y} \sigma_{xy} - w_1 f_x + \rho w_1 \ddot{u}_x \right) dxdy$$

$$- \oint_{\Gamma_e} h_e w_1 \left(\sigma_{xx} n_x + \sigma_{xy} n_y \right) ds$$

Plane elasticity

$$0 = \int_{\Omega_e} h_e \left(\frac{\partial w_2}{\partial x} \sigma_{xy} + \frac{\partial w_2}{\partial y} \sigma_{yy} - w_2 f_y + \rho w_2 \ddot{u_y} \right) dxdy$$

$$- \oint_{\Gamma_e} h_e w_2 (\sigma_{xy} n_x + \sigma_{yy} n_y) ds$$

where

$$\sigma_{xx} = c_{11} \frac{\partial u_x}{\partial x} + c_{12} \frac{\partial u_y}{\partial y} \qquad \sigma_{xy} = c_{66} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\sigma_{yy} = c_{12} \frac{\partial u_x}{\partial x} + c_{22} \frac{\partial u_y}{\partial y}$$

The last step of the development is to identify the primary and secondary variables of the formulation and rewrite the boundary integrals in terms of the secondary variables

Plane elasticity

By comparing B. C, it follows that the boundary forces tr and ty are the secondary variables. The weight functions w_1 and w_2 are the first variations of u_x and u_y, respectively

The final weak forms are given by

$$0 = \int_{\Omega_e} h_e \left[\frac{\partial w_1}{\partial x}\left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y}\right) + c_{66}\frac{\partial w_1}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \rho w_1 \ddot{u}_x\right] dxdy$$

$$- \oint_{\Omega_e} h_e w_1 f_x dxdy - \oint_{\Gamma_e} h_e w_1 t_x ds$$

$$0 = \int_{\Omega_e} h_e \left[c_{66}\frac{\partial w_2}{\partial x}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{\partial w_2}{\partial y}\left(c_{12}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y}\right) + \rho w_2 \ddot{u}_y\right] dxdy$$

$$- \oint_{\Omega_e} h_e w_2 f_y dxdy - \oint_{\Gamma_e} h_e w_2 t_y ds$$

Plane elasticity

An examination of the weak forms reveals that:

- u_x **and** u_y **are the primary variables, which must be carried as the primary nodal degrees of freedom**

- **Only first derivatives of** u_x **and** u_y **with respect to** x **and** y**, respectively, appear. Therefore,** u_x **and** u_y **must be approximated by the Lagrange family of interpolation functions, and at least bilinear (i.e, linear both in** x **and** y**) interpolation is required**

The simplest elements that satisfy those requirements are the linear triangular and linear quadrilateral elements

- **Although** u_x **and** u_y **are independent of each other, they are the components of the displacement vector. Therefore, both components should be approximated using the same type and degree of interpolation**

Finite element model

General Model:

Let u_x and u_y be approximated by the finite element interpolations (the element label e is omitted in the interest of brevity)

$$u_x \approx \sum_{j=1}^{n} u_x^j \psi_j(x,y) \qquad u_y \approx \sum_{j=1}^{n} u_y^j \psi_j(x,y)$$

or

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \psi\Delta \qquad \mathbf{w} = \delta\mathbf{u} = \begin{Bmatrix} w_1 = \delta u_x \\ w_2 = \delta u_y \end{Bmatrix} = \boldsymbol{\psi}\delta\Delta$$

where

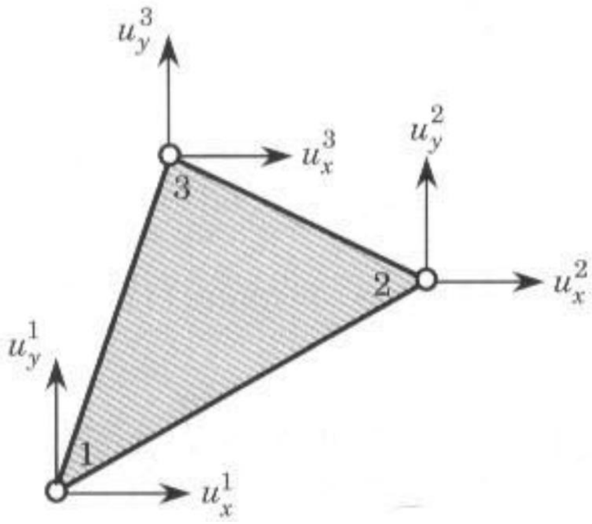
$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 & 0 & \psi_2 & 0 & \cdots & \psi_n & 0 \\ 0 & \psi_1 & 0 & \psi_2 & \cdots & 0 & \psi_n \end{bmatrix}$$

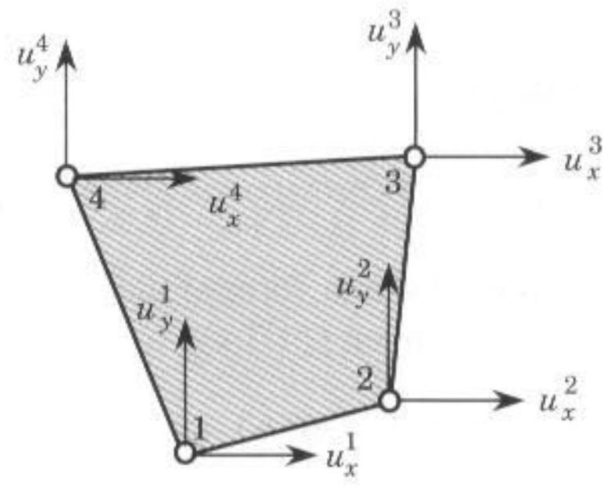
$$\Delta = \{u_x^1 \quad u_y^1 \quad u_x^2 \quad u_y^2 \quad \cdots \quad u_x^n \quad u_y^n\}^{\mathrm{T}}$$

Plane elasticity

At the moment, we will not restrict ψ_i to any specific element so that the finite element formulation to be developed is valid for any admissible element

- **If a linear triangular element ($n = 3$) is used, we have two (u_x^i, u_y^i) ($i = 1, 2, 3$) degrees of freedom per node and a total of six nodal displacements per element**
- **For a linear quadrilateral element, there are a total of eight nodal displacements per element**

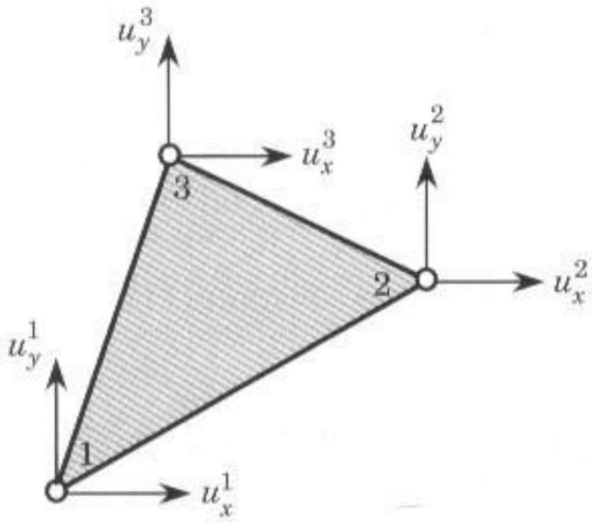


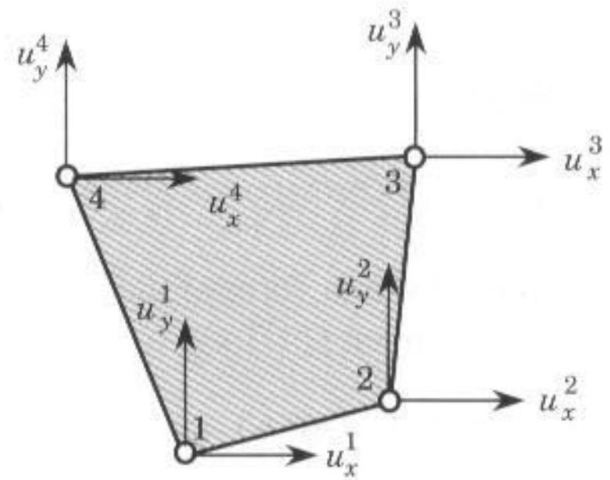


Plane elasticity

Since the first derivatives of ψ_i^e for a triangular element are elementwise constant, all the strains (ε_{xx} , ε_{yy} , ε_{xy}) computed for the linear triangular element are elementwise constant

- The linear triangular element for plane elasticity problems is known as the constant-strain triangular (CST) element
- For a quadrilateral element the first derivatives of ψ_i^e are not constant: $\partial\psi_i^e/\partial\xi$ is linear in η and constant in ξ, and $\partial\psi_i^e/\partial\eta$ is linear in ξ and constant in η)





Plane elasticity

The strains are

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} = \mathbf{D}\boldsymbol{\psi}\Delta \equiv \mathbf{B}\Delta, \qquad \boldsymbol{\sigma} = \mathbf{C}\mathbf{B}\Delta$$

$$\mathbf{B} = \mathbf{D}\boldsymbol{\psi} = \begin{bmatrix} \frac{\partial\psi_1}{\partial x} & 0 & \frac{\partial\psi_2}{\partial x} & 0 & \cdots & \frac{\partial\psi_n}{\partial x} & 0 \\ 0 & \frac{\partial\psi_1}{\partial y} & 0 & \frac{\partial\psi_1}{\partial y} & \cdots & 0 & \frac{\partial\psi_n}{\partial y} \\ \frac{\partial\psi_1}{\partial y} & \frac{\partial\psi_1}{\partial x} & \frac{\partial\psi_2}{\partial y} & \frac{\partial\psi_2}{\partial x} & \cdots & \frac{\partial\psi_n}{\partial y} & \frac{\partial\psi_n}{\partial x} \end{bmatrix}$$

$$\mathbf{D} = (\mathbf{D}^*)^{\mathrm{T}} \qquad \mathbf{D}^* = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

Substituting approximation for u_x and u_y to weak form, setting $w_i = \psi_i$ and $w_2 = \psi_i$ to obtain the ith algebraic equation associated with each of the weak statements and writing the resulting algebraic equations in matrix form, we obtain

Plane elasticity

$$u_x \approx \sum_{j=1}^{n} u_x^j \psi_j(x,y) \qquad u_y \approx \sum_{j=1}^{n} u_y^j \psi_j(x,y)$$

$$0 = \int_{\Omega_e} h_e \left[\frac{\partial w_1}{\partial x}\left(c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y} \right) + c_{66}\frac{\partial w_1}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \rho w_1 \ddot{u}_x \right] dxdy$$

$$- \oint_{\Omega_e} h_e w_1 f_x dxdy - \oint_{\Gamma_e} h_e w_1 t_x ds$$

$$0 = \int_{\Omega_e} h_e \left[c_{66}\frac{\partial w_2}{\partial x}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \frac{\partial w_2}{\partial y}\left(c_{12}\frac{\partial u_x}{\partial x} + c_{22}\frac{\partial u_y}{\partial y} \right) + \rho w_2 \ddot{u}_y \right] dxdy$$

$$- \oint_{\Omega_e} h_e w_2 f_y dxdy - \oint_{\Gamma_e} h_e w_2 t_y ds$$

$$\begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}_x\} \\ \{\ddot{u}_y\} \end{Bmatrix} + \begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u_x\} \\ \{u_y\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

Plane elasticity

Matrix form

$$\begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}_x\} \\ \{\ddot{u}_y\} \end{Bmatrix} + \begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u_x\} \\ \{u_y\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$M_{ij} = \int_{\Omega_e} \rho h \psi_i \psi_j dxdy$$

$$K_{ij}^{11} = \int_{\Omega_e} h(c_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + c_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} dxdy$$

$$K_{ij}^{12} = K_{ji}^{21} = \int_{\Omega_e} h(c_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + c_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} dxdy$$

$$K_{ij}^{22} = \int_{\Omega_e} h(c_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + c_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} dxdy$$

$$F_i^1 = \int_{\Omega_e} h \psi_i f_x dxdy + \oint_{\Gamma_e} h \psi_i t_x ds\,, \qquad F_i^2 = \int_{\Omega_e} h \psi_i f_y dxdy + \oint_{\Gamma_e} h \psi_i t_y ds$$

Plane elasticity

Vector form

$$\mathbf{M}^e\ddot{\Delta}^e + \mathbf{K}^e\Delta^e = \mathbf{F}^e + \mathbf{Q}^e$$

$$\mathbf{K}^e = \int_{\Omega_e} h_e \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}d\mathbf{x} \qquad \mathbf{M}^e = \int_{\Omega_e} \rho h_e \mathbf{\Psi}^{\mathrm{T}}\mathbf{\Psi}d\mathbf{x}$$

$$\mathbf{F}^e = \int_{\Omega_e} h_e \mathbf{\Psi}^{\mathrm{T}}\mathbf{f}d\mathbf{x} \qquad \mathbf{Q}^e = \int_{\Omega_e} h_e \mathbf{\Psi}^{\mathrm{T}}\mathbf{t}d\mathbf{x}$$

The element mass matrix $\mathrm{M}^e$and stiffness matrix $\mathrm{K}^e$are of order $2n \times 2n$, and the element load vector Fe and the vector of internal forces Q^e are of order $2n \times 1$ (n is the number of nodes in a Lagrange finite element)

Transient Problems

For transient analysis, using the time-approximation method

$$\mathbf{M}^e\ddot{\Delta}^e + \mathbf{K}^e\Delta^e = \mathbf{F}^e + \mathbf{Q}^e$$

$$\boxed{\hat{\mathbf{K}}^e_{s+1}\Delta^e_{s+1} = \hat{\mathbf{F}}^e_{s,s+1}}$$

$$\hat{\mathbf{K}}^e_{s+1} = \mathbf{K}^e_{s+1} + a_3\mathbf{M}^e_{s+1}$$

where

$$\hat{\mathbf{F}}^e_{s,s+1} = \bar{\mathbf{F}}^e_{s+1} + \mathbf{M}^e_{s+1}(a_3\Delta^e_s + a_4\dot{\Delta}^e_s + a_5\ddot{\Delta}^e_s)$$

$$a_3 = \frac{2}{\gamma(\Delta t)^2}, \quad a_4 = \Delta t a_3, \quad a_5 = \frac{1}{\gamma} - 1$$

γ **is parameter in NewMark method**

$$\bar{\mathbf{F}}^e = \mathbf{F}^e + \mathbf{Q}^e$$

$\mathbf{K}^e$, $\mathbf{M}^e$ and $\bar{\mathbf{F}}^e$ **is the vector appeared before**

Evaluation of integrals

For the linear triangular (i.e, CST) element, the ψ_i^e and its derivatives are given by

$$\psi_i^e = \frac{1}{2A_e}(\alpha_i^e + \beta_i^e x + \gamma_i^e y), \qquad \frac{\partial \psi_i^e}{\partial x} = \frac{\beta_i^e}{2A_e}, \qquad \frac{\partial \psi_i^e}{\partial y} = \frac{\gamma_i^e}{2\mathrm{A}_e}$$

Since the derivatives of ψ_i^e are constant, we have

$$\mathbf{B}^e = \frac{1}{2A_e}\begin{bmatrix} \beta_1^e & 0 & \beta_2^e & 0 & \cdots & \beta_n^e & 0 \\ 0 & \gamma_1^e & 0 & \gamma_2^e & \cdots & 0 & \gamma_n^e \\ \gamma_1^e & \beta_1^e & \gamma_2^e & \beta_2^e & \cdots & \gamma_n^e & \beta_n^e \end{bmatrix} \quad (3 \times 2n)$$

where A_e, is the area of the triangular element. Since B and C are independent of x and y, the element stiffness matrix for the CST element is given by

$$\mathbf{K}^e = h_e \mathrm{A}_e (\mathbf{B}^e)^{\mathrm{T}} \mathbf{C}^e \mathbf{B}^e \qquad (2n \times 2n)$$

Plane elasticity

For the case in which the body force components f_x and f_y, are elementwise constant（say, equal to f_{x0}^e and f_{y0}^e, respectively ）the load vector $\mathbf{F}^e$ has the form

$$\mathbf{F}^e = \int_{\Omega_e} h_e (\Psi^e)^{\mathrm{T}} \mathbf{f}_0^e \, d\mathbf{x} = \frac{A_e h_e}{3} \begin{Bmatrix} f_{x0}^e \\ f_{y0}^e \\ f_{x0}^e \\ f_{y0}^e \\ f_{x0}^e \\ f_{y0}^e \end{Bmatrix} (6 \times 1)$$

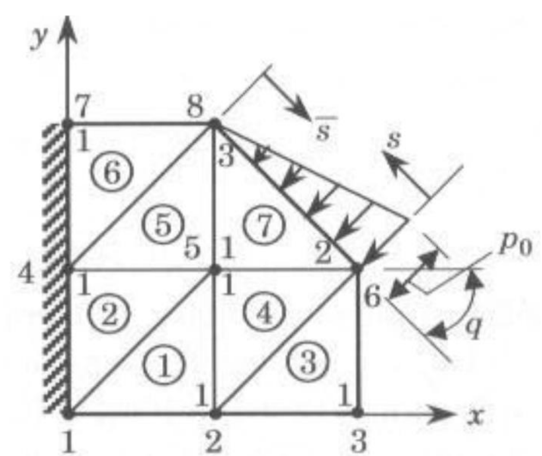
For a general quadrilateral element, it is not easy to compute the coefficients of the stiffness matrix by hand. In such cases we use the numerical integration method

Plane elasticity

Assembly of finite element equations

The assembly procedure for problems with many degrees of freedom is the same as that used for a single degree of freedom problem, except that the procedure should be applied to both degrees of freedom at each node

For example, consider the plane elastic structure and the finite element mesh used in Fig.

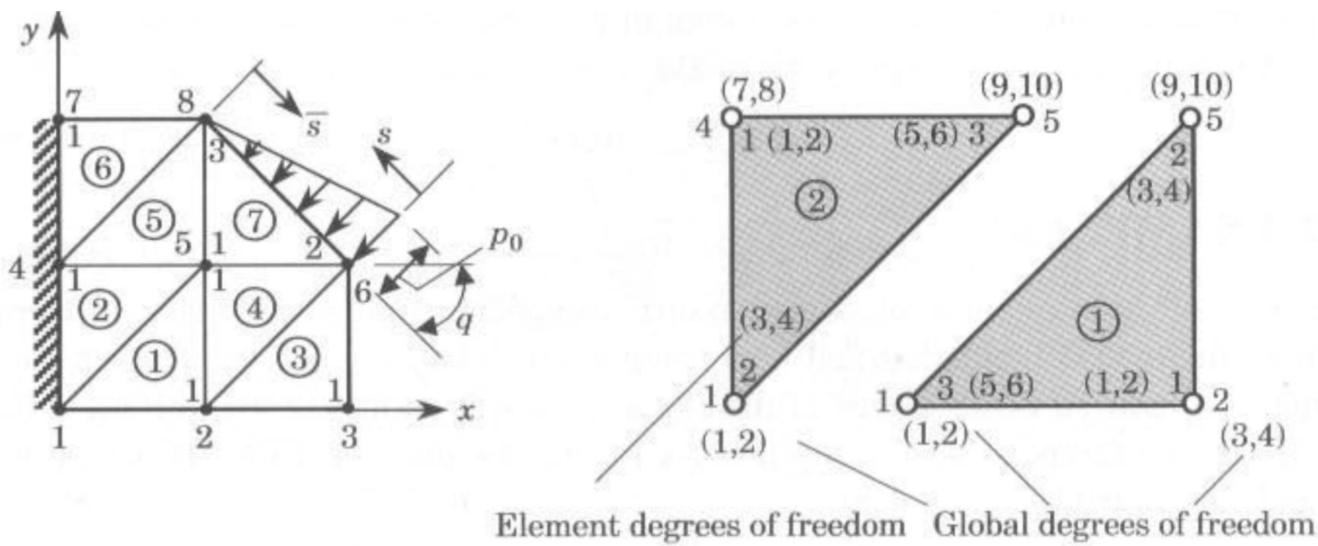


There are 8 nodes in the mesh: hence the total size of the assembled stiffness matrix will be 16 x 16, and the force vector will be 16 x 1.

Plane elasticity

The first two rows and columns of the global stiffness matrix, correspond to the global degrees (1,2) of freedom at global node 1, which has contributions from nodes 2 and 3 of elements 1 and 2, respectively

- **Thus, the contributions to global coefficients K_{IJ} $(I,J = 1,2)$ come from K_{ij}^1 $(i,j = 3,4)$ and K_{ij}^2 $(i,j = 5,6)$**

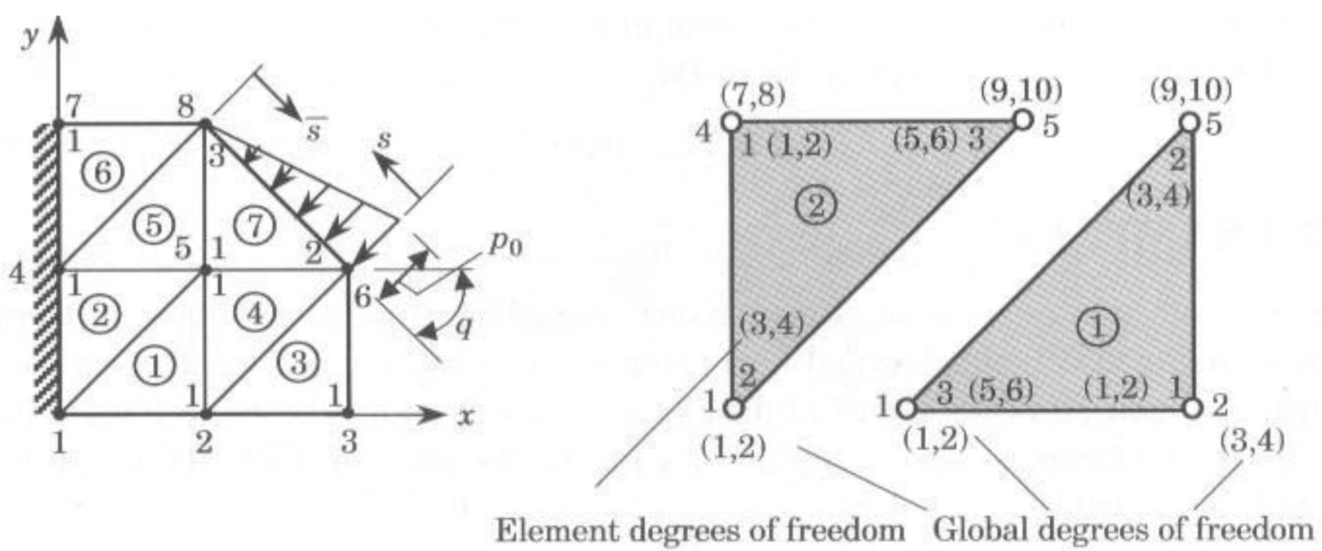


Plane elasticity

The global stiffness matrix coefficients K_{11}, K_{12}, K_{13}, K_{15}, K_{22}, K_{33} and K_{34} are known in terms of the element coefficients as follows

$$K_{11} = K_{55}^1 + K_{33}^2, \quad K_{22} = K_{66}^1 + K_{44}^2, \quad K_{12} = K_{56}^1 + K_{34}^2, \quad K_{13} = K_{51}^1$$

$$K_{33} = K_{11}^1 + K_{55}^3 + K_{33}^4, \quad K_{34} = K_{12}^1 + K_{56}^3 + K_{34}^4, \quad K_{15} = 0$$

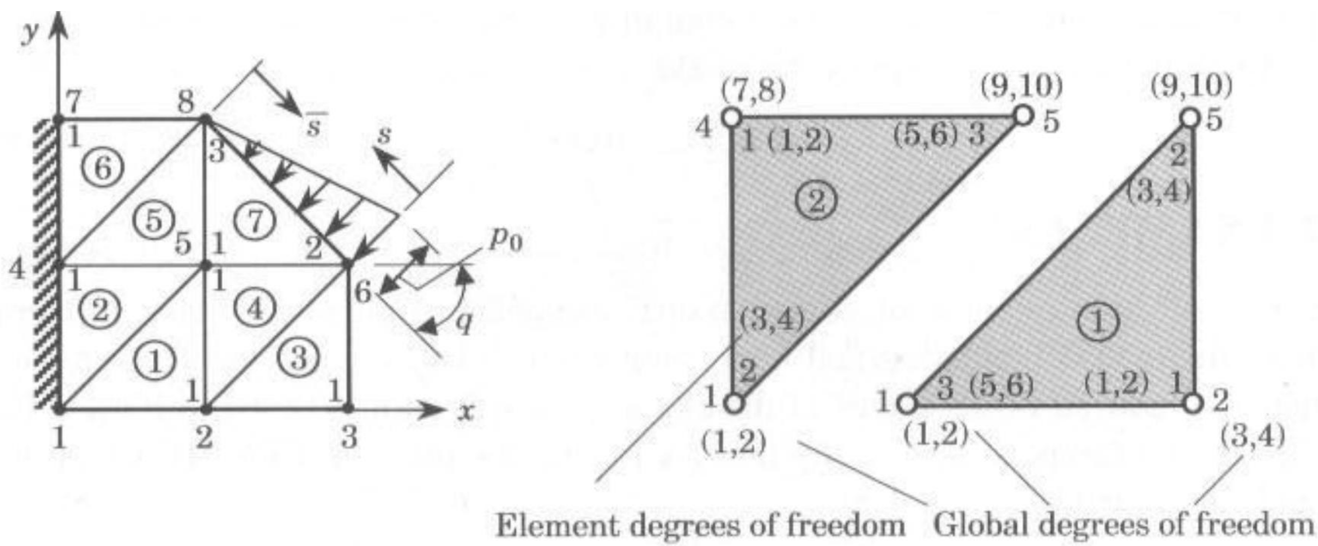


Plane elasticity

$$K_{34} = K_{12}^{1} + K_{56}^{3} + K_{34}^{4} \qquad K_{13} = K_{51}^{1}$$

- K_{34}**denotes the coupling stiffness coefficient between the third (u_x) and fourth (u_y) global displacement degrees of freedom, both of which are at global node 2**

- K_{13} **denotes the coupling coefficient between first displacement degree of freedom (u_x) at global node 1 and third global displacement degree (u_x) of freedom at global node 2**



Plane elasticity

With regard to the specification of the displacements（the primary degrees of freedom）and forces（the secondary degrees of freedom） in a finite element mesh, we have the fol lowing four distinct possibilities

Case1: u_x and u_y are specified(and t_x and t_y and unknown)
Case2: u_x and t_y are specified(and t_x and u_y and unknown)
Case3: t_x and u_y are specified(and u_x and t_y and unknown)
Case4: t_x and t_y are specified(and u_x and u_y and unknown)

In general, only one of the quantities of each of the pairs (u_x, t_x) and (u_y, t_y) is known at a nodal point in the mesh. We are required to make a decision as to which degree of freedom is known when singular points are encountered

谢 谢！

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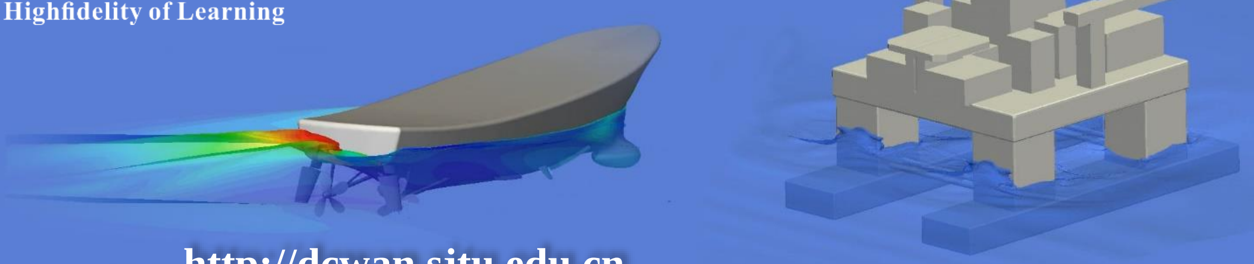

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