上海交通大學 SHANGHAI JIAO TONG UNIVERSITY

CMHL SJTU COMPUTATIONAL MARINE HYDRODYNAMICS LAB 上海交大船舶与海洋工程计算水动力学研究中心

Class-5

NA26018

Finite Element Analysis of Solids and Fluids

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Single Variable Problem (2D)

Linear Rectangular Element

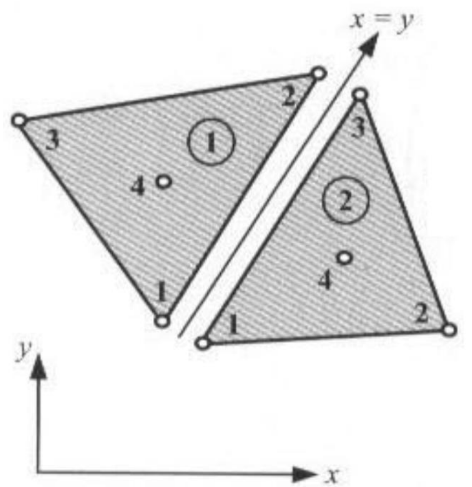
Next, consider the complete polynomial

$$u_h^e(x,y) = c_1^e + c_2^e x + c_3^e y + c_4^e xy$$

which contains 4 linearly independent terms and is linear in x and y, with a bilinear term x and y. This polynomial requires an element with four nodes.

Two possible geometric shapes:

- **a triangle with the 4 node at the center (or centroid) of the triangle**
- **a rectangle with the nodes at the vertices.**

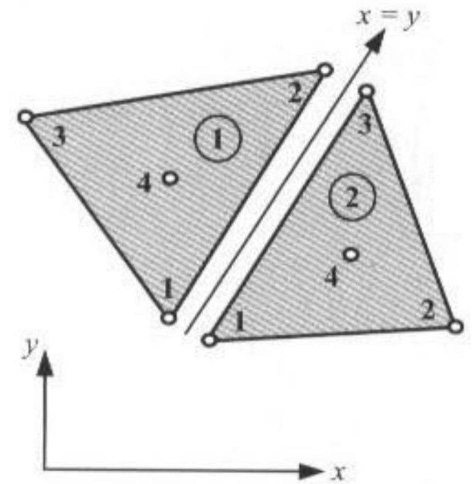


Single Variable Problem (2D)

Note:
A triangle with a fourth node at the center does not provide a single-valued variation of u at interelement boundaries, resulting in incompatible variation of u at interelement boundaries and is therefore not admissible

$$u_h^e(x,y) = c_1^e + c_2^e x + c_3^e y + c_4^e xy$$

$$= c_1^e + (c_2^e + c_3^e)x + c_4^e x^2 (e = 1,2)$$



Thus, the quadratic variation of u_h along $x = y$ cannot be defined uniquely with only two nodal values

Single Variable Problem (2D)

$$u_h^e(x,y) = c_1^e + c_2^e x + c_3^e y + c_4^e xy$$

Here we consider an approximation of the form use a rectangular element with sides a and b. For the sake of convenience, we choose a local coordinate system $(\bar{x}, \bar{y})$ to derive the interpolation functions. We assume that (the element label is omitted)

$$u_h^e(\bar{x},\bar{y}) = c_1 + c_2\bar{x} + c_3\bar{y} + c_4\bar{x}\bar{y}$$

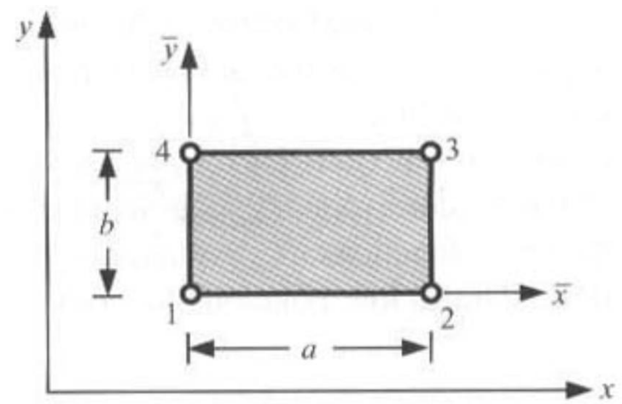
This require

$$u_1 = u_h(0,0) = c_1$$

$$u_2 = u_h(a,0) = c_1 + c_2 a$$

$$u_3 = u_h(a,b) = c_1 + c_2 a + c_3 b + c_4 ab$$

$$u_4 = u_h(0,b) = c_1 + c_3 b$$



Single Variable Problem (2D)

Solving for $c_i (i = 1,2,3,4)$ we obtain

$$c_1 = u_1, c_2 = \frac{u_2 - u_1}{a}$$

$$c_3 = \frac{u_4 - u_1}{b}, c_4 = \frac{u_3 - u_4 + u_1 - u_2}{ab}$$

$$u_h(\bar{x}, \bar{y}) = c_1 + c_2\bar{x} + c_3\bar{y} + c_4\bar{x}\bar{y}$$

$$u_h(\bar{x}, \bar{y}) = u_1\left(1 - \frac{\bar{x}}{a} - \frac{\bar{y}}{b} + \frac{\bar{x}}{a}\frac{\bar{y}}{b}\right) + u_2\left(\frac{\bar{x}}{a} - \frac{\bar{x}}{a}\frac{\bar{y}}{b}\right) + u_3\frac{\bar{x}}{a}\frac{\bar{y}}{b} + u_4\left(\frac{\bar{y}}{b} - \frac{\bar{x}}{a}\frac{\bar{y}}{b}\right)$$

$$= u_1^e\psi_1^e + u_2^e\psi_2^e + u_3^e\psi_3^e + u_4^e\psi_4^e$$

where

$$\psi_1^e = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right), \psi_2^e = \frac{\bar{x}}{a}\left(1 - \frac{\bar{y}}{b}\right)$$

$$\psi_3^e = \frac{\bar{x}}{a}\frac{\bar{y}}{b}, \psi_4^e = \left(1 - \frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$

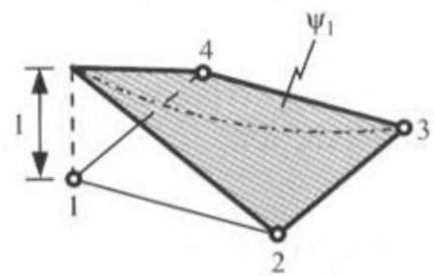
Single Variable Problem (2D)

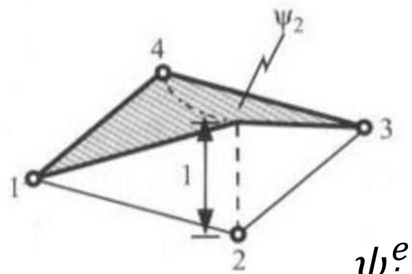
$$\psi_1^e = \left(1-\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right), \psi_2^e = \frac{\bar{x}}{a}\left(1-\frac{\bar{y}}{b}\right)$$

$$\psi_3^e = \frac{\bar{x}}{a}\frac{\bar{y}}{b}, \psi_4^e = \left(1-\frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$

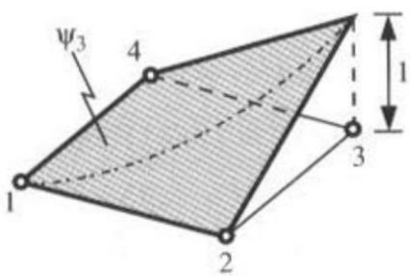
In concise form,

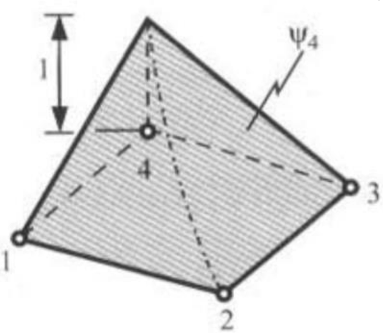
$$\psi_i^e(\bar{x},\bar{y}) = (-1)^{i+1}\left(1-\frac{\bar{x}+\bar{x}_i}{a}\right)\left(1-\frac{\bar{y}+\bar{y}_i}{b}\right)$$





$$\psi_i^e\left(\bar{x}_j,\bar{y}_j\right) = \delta_{i,j}\ (i,j = 1,\cdots,4), \sum_{i=1}^{4}\psi_i^e = 1$$



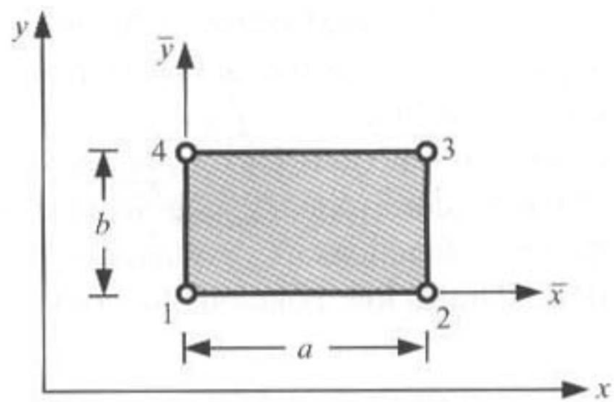


Single Variable Problem (2D)

The procedure given above for the construction of the interpolation functions involves the inversion of an $n \times n$ matrix, where n is the number of nodes in the element. When n is large, the inversion becomes very tedious

Alternatively, the interpolation functions for rectangular element can also be obtained by taking the tensor product of the corresponding one-dimensional interpolation functions. To obtain the linear interpolation functions of a rectangular element, we take the "tensor product" of the one-dimensional linear interpolation functions associated with sides 1-2 and 1-3:

$$\begin{pmatrix} 1-\frac{\bar{x}}{a} \\ \frac{\bar{x}}{a} \end{pmatrix} \begin{pmatrix} 1-\frac{\bar{y}}{b} \\ \frac{\bar{y}}{b} \end{pmatrix}^{\mathrm{T}} = \begin{bmatrix} \psi_1 & \psi_4 \\ \psi_2 & \psi_3 \end{bmatrix}$$



Single Variable Problem (2D)

$$\psi_i^e(\bar{x}_j, \bar{y}_j) = \delta_{i,j}\ (i,j = 1,\cdots,4),\ \sum_{i=1}^{4} \psi_i^e = 1$$

The alternative procedure that makes use of the interpolation properties can also be used. Here we illustrate the alternative procedure for the 4-node rectangular element Equation requires that

$$\psi_1^e(\bar{x}_i, \bar{y}_i) = 0\ (i = 2,3,4), \qquad \psi_1^e(\bar{x}_1, \bar{y}_1) = 1$$

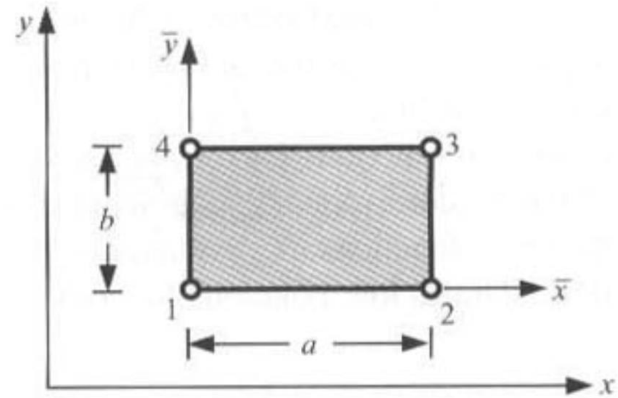
That is, ψ_i^e is identically zero on lines $x = a$ and $y = b$. Hence, $\psi_i^e(\bar{x}, \bar{y})$ must be of the form

$$\psi_1^e(\bar{x}, \bar{y}) = c_1(a - \bar{x})(b - \bar{y}) \text{ for any } c_1 \neq 0$$

Using the condition $\psi_i^e(\bar{x}, \bar{y}) = \psi_i^e(0,0) = 1$, we obtain $c_1 = 1/ab$. Hence,

$$\psi_1^e(\bar{x}, \bar{y}) = \frac{1}{ab}(a - \bar{x})(b - \bar{y}) = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right)$$

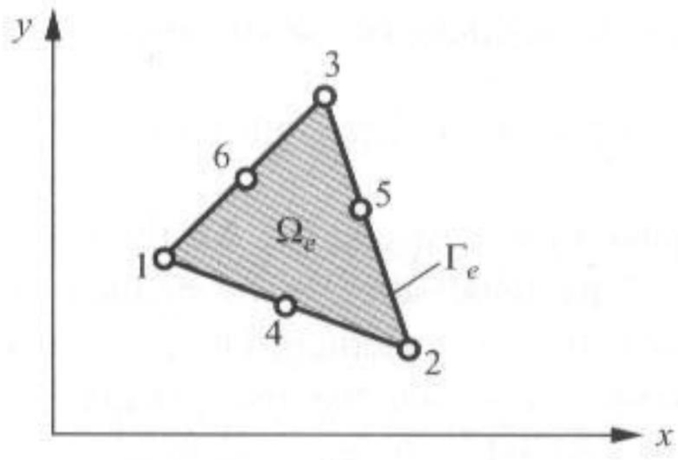
Likewise, we can obtain the remaining three interpolation functions



Quadratic Elements

A quadratic triangular element must have 3 nodes per side in order to define a unique quadratic variation along that side. Thus, there are a total of six nodes in a quadratic triangular element. A six-term complete polynomial that includes both x and y is

$$u_h^e(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2$$



- **The constants ci may be derived in terms of the six nodal values by the procedure outlined for the three-node triangular element and four-node rectangular element**
- **However, in practice the interpolation functions of higher-order elements are derived using the alternative procedure**

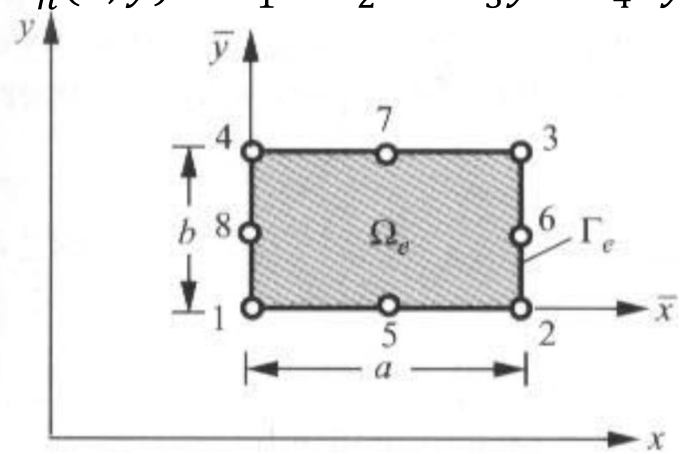
Single Variable Problem (2D)

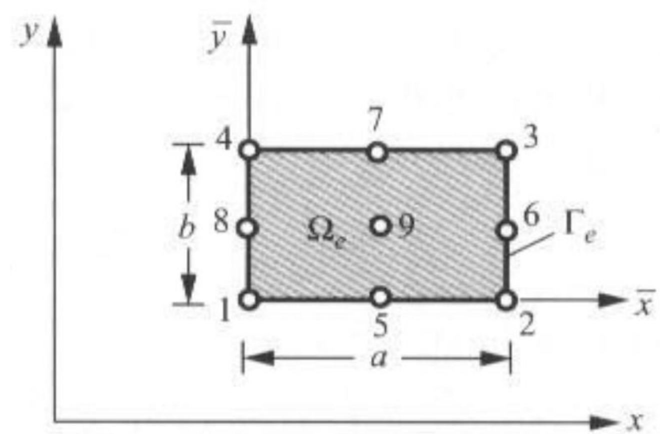
Similarly, a quadratic rectangular element has 3 nodes per side, resulting in an eight-node rectangular. The eight-term polynomial is

$$u_h^e(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2 + c_7 xy^2 + c_8 yx^2$$

The interpolation functions of this element cannot be generated by the tensor product of one-dimensional quadratic. Indeed, the two-dimensional interpolation functions associated with the tensor product of one-dimensional quadratic functions correspond to the nine-node rectangular element.
The nine-term polynomial is given by

$$u_h^e(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2 + c_7 xy^2 + c_8 yx^2 + c_9 x^2 y^2$$





Evaluation of Element Matrices and Vectors

- The exact evaluation of the element matrices $\{K_e\}$ and $\{f_e\}$ is, in general not easy. In general, they are evaluated using numerical integration techniques. However, when aij, a00, and f are elementwise constant, it is possible to evaluate the integrals exactly over the linear triangular and rectangular elements discussed in the previous section.

- The boundary integral in $\{Q_e\}$ of weak form can be evaluated whenever qn is known. For an interior element (i.e an element that does not have any of its sides on the boundary of the problem), the contribution from the boundary integral cancels with similar contributions from adjoining elements of the mesh

Single Variable Problem (2D)

$$K_{ij}^{e}=\int_{\Omega_e}\left[\frac{\partial\psi_i^e}{\partial x}\left(a_{11}\frac{\partial\psi_j^e}{\partial x}+a_{12}\frac{\partial\psi_j^e}{\partial y}\right)+\frac{\partial\psi_j^e}{\partial y}\left(a_{21}\frac{\partial\psi_j^e}{\partial x}+a_{22}\frac{\partial\psi_j^e}{\partial y}\right)+a_{00}\psi_i^e\psi_j^e\right]dxdy$$

For the sake of brevity, we rewrite $\{K_e\}$ as the sum of five basic matrices

$$[K^e]=a_{00}[S^{00}]+a_{11}[S^{11}]+a_{12}[S^{12}]+a_{21}[S^{12}]^T+a_{22}[S^{22}]$$

$$S_{ij}^{\alpha\beta}=\int_{\Omega}\psi_{i,\alpha}\psi_{j,\beta}dxdy$$

$$\psi_{i,\alpha}\equiv\partial\psi_i/\partial x_\alpha,\ x_1=x,\quad and\quad x_2=y;\ \psi_{i,0}=\psi_i$$

All the matrices and interpolation functions are understood to be defined over an element, i.e. all expressions and quantities should have the element label e, but these are omitted in the interest of brevity

Single Variable Problem (2D)

Now proceed to compute the using the linear interpolation functions derived in the previous section

Element Matrices of a Linear Triangular Element

First, we note that integrals of polynomials over arbitrary shaped triangular domains can be evaluated exactly. To this end, let I_{mn} denote the integral of the expression $x^m y^n$ over an arbitrary triangle Δ

$$I_{mn} \equiv \int_{\Delta} x^m y^n dxdy$$

Single Variable Problem (2D)

$$I_{00} = \int_{\triangle} x^0 y^o dxdy = \int_{\triangle} 1 \cdot dxdy = A$$

area of the triangle

$$I_{10} = \int_{\triangle} x^1 y^o dxdy = \int_{\triangle} x\, dxdy = A\hat{x}$$

$$\hat{x} = \frac{1}{3}\sum_{i=1}^{3} x_i$$

$$I_{01} = \int_{\triangle} x^0 y^1 dxdy = \int_{\triangle} y\, dxdy = A\hat{y}$$

$$\hat{y} = \frac{1}{3}\sum_{i=1}^{3} y_i$$

$$I_{11} = \int_{\triangle} xy\, dxdy = \frac{A}{12}\left(\sum_{i=1}^{3} x_i y_i + 9\hat{x}\hat{y}\right)$$

$$I_{20} = \int_{\triangle} x^2\, dxdy = \frac{A}{12}\left(\sum_{i=1}^{3} x_i^2 + 9\hat{x}^2\right)$$

$$I_{02} = \int_{\triangle} y^2\, dxdy = \frac{A}{12}\left(\sum_{i=1}^{3} y_i^2 + 9\hat{y}^2\right)$$

Where (x_i, y_i) are the coordinates of the vertices of the triangle. We can use the above results to evaluate integrals defined over triangular elements

Single Variable Problem (2D)

Next, we evaluate $\{K^e\}$ and $\{f^e\}$ for linear triangular element under the assumption that aij and f are elementwise constant. Also, note that

$$\psi_i^e = \frac{1}{2A_e}(\alpha_i^e + \beta_i^e x + \gamma_i^e y) \qquad (i = 1,2,3)$$

$$\sum_{i=1}^{3} \alpha_i^e = 2A_e \quad \sum_{i=1}^{3} \beta_i^e = 0 \quad \sum_{i=1}^{3} \gamma_i^e = 0$$

$$\alpha_i^e + \beta_i^e \hat{x}_e + \gamma_i^e \hat{y}_e = \frac{2}{3} A_e$$

$$\frac{\partial \psi_i}{\partial x} = \frac{\beta_i^e}{2A_e} \qquad \frac{\partial \psi_i}{\partial y} = \frac{\gamma_i^e}{2A_e}$$

$$\hat{x} = \frac{1}{3}\sum_{i=1}^{3} x_i$$

$$\hat{y} = \frac{1}{3}\sum_{i=1}^{3} y_i$$

$$S_{ij}^{11} = \frac{1}{4A}\beta_i\beta_j \qquad S_{ij}^{12} = \frac{1}{4A}\beta_i\gamma_j \qquad S_{ij}^{22} = \frac{1}{4A}\gamma_i\gamma_j$$

$$S_{ij}^{00} = \frac{1}{4A}\{[\alpha_i\alpha_j + (\alpha_i\beta_j + \alpha_j\beta_i)\hat{x} + (\alpha_i\gamma_j + \alpha_j\gamma_i)\hat{y}]$$
$$+ \frac{1}{A}[I_{20}\beta_i\beta_j + I_{11}(\gamma_i\beta_j + \gamma_j\beta_i) + I_{02}\gamma_i\gamma_j]\}$$

Single Variable Problem (2D)

for an elementwise constant value of $f = f^e$,

$$f_i^e = \int_{\triangle_e} f_e \psi_i^e(x,y)dxdy = \frac{f_e}{2A_e}\int_{\triangle_e}(\alpha_i^e + \beta_i^e x + \gamma_i^e y)dxdy$$

$$= \frac{f_e}{2A_e}(\alpha_i^e I_{00} + \beta_i^e I_{10} + \gamma_i^e I_{01})$$

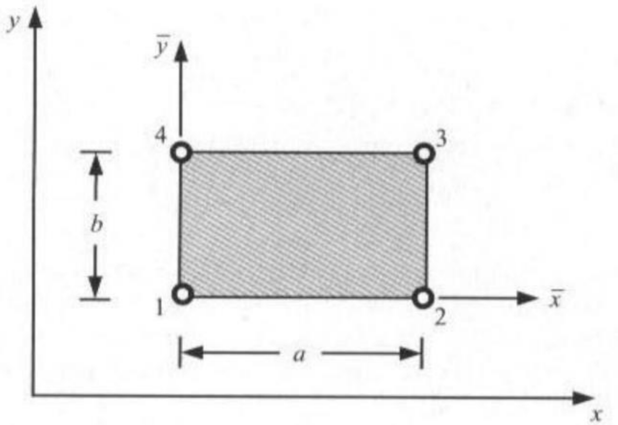
$$= \frac{f_e}{2A_e}(\alpha_i^e A_e + \beta_i^e A_e \hat{x}_e + \gamma_i^e A_e \hat{y}_e)$$

$$= \frac{1}{2} f_e(\alpha_i^e + \beta_i^e \hat{x}_e + \gamma_i^e \hat{y}_e) = \frac{1}{3} f_e A_e$$

The result should be obvious because for a constant source f_e the total magnitude of the source（heat） on the element is equal to $f_e A_e$, which is then distributed equally among the three nodes, giving a nodal value of $f_e A_e/3$

Single Variable Problem (2D)

Element Matrices of a Linear Rectangular Element



$$\psi_1^e = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right) \qquad \psi_2^e = \frac{\bar{x}}{a}\left(1 - \frac{\bar{y}}{b}\right)$$

$$\psi_3^e = \frac{\bar{x}}{a}\frac{\bar{y}}{b} \qquad \psi_4^e = \left(1 - \frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$

When the data a_{ij} $(i,j = 0,1,2)$ and f of the problem is not a function of x and y, we can use the interpolation functions, expressed in the local coordinates $(\bar{x},\ \bar{y})$ that are mere translation of (x, y), to compute the element coefficients $S_{ij}^{\alpha\beta}$ e.g.:

$$S_{ij}^{00} = \int_{\Omega_e} \psi_i(x,y)\psi_j(x,y)dxdy = \int_0^a \int_0^b \psi_i\ \psi_j d\bar{x}d\bar{y}$$

where a and b are the lengths along the x and y axes of the element

Single Variable Problem (2D)

Since the integration with respect to x and y can be carried out independent of each other, integration over a rectangular element becomes a pair of line integrals. We have

$$S_{11}^{00} = \int_0^a \int_0^b \psi_1\, \psi_1 d\bar{x} d\bar{y} = \int_0^a \int_0^b \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right)\left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right) d\bar{x} d\bar{y}$$

$$= \int_0^a \left(1 - \frac{\bar{x}}{a}\right)^2 d\bar{x} \int_0^b \left(1 - \frac{\bar{y}}{b}\right)^2 d\bar{y} = \frac{a}{3}\frac{b}{3} = \frac{ab}{9}$$

With the aid of

$$\int_0^a \left(1 - \frac{s}{a}\right) ds = \frac{a}{2} \qquad \int_0^a \frac{s}{a} ds = \frac{a}{2}$$

$$\int_0^a \left(1 - \frac{s}{a}\right)^2 ds = \frac{a}{3} \qquad \int_0^a \left(1 - \frac{s}{a}\right) ds = \frac{a}{6} \qquad \int_0^a \left(\frac{s}{a}\right)^2 ds = \frac{a}{3}$$

Single Variable Problem (2D)

In summary, the element matrices for a rectangular element are

$$[S^{11}] = \frac{b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} \qquad [S^{12}] = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

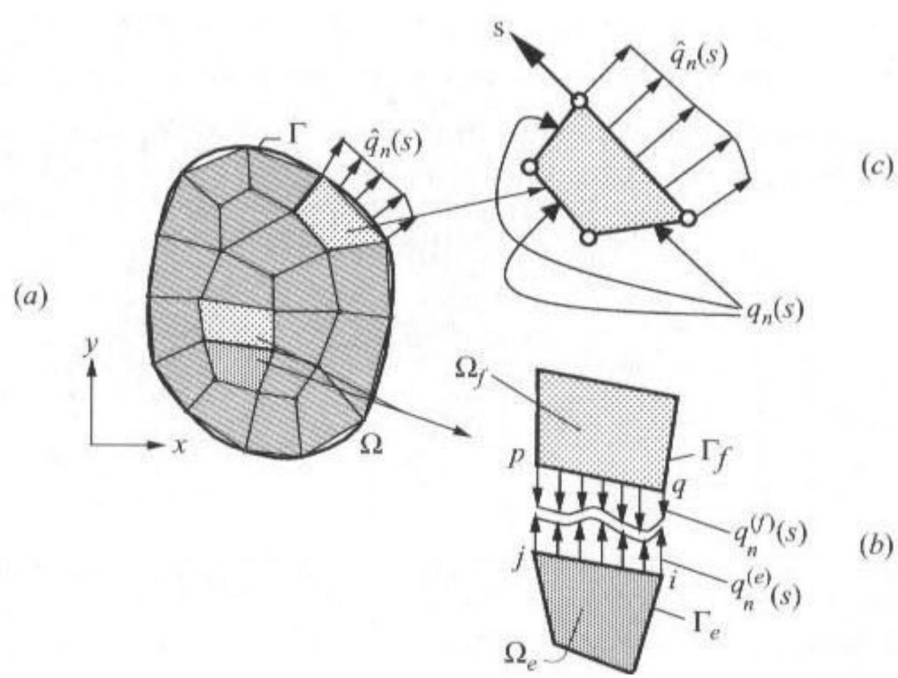
$$[S^{22}] = \frac{b}{6a}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -2 & 1 & 2 \end{bmatrix} \qquad [S^{00}] = \frac{b}{6a}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

$$\{f\} = \frac{1}{4} f_e ab \{1 \quad 1 \quad 1 \quad 1\}^{\mathrm{T}}$$

Evaluation of Boundary Integrals

Here, we consider the evaluation of boundary integrals of the type

$$Q_i^e = \oint_{\Gamma_e} q_n^e \psi_i^e(s)ds$$



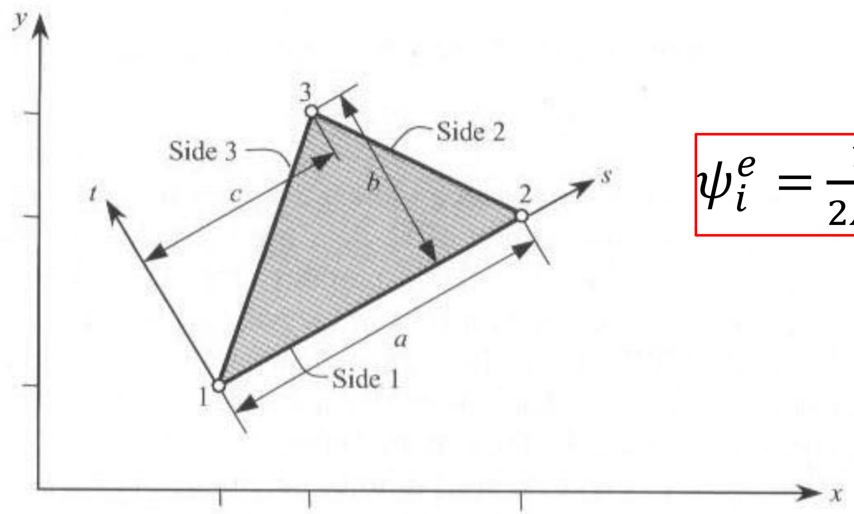
where qn is a known function of the distance s along the boundary Γ_e. It is not necessary to compute such integrals when a portion of Γ_e does not coincide with the boundary Γ of the total domain Ω

Single Variable Problem (2D)

The boundary Γ_e of a two-dimensional element consist of line segments, which can be viewed as one-dimensional elements

- **Thus, the evaluation of the boundary integrals on two-dimensional problems amounts to evaluating line integrals**
- **When two-dimensional interpolation functions are evaluated on the boundary of an element, we obtain the corresponding one-dimensional interpolation functions**

For example, consider a linear triangular element, The linear interpolation functions for this element are given by



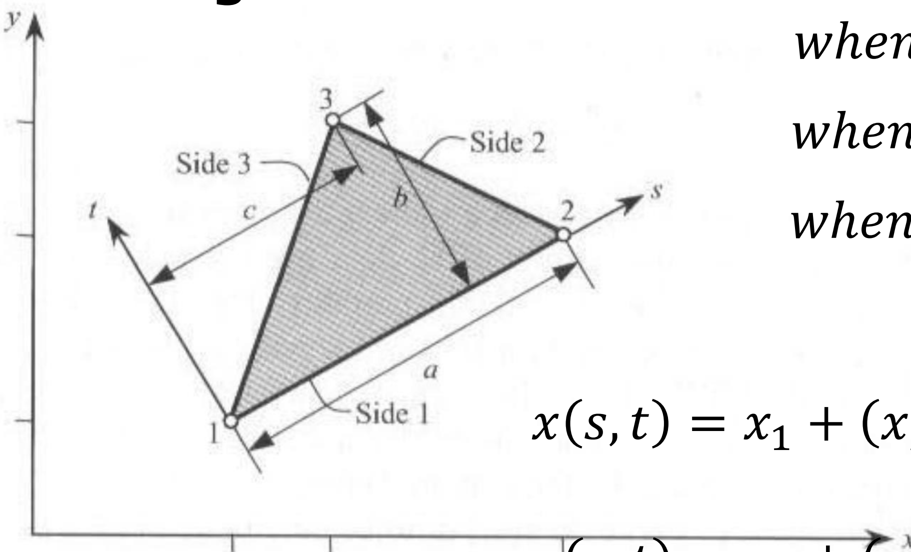
$$\psi_i^e = \frac{1}{2A_e}(\alpha_i^e + \beta_i^e x + \gamma_i^e y) \qquad (i = 1,2,3)$$

Single Variable Problem (2D)

Now let us choose a coordinate system (s,t) with its origin at node 1 and the coordinate s parallel to the side connecting nodes 1 and 2. The two coordinate systems (x,y) and (s,t) are related as follows

$$x = a_1 + b_1 s + c_1 t$$
$$y = a_2 + b_2 s + c_2 t$$

The constants a_1, b_1, c_1, a_2, b_2 and c_2 can be determined with the following conditions



$$when\ \ s = 0, t = 0, \qquad x = \ x_1, y = y_1$$
$$when\ \ s = a, t = 0, \qquad x = \ x_2, y = y_2$$
$$when\ \ s = c, t = b, \qquad x = \ x_3, y = y_3$$

$$x(s,t) = x_1 + (x_2 - x_1)\frac{s}{a} + \left[\left(\frac{c}{a} - 1\right)x_1 - \frac{c}{a}x_2 + x_3\right]\frac{t}{b}$$
$$y(s,t) = y_1 + (y_2 - y_1)\frac{s}{a} + \left[\left(\frac{c}{a} - 1\right)y_1 - \frac{c}{a}y_2 + y_3\right]\frac{t}{b}$$

Single Variable Problem (2D)

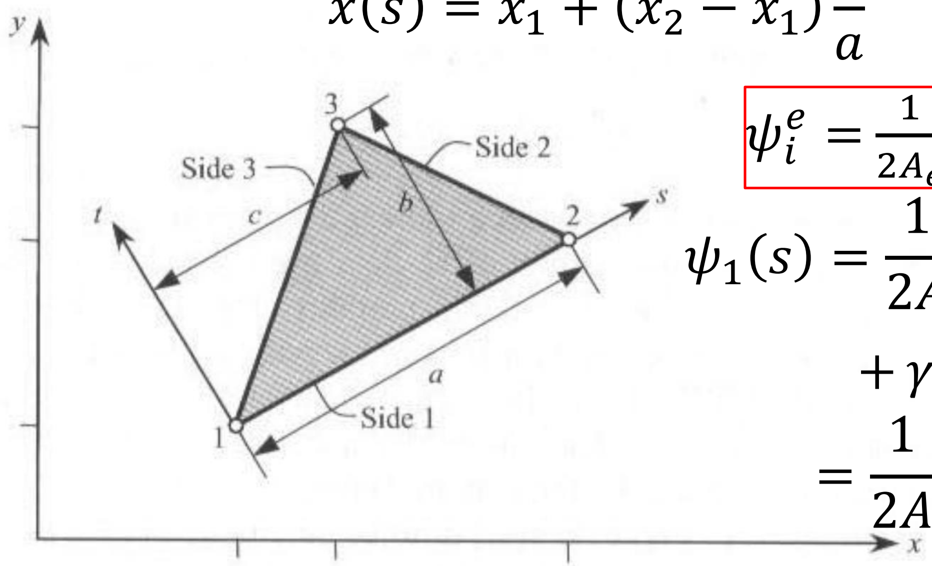
$$x(s,t) = x_1 + (x_2 - x_1)\frac{s}{a} + \left[\left(\frac{c}{a} - 1\right)x_1 - \frac{c}{a}x_2 + x_3\right]\frac{t}{b}$$

$$y(s,t) = y_1 + (y_2 - y_1)\frac{s}{a} + \left[\left(\frac{c}{a} - 1\right)y_1 - \frac{c}{a}y_2 + y_3\right]\frac{t}{b}$$

This allow us to express $\psi_i\,(x,y)$ as $\psi_i\,(s,t)$, which can be evaluated on the side connecting nodes 1 and 2 by setting $t=0$ in $\psi_i\,(s,t)$

$$\psi_i(s) \equiv \psi_i(s,0) = \psi_i\big(x(s,0), y(s,0)\big)$$

$$x(s) = x_1 + (x_2 - x_1)\frac{s}{a} \qquad y(s) = y_1 + (y_2 - y_1)\frac{s}{a}$$



$$\psi_i^e = \frac{1}{2A_e}(\alpha_i^e + \beta_i^e x + \gamma_i^e y) \qquad (i = 1,2,3)$$

$$\psi_1(s) = \frac{1}{2A}\left\{\alpha_1 + \beta_1\left[\left(1 - \frac{s}{a}\right)x_1 + \frac{s}{a}x_2\right] + \gamma_1\left[\left(1 - \frac{s}{a}\right)y_1 + \frac{s}{a}y_2\right]\right\}$$

$$= \frac{1}{2A}(\alpha_1 + \alpha_2 + \alpha_3)\left(1 - \frac{s}{a}\right) = 1 - \frac{s}{a}$$

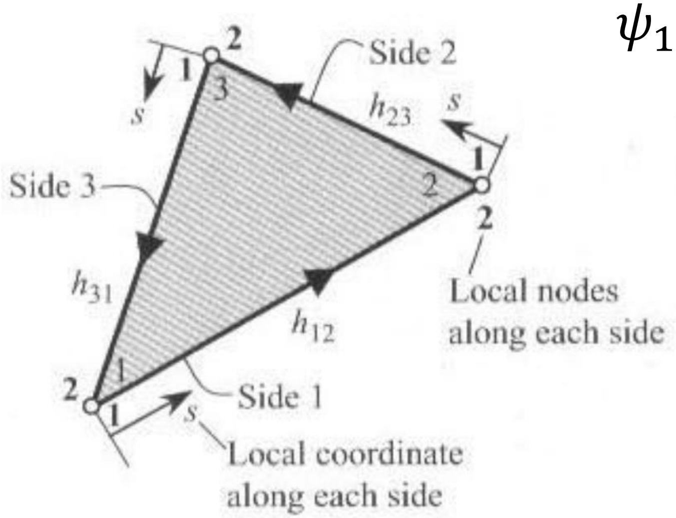
Single Variable Problem (2D)

Similarly, we have

$$\psi_2(s) = \frac{s}{a} \qquad \psi_3(s) = 0 \qquad \psi_1(s) = 1 - \frac{s}{a}$$

We note that $\psi_1(s)$ and $\psi_2(s)$ are precisely the linear, one-dimensional, interpolation functions associated with the line element connecting nodes 1 and 2 ($a = h_{12}$)
Similarly, when $\psi_i(x, y)$ are evaluated on side 3-1 of the element, we obtain

$$\psi_1(s) = \frac{s}{h_{13}} \qquad \psi_2 = 0 \qquad \psi_3(s) = 1 - \frac{s}{h_{13}}$$



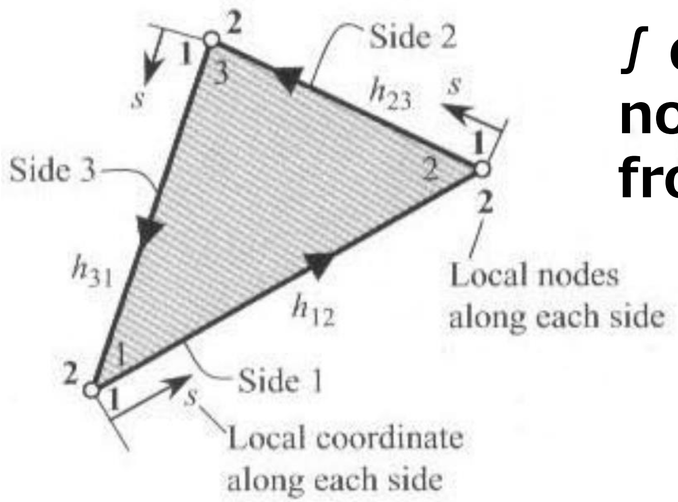
where the s coordinate is taken along the side 3-1, with origin at node 3, and h_{13} is the length of side 1-3. Thus, evaluation of ψ_i^e involves the use of appropriate one-dimensional interpolation functions and the known variation of q_n on the boundary

Single Variable Problem (2D)

$$Q_i^e = \oint_{\Gamma_e} q_n^e \psi_i^e(s)ds$$

In general, the integral over the boundary of a linear triangular element can be expressed as

$$Q_i^e = \int_{1-2} \psi_i(s) q_n(s) ds + \int_{2-3} \psi_i(s) q_n(s) ds + \int_{3-1} \psi_i(s) q_n(s) ds$$
$$\equiv Q_{i1}^e + Q_{i2}^e + Q_{i3}^e$$



$\int$ denotes integral over line connecting node i to node j, the s coordinate is taken from node i to node j, with origin at node i

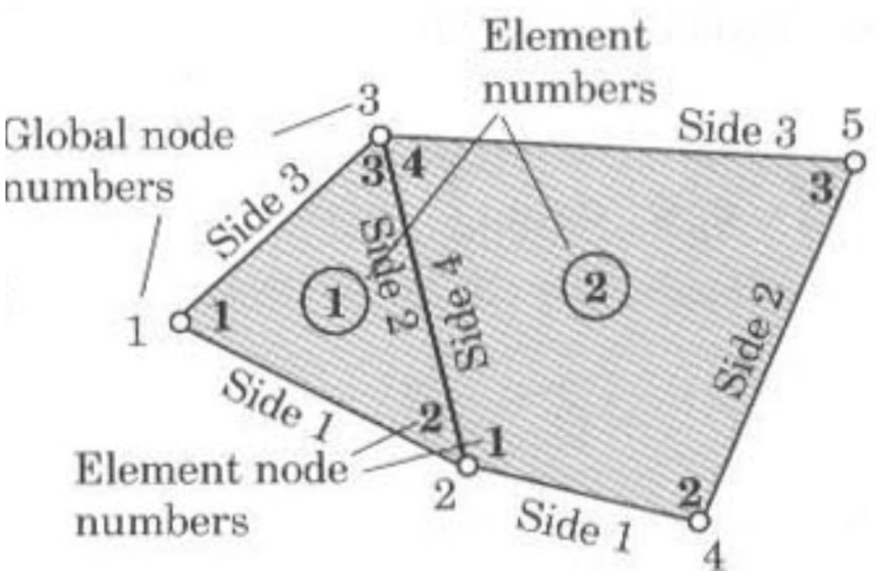
Single Variable Problem (2D)

Assembly of Element Equations

The assembly of finite element equations is based on the same two principles that were used in one-dimensional problems

1. **Continuity of primary variables**
2. **“Equilibrium“ (or “balance”) of secondary variables**

We illustrate the procedure by considering a finite element mesh consisting of a triangular element and a quadrilateral element



Let K_{ij}^{1} $(i,j = 1,2,3)$ denote the coefficient matrix corresponding to the triangular element, and let K_{ij}^{2} $(i,j = 1,2,3,4)$ denote the coefficient matrix corresponding to the quadrilateral element

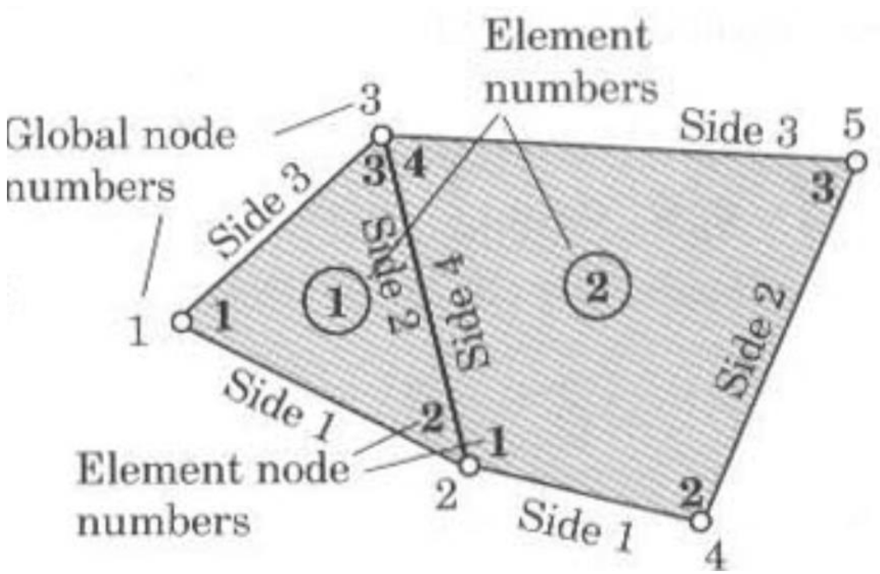
Single Variable Problem (2D)

From the finite element mesh shown in Fig, we note the following correspondence (i.e, connectivity relations) between the global and element nodes

$$[B] = \begin{bmatrix} 1 & 2 & 3 & \times \\ 2 & 4 & 5 & 3 \end{bmatrix}$$

where x indicates that there is no entry. The correspondence between the local and global nodal values is

$$u_1^1 = U_1 \qquad u_2^1 = u_1^2 = U_2 \qquad u_3^1 = u_4^2 = U_3 \qquad u_2^2 = U_4 \qquad u_3^2 = U_5$$



which amounts to imposing the continuity of the primary variables at the nodes common to elements 1 and 2

Single Variable Problem (2D)

Note that the continuity of the primary variables at the interelement nodes guarantees the continuity of the primary variable along the entire interelement boundary.

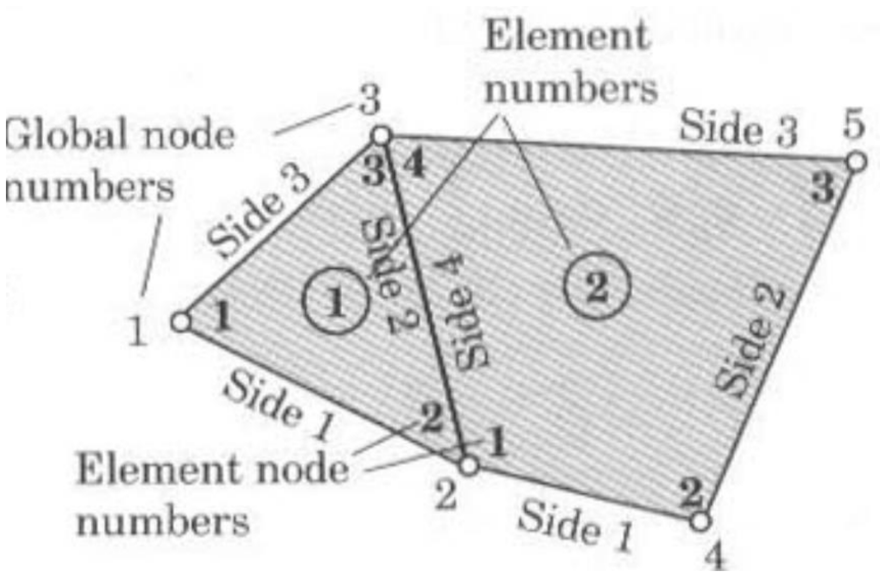
- **For the case in Fig., the requirement $u_2^1 = u_1^2$ and $u_3^1 = u_4^2$ guarantees $u_h^1(s) = u_h^2(s)$ on the side connecting global nodes 2 and 3**

This can be shown as follows:

- **The solution $u_h^1(s)$ along the line connecting global nodes 2 and 3 is linear, and it is given by**

$$u_h^1(s) = u_2^1\left(1 - \frac{s}{h}\right) + u_3^1\frac{s}{h}$$

where s is the local coordinate with its origin at global node 2 and h is the length of th side 2-3(or side 2)

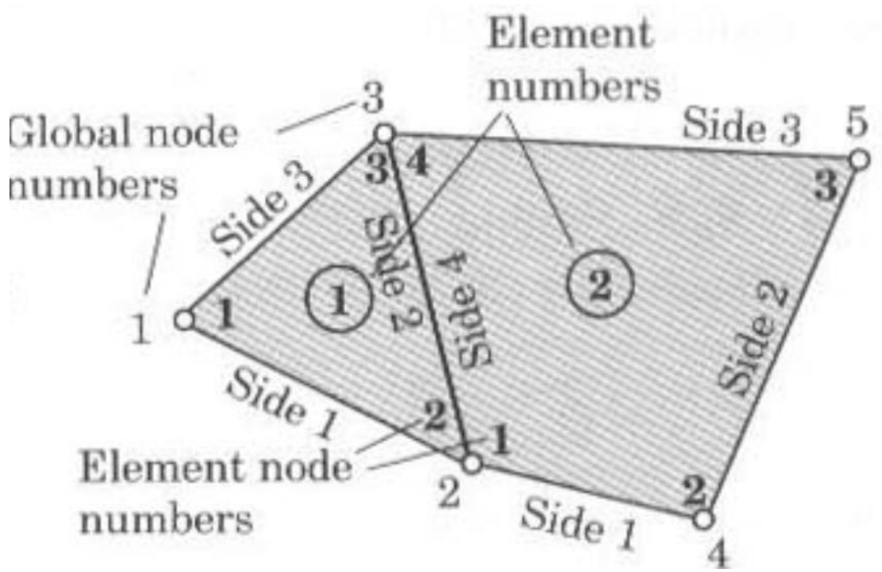


Single Variable Problem (2D)

Similarly, the finite element solution along the same line but from element 2 is

$$u_h^2(s) = u_1^2\left(1 - \frac{s}{h}\right) + u_4^2\frac{s}{h}$$

Since u_2^1=u_1^2 and u_3^1=u_4^2, it follows that $\boldsymbol{u_h^1(s) = u_h^2(s)}$, for every value of s along the interface of the two elements

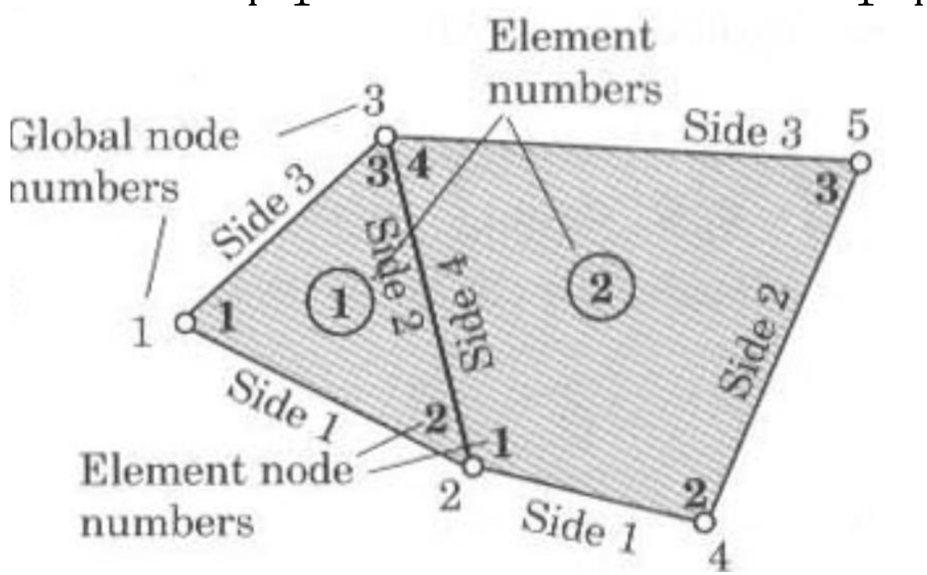


Single Variable Problem (2D)

Next we use the balance of secondary variables. At the intertace between elements, the flux from the two elements should be equal in magnitude and opposite in sign

For the two elements in Fig., the interface is along the side connecting global nodes 2 and 3. Hence, the internal flux q_n^1 on side 2-3 of element 1 should balance the flux q_n^2 on side 4-1 of element 2 (recall the sign convention on qen)

$$(q_n^1)_{2-3} = \left(q_n^2\right)_{4-1} \quad \textbf{or} \quad (q_n^1)_{2-3} = \left(-q_n^2\right)_{1-4}$$



Single Variable Problem (2D)

$$(q_n^1)_{2-3} = \left(q_n^2\right)_{4-1} \quad \text{or} \quad (q_n^1)_{2-3} = \left(-q_n^2\right)_{1-4}$$

In the finite element method we impose the above relation in a weighted integral sense

$$\int_{h_{23}^1} q_n^1 \psi_2^1 ds = -\int_{h_{14}^2} q_n^2 \psi_1^2 ds \qquad \int_{h_{23}^1} q_n^1 \psi_3^1 ds = -\int_{h_{14}^2} q_n^2 \psi_4^2 ds$$

where h_{pq}^e denotes length of the side connecting node p to node q of element Ω^e. The above equations can be written in the form

$$\int_{h_{23}^1} q_n^1 \psi_2^1 ds + \int_{h_{14}^2} q_n^2 \psi_1^2 ds = 0 \qquad \int_{h_{23}^1} q_n^1 \psi_3^1 ds + \int_{h_{14}^2} q_n^2 \psi_4^2 ds = 0$$

$$Q_{22}^1 + Q_{14}^2 = 0 \qquad Q_{32}^1 + Q_{44}^2 = 0$$

$$Q_{i,J}^e = \int_{side\ J} q_n^e \psi_i^e ds$$

where $Q_{i,J}^e$, denotes the part of Q_i^e that comes from side J of element e, $Q_{i,J}^e$ is only a portion of Q_i^e

Single Variable Problem (2D)

The element equations of the two-element mesh shown in Fig. are written first. For the model problem at hand, there is only one primary degree of freedom (NDF=1) per node. For the triangular element, the element equations are of the form

$$K_{11}^1 u_1^1 + K_{12}^1 u_2^1 + K_{13}^1 u_3^1 = f_1^1 + Q_1^1$$

$$K_{21}^1 u_1^1 + K_{22}^1 u_2^1 + K_{23}^1 u_3^1 = f_2^1 + Q_2^1$$

$$K_{31}^1 u_1^1 + K_{32}^1 u_2^1 + K_{33}^1 u_3^1 = f_3^1 + Q_1^1$$

For the quadrilateral element, the element equations are given by

$$K_{11}^2 u_1^2 + K_{12}^2 u_2^2 + K_{13}^2 u_3^2 + K_{14}^2 u_4^2 = f_1^2 + Q_1^2$$

$$K_{21}^2 u_1^2 + K_{22}^2 u_2^2 + K_{23}^2 u_3^2 + K_{24}^2 u_4^2 = f_2^2 + Q_2^2$$

$$K_{31}^2 u_1^2 + K_{32}^2 u_2^2 + K_{33}^2 u_3^2 + K_{34}^2 u_4^2 = f_3^2 + Q_3^2$$

$$K_{41}^2 u_1^2 + K_{42}^2 u_2^2 + K_{43}^2 u_3^2 + K_{44}^2 u_4^2 = f_4^2 + Q_4^2$$

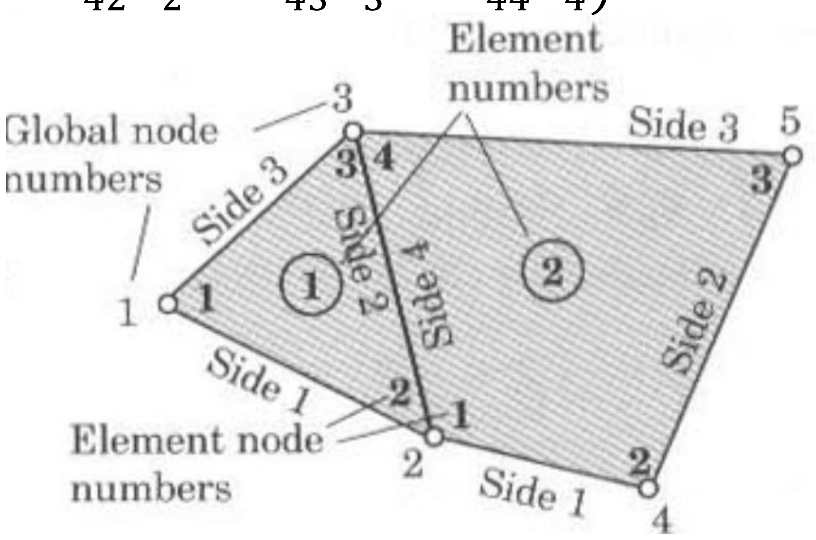
Single Variable Problem (2D)

In order to impose the balance of secondary variables, it is required that we

- **add the second equation of element 1 to the first equation of element 2,**
- **and also add the third equation of element 1 to the fourth equation of element 2**

$$\left(K_{21}^1 u_1^1 + K_{22}^1 u_2^1 + K_{23}^1 u_3^1\right) + \left(K_{11}^2 u_1^2 + K_{12}^2 u_2^2 + K_{13}^2 u_3^2 + K_{14}^2 u_4^2\right)$$
$$= \left(f_2^1 + Q_2^1\right) + \left(f_1^2 + Q_1^2\right)$$

$$\left(K_{31}^1 u_1^1 + K_{32}^1 u_2^1 + K_{33}^1 u_3^1\right) + \left(K_{41}^2 u_1^2 + K_{42}^2 u_2^2 + K_{43}^2 u_3^2 + K_{44}^2 u_4^2\right)$$
$$= \left(f_3^1 + Q_3^1\right) + \left(f_4^2 + Q_4^2\right)$$



Single Variable Problem (2D)

Using the global-variable notation, one can rewrite the above equations as (which amounts to imposing continuity of the primary variables)

$$u_1^1 = U_1 \quad u_2^1 = u_1^2 = U_2 \quad u_3^1 = u_4^2 = U_3 \quad u_2^2 = U_4 \quad u_3^2 = U_5$$

$$\left(K_{21}^1 u_1^1 + K_{22}^1 u_2^1 + K_{23}^1 u_3^1\right) + \left(K_{11}^2 u_1^2 + K_{12}^2 u_2^2 + K_{13}^2 u_3^2 + K_{14}^2 u_4^2\right)$$
$$= \left(f_2^1 + Q_2^1\right) + \left(f_1^2 + Q_1^2\right)$$

$$\left(K_{31}^1 u_1^1 + K_{32}^1 u_2^1 + K_{33}^1 u_3^1\right) + \left(K_{41}^2 u_1^2 + K_{42}^2 u_2^2 + K_{43}^2 u_3^2 + K_{44}^2 u_4^2\right)$$
$$= \left(f_3^1 + Q_3^1\right) + \left(f_4^2 + Q_4^2\right)$$

$$K_{21}^1 U_1 + \left(K_{22}^1 + K_{11}^2\right) U_2 + \left(K_{23}^1 + K_{14}^2\right) U_3 + K_{12}^2 U_4 + K_{13}^2 U_5$$
$$= f_2^1 + f_1^2 + \left(Q_2^1 + Q_1^2\right)$$

$$K_{31}^1 U_1 + \left(K_{32}^1 + K_{41}^2\right) U_2 + \left(K_{33}^1 + K_{44}^2\right) U_3 + K_{42}^2 U_4 + K_{43}^2 U_5$$
$$= f_3^1 + f_4^2 + \left(Q_3^1 + Q_4^2\right)$$

Single Variable Problem (2D)

Now we can impose the conditions

$$Q_{22}^1 + Q_{14}^2 = 0 \qquad Q_{32}^1 + Q_{44}^2 = 0$$

by setting appropriate portions of expressions in parenthesis on the right-hand side of the above equations to zero

$$Q_2^1 + Q_1^2 = \left(Q_{21}^1 + Q_{22}^1 + Q_{23}^1\right) + \left(Q_{11}^2 + Q_{12}^2 + Q_{13}^2 + Q_{14}^2\right)$$
$$= Q_{21}^1 + Q_{23}^1 + \left(\underline{Q_{22}^1 + Q_{14}^2}\right) + Q_{11}^2 + Q_{12}^2 + Q_{13}^2$$

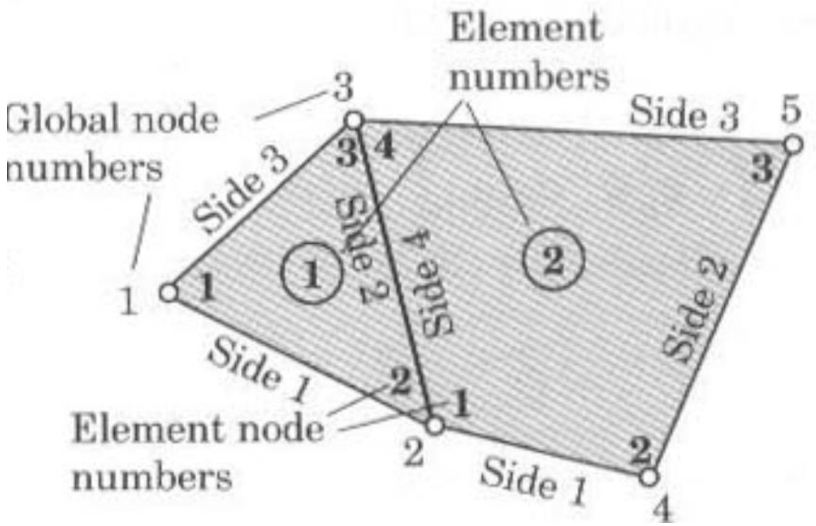
$$Q_3^1 + Q_4^2 = \left(Q_{31}^1 + Q_{32}^1 + Q_{33}^1\right) + \left(Q_{41}^2 + Q_{42}^2 + Q_{43}^2 + Q_{44}^2\right)$$
$$= Q_{31}^1 + Q_{33}^1 + \left(\underline{Q_{32}^1 + Q_{44}^2}\right) + Q_{41}^2 + Q_{42}^2 + Q_{43}^2$$

The underlined terms are zero by the balance requirement. The remaining terms of each equation will be either known because qn is known on the boundary or remain unknown because the primary variable is specified on the boundary

Single Variable Problem (2D)

In general, when several elements are connected, the assembly of the elements is carried out by putting element coefficients K_{ij}^{e}, f_i^e, and Q_i^e into proper locations of the global coefficient matrix and right-hand column vectors. This is done by means of the connectivity relations (i.e., correspondence of the local node number to the global node number). For example, if global node number 3 corresponds to node 3 of element 1 and node 4 of element 2, then we have

$$F_3 = F_3^1 + F_4^2 \equiv f_3^1 + f_4^2 + Q_3^1 + Q_4^2 \qquad K_{33} = K_{33}^1 + K_{44}^2$$



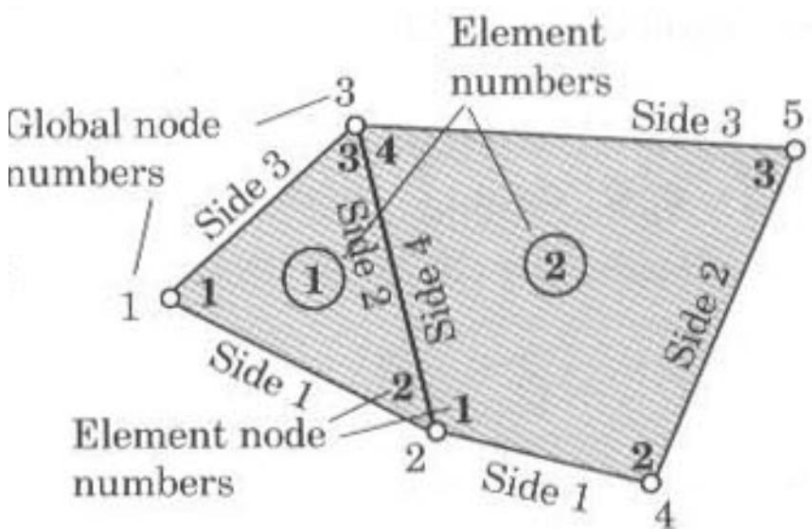
Single Variable Problem (2D)

If global node numbers 2 and 3 correspond, respectively, to nodes 2 and 3 of element 1 and nodes 1 and 4 of element 2, then global coefficients K_{22}, K_{23}, and K_{33} are given by

$$K_{22} = K_{22}^1 + K_{11}^2 \qquad K_{23} = K_{23}^1 + K_{14}^2 \qquad K_{33} = K_{33}^1 + K_{44}^2$$

Similarly, the source components of global nodes 2 and 3 are added

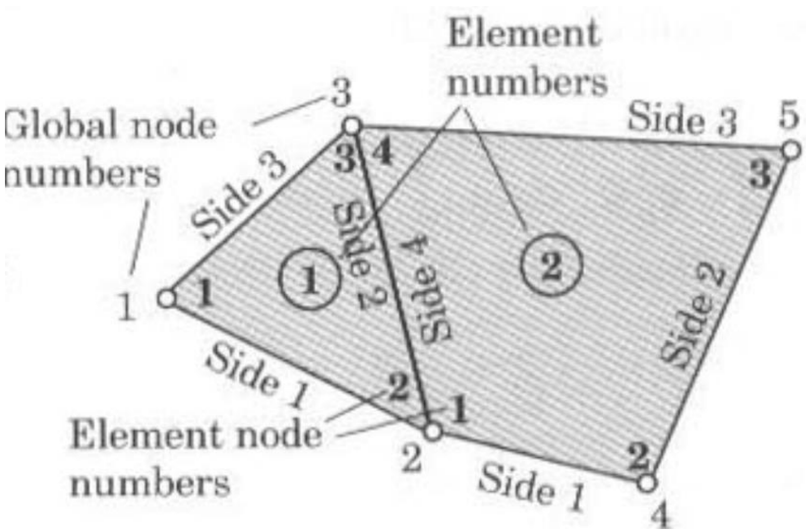
$$F_2 = F_2^1 + F_1^2 \qquad F_3 = F_3^1 + F_4^2$$



Single Variable Problem (2D)

For the two-element mesh shown in Fig., the assembled equations are given by

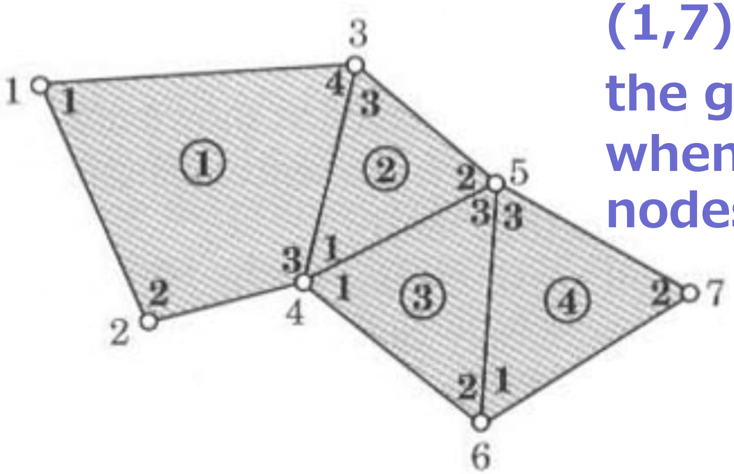
$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{23}^1 + K_{14}^2 & K_{12}^2 & K_{13}^2 \\ K_{31}^1 & K_{32}^1 + K_{41}^1 & K_{33}^1 + K_{44}^2 & K_{42}^2 & K_{43}^2 \\ 0 & K_{21}^2 & K_{24}^2 & K_{22}^2 & K_{23}^2 \\ 0 & K_{31}^2 & K_{34}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_3^1 + F_4^2 \\ F_2^2 \\ F_3^2 \end{Bmatrix}$$



Single Variable Problem (2D)

- The assembly procedure described above can be used to assemble elements of any shape and type
- The procedure can be implemented in a computer program, as described for one-dimensional problems, with the help of the array $[B]$
- For hand calculations, we can use the procedure described above

For example, consider the finite element mesh shown in Fig. The location (4, 4) of the global coefficient matrix contains $K_{33}^1 + K_{11}^2 + K_{11}^3$. The location 4 in the assembled column vector contains $f_3^1 + f_1^2 + f_1^3$. Locations (1,5), (1,6), (1,7), (2,5), (2,6), (2,7), (3,6), (3,7), (4,7) of the global matrix contain zeros because $K_{ij} = 0$ when global nodes I and J do not correspond to nodes of the same element in the mesh



Single Variable Problem (2D)

This completes the first five steps in the finite element modeling of the model equation. The next two steps of the analysis, namely, the imposition of boundary conditions and solution of equations will remain the same as for one-dimensional problems. The postprocessing of the solution for two-dimensional problems is discussed next

Postcomputations

The finite element solution at any point (x, y) in an element Ω_e is given by

$$u_h^e(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x, y)$$

and its derivatives are computed from above as

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} u_j^e \frac{\partial \psi_j^e}{\partial x} \qquad \frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} u_j^e \frac{\partial \psi_j^e}{\partial y}$$

Single Variable Problem (2D)

The **derivatives of u** will not be continuous at interelement boundaries because continuity of the derivatives is not imposed during the assembly procedure

- The weak form of the equations suggests that the primary variable is u, which is to be carried as the nodal variable
- If additional variables, such as higher-order derivatives of the dependent unknown, are carried as nodal variables in the interest of making them continuous across interelement boundaries, the degree of interpolation (or order of the element) increases
- In addition, the continuity of higher-order derivatives that are not identified as the primary variables may violate the physical principles of the problem

For example, making du/dx continuous will violate the requirement that q_x $(=a_{11}\partial u/\partial x)$ be continuous at the interface of two **dissimilar materials** because a_{11} is different for the two materials at the interface

Single Variable Problem (2D)

For the linear triangular element, the derivatives are constants within each element

$$\psi_j^e = \frac{1}{2A_e}(\alpha_i + \beta_j x + \gamma_i y) \qquad \frac{\partial \psi_j^e}{\partial x} = \frac{1}{2A_e}\beta_j \qquad \frac{\partial \psi_j^e}{\partial y} = \frac{1}{2A_e}\gamma_j$$

$$\frac{\partial u_h^e}{\partial x} = \sum_{j=1}^{n} \frac{u_j^e \beta_j}{2A_e} \qquad \frac{\partial u_h^e}{\partial y} = \sum_{j=1}^{n} \frac{u_j^e \gamma_j}{2A_e}$$

For linear rectangular elements, $\partial u_h^e/\partial \bar{x}$ is linear in $\bar{y}$ and $\partial u_h^e/\partial \bar{y}$ is linear in $\bar{x}$

$$\frac{\partial \psi_j^e}{\partial \bar{x}} = -\frac{1}{a}\left(1 - \frac{\bar{y} + \bar{y}_j}{b}\right) \qquad \frac{\partial \psi_j^e}{\partial \bar{y}} = -\frac{1}{b}\left(1 - \frac{\bar{x} + \bar{x}_j}{a}\right)$$

$$\frac{\partial u_h^e}{\partial \bar{x}} = \frac{1}{a}\sum_{j=1}^{n}(-1)^{j+2} u_j^e \left(1 - \frac{\bar{y} + \bar{y}_j}{b}\right) \qquad \frac{\partial u_h^e}{\partial \bar{y}} = \frac{1}{b}\sum_{j=1}^{n}(-1)^{j+2} u_j^e \left(1 - \frac{\bar{x} + \bar{x}_j}{a}\right)$$

Single Variable Problem (2D)

- Although $\frac{\partial u_h^e}{\partial \bar{x}}, \frac{\partial u_h^e}{\partial \bar{y}}$ are linear functions of $\bar{y}$ and $\bar{x}$, respectively, in each element, they are discontinuous at interelement boundaries
- Consequently, quantities computed using derivatives of the finite element solution u_h^e are discontinuous at interelement boundaries

For example, if we compute $q_x^e = a_{11}\partial u^e / \partial \bar{x}$, at a node shared by three different elements three different values of q_x^e are expected

- The difference between the three values will diminish as the mesh is refined
- Some commercial finite element software give a single value of q_x at the node by averaging the values obtained from various elements connected at the node

谢 谢！

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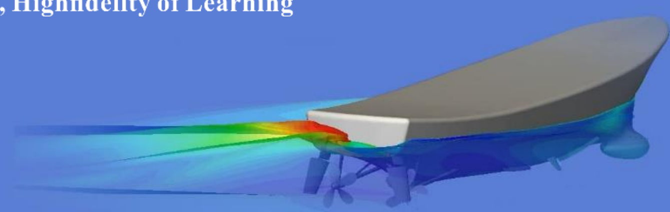

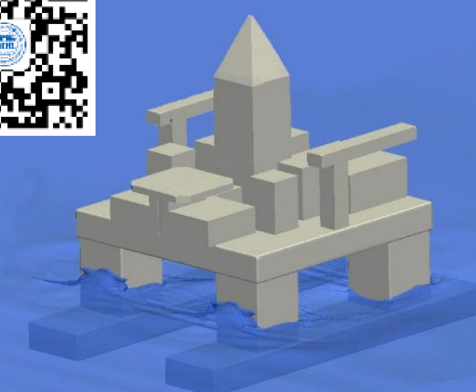

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