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# Finite Element Analysis of Solids and Fluids


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# Time-dependent problems

## Fully Discretized Finite Element Equations

We now have the tools necessary to convert the set of ordinary differential equations to a set of algebraic equations in much the same way we converted a single differential equation to an algebraic equation

Here, we start with matrix equation of semidiscrete finite element models

$$[K]\{u\} + [M^1]\{\dot{u}\} + [M^2]\{\ddot{u}\} = \{F\}$$

with $[M^1] = [M]\ and\ [M^2] = [0]$

$$[M]\ \{\dot{u}\} + [K]\{u\} = \{F\} \qquad \textbf{(a)}$$

subject to the initial conditions

$$\{u\}_0 = \{u_0\} \qquad \textbf{(b)}$$

$\{u\}_0$ denotes the vector of nodal values of $u$ at time $t = 0$, whereas $\{u_0\}$ denotes the column of nodal values $u_{j0}$

# Time-dependent problems

**As applied to a vector of time derivatives of the nodal values the $\alpha$-family of approximation reads as**

$$\Delta t_{s+1}[(1-\alpha)\{\dot{u}\}_s + \alpha\{\dot{u}\}_{s+1}] = \{u\}_{s+1} - \{u\}_s \quad \text{for } 0 \le \alpha \le 1 \qquad \textbf{(c)}$$

$$\{u\}_{s+1} = \{u\}_s + \Delta t\{\dot{u}\}_{s+\alpha}$$
$$\{\dot{u}\}_{s+\alpha} = (1-\alpha)\{\dot{u}\}_s + \alpha\{\dot{u}\}_{s+1} \qquad \text{for } 0 \le \alpha \le 1 \qquad \textbf{(d)}$$

**Equation (c) can be used to reduce the ordinary differential equations (a) to algebraic equations among the uj at time $t_{s+1}$. Since (a) is valid for any $t > 0$, we can write it for times $t = t_s$ and $t = t_{s+1}$**

$$[M]\{\dot{u}\}_s + [K]_s\{u\}_s = \{F\}_s \qquad \textbf{(e)}$$
$$[M]\{\dot{u}\}_{s+1} + [K]_{s+1}\{u\}_{s+1} = \{F\}_{s+1} \qquad \textbf{(f)}$$

# Time-dependent problems

$$[M]\{\dot{u}\}_s = \{F\}_s - [K]_s\{u\}_s \quad \textbf{(g)}$$

$$[M]\{\dot{u}\}_{s+1} = \{F\}_{s+1} - [K]_{s+1}\{u\}_{s+1} \quad \textbf{(h)}$$

**It is assumed that the matrix $[M]$ is independent of time. Premultiplying both sides of (c) with $[M]$ we obtain**

$$\Delta t_{s+1}\alpha[M]\{\dot{u}\}_{s+1} + \Delta t_{s+1}(1-\alpha)[M]\{\dot{u}\}_s = [M](\{u\}_{s+1} - \{u\}_s)$$

**By (g) (h) we obtain**

$$\Delta t_{s+1}\alpha(\{F\}_{s+1} - [K]_{s+1}\{u\}_{s+1}) + \Delta t_{s+1}(1-\alpha)(\{F\}_s - [K]_s\{u\}_s)$$
$$= [M](\{u\}_{s+1} - \{u\}_s)$$

**Solving for the vector $\{u\}_{s+1}$, we obtain**

$$\left[\widehat{K}\right]_{s+1}\{u\}_{s+1} = [\overline{K}]_s\{u\}_s + \{\overline{F}\}_{s,s+1}$$

# Time-dependent problems

$$\left[\widehat{K}\right]_{s+1}\{u\}_{s+1} = [\bar{K}]_s\{u\}_s + \{\bar{F}\}_{s,s+1} \qquad \textbf{(i)}$$

**where**

$$\left[\widehat{K}\right]_{s+1} = [M] + a_1[K]_{s+1}, [\bar{K}]_s = [M] - a_2[K]_s$$

$$\{\bar{F}\}_{s,s+1} = \Delta t_{s+1}[\alpha\{F\}_{s+1} + (1-\alpha)\{F\}_s] \qquad \textbf{(j)}$$

$$a_1 = \alpha\Delta t_{s+1}, a_1 = (1-\alpha)\Delta t_{s+1}$$

**Note that, in deriving Eqs. (i) and (j), it has been assumed that $[M]$ is independent of time and that the time step is nonuniform Equations (i) and (j) are valid for a typical finite element whose semidiscretized equations are of the form Eq. (a):**

$$[M]\{\dot{u}\} + [K]\{u\} = \{F\}$$

# Time-dependent problems

**NOTE:**

- **Eqs. (i) and (j) hold for any problem, independent of the dimension and method of spatial approximation as long as the end result is Eq. (a)**

- **The assembly, imposition of boundary conditions, and solution of the assembled equations are the same as described before for steady-state problems. Calculation of $[K]$ and $\{F\}$ at time $t = 0$ requires knowledge of the initial conditions $\{u\}_0$ and the time variation of $\{F\}$**

- **For $\alpha = 0$ (the forward difference scheme), we obtain $[\overline{K}] = [M]$. If matrix $[M]$ is diagonal, (i) becomes**

$$u_i^{s+1} = \frac{1}{M_{(ii)}} \left( \sum_{j=1}^{n} \overline{K}_{ij}^{s} u_j^{s} + \hat{F}_i^{s,s+1} \right), \text{(no sum on i)}$$

# Time-dependent problems

$$u_i^{s+1} = \frac{1}{M_{(ii)}} \left( \sum_{j=1}^{n} \bar{K}_{ij}^{s} u_j^{s} + \hat{F}_i^{s,s+1} \right), \text{(no sum on i)}$$

**Thus, when $\alpha = 0$, no inversion of the coefficient matrix is required in solving for $u_i^{s+1}$. Such a scheme is called explicit**

**A scheme is said to be implicit when it is not explicit (i.e an implicit scheme requires the inversion of a coefficient matrix)**

- **explicit schemes are computational less expensive compared to implicit schemes**
- **implicit schemes are more accurate and have larger critical time steps**

# Time-dependent problems

$$u_i^{s+1} = \frac{1}{M_{(ii)}} \left( \sum_{j=1}^{n} \overline{K}_{ij}^{s} u_j^{s} + \hat{F}_i^{s,s+1} \right), \text{(no sum on i)}$$

**In conventional finite element formulations, $[M]$ is seldom diagonal. Therefore, explicit schemes in finite element analysis can exist only if the time-approximation scheme is such that $[\overline{K}] = [M]$ and $[M]$ is diagonal**

**The matrix $[M]$ is called the consistent (mass) matrix, and it is not diagonal unless $\psi_i$ are orthogona functions over the element domain**

**There are several ways to diagonalize mass matrices $[M]$**

# Time-dependent problems

## Consistency, Accuracy, and Stability

**Error of the approximating:**

- Truncation error introduced in approximating the derivative
- Round-off errors can be introduced because of the finite arithmetic used in the computations
- Since the solution at time $t_{s+1}$ depends on the solution at time $t_s$, the error can grow with time

**Consistency:**

- The numerical scheme is said to be consistent with the continuous problem if the round-off and truncation errors go to zero as $\Delta t \sim 0$

**Stability:**

Stability of a solution is a measure of the boundedness of the approximate solution with time

- As discussed earlier, if the error is bounded, the solution scheme is said to be stable

# Time-dependent problems

**Accuracy:**

**Accuracy of a numerical scheme is a measure of the closeness between the approximate solution and the exact solution. The size of the time step can influence both accuracy and stability. When we construct an approximate solution, we like it to converge to the true solution when the number of elements or the degree of approximation is increased and the time step $\Delta t$ is decreased**

- **A time-approximation scheme is said to be convergent if, for fixed $ts$, the numerical value $\{u\}$s converges to its true value $\{u(ts)\}$ as $\Delta t \sim 0$. If a numerical scheme is stable and consistent it is also convergent**

# Time-dependent problems

## Hyperbolic Equations Time Approximation

Consider matrix equations of the form

$$[K]\{u\} + [C]\{\dot{u}\} + [M]\{\ddot{u}\} = \{F\} \qquad \textbf{(a)}$$

subjected to initial conditions

$$\{u(0)\} = \{u_0\}, \{\dot{u}(0)\} = \{v_0\} \qquad \textbf{(b)}$$

- Such equations arise in structural dynamics, where $[M]$ denotes **mass** matrix, $[C]$ the **damping** matrix, and $[K]$ the **stiffness** matrix
- The damping matrix $[C]$ is often taken to be a linear combination of the mass and stiffness matrices, $[C] = \beta_1[M] + \beta_2[K]$, where $\beta_1$ and $\beta_2$ are determined from physical experiments
- In present study, we will not consider damping (i. e, $[C] = 0$).
- Transient analysis of both bars and beams lead to equations of the type given in (a) and (b)

# Time-dependent problems

$$[K]\{u\} + [C]\{\dot{u}\} + [M]\{\ddot{u}\} = \{F\} \quad \textbf{(a)}$$

$$\{u(0)\} = \{u_0\}, \{\dot{u}(0)\} = \{v_0\} \quad \textbf{(b)}$$

**Axial Motion of Bars,** $[C] = 0$

$$K_{ij}^e = \int_{x_a}^{x_a} \left[ a(x) \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c(x)\psi_i^e \psi_j^e \right] dx, M_{ij}^e = \int_{x_a}^{x_a} c_0(x)\psi_i^e \psi_j^e dx,$$

**Transverse Motion of Euler-Bernoulli Beams,** $[C] = 0$

$$K_{ij}^e = \int_{x_a}^{x_a} EI \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx, M_{ij}^e = \int_{x_a}^{x_a} \left( \rho A \phi_i^e \phi_j^e dx + \rho I \frac{d\phi_i^e}{dx} \frac{d\phi_j^e}{dx} \right) dx,$$

**Transverse Motion of Timoshenko Beams**

**…**

# Time-dependent problems

**Numerical Methods For Second-order Time Derivatives:**

**There are several numerical methods available to approximate the second-order time derivatives and convert differential equations to algebraic equations**

- **Newmark family of time-approximation schemes is widely used in structural dynamics**
- **Other methods, such as the Wilson method, Houbolt method can also be used to develop the algebraic equations from the second-order differential equations**

**Here we consider the Newmark family of approximation schemes**

# Time-dependent problems

## Newmark's scheme

**In the Newmark method, the function and its first time derivative are approximated according to**

$$\{u\}_{s+1} = \{u\}_s + \Delta t\{\dot{u}\}_s + \frac{1}{2}(\Delta t)^2\{\ddot{u}\}_{s+\gamma} \quad \textbf{(c)}$$

$$\{\dot{u}\}_{s+1} = \{\dot{u}\}_s\{\ddot{u}\}_{s+\alpha}\Delta t \quad \textbf{(d)}$$

$$\{\ddot{u}\}_{s+\theta} = (1-\theta)\{\ddot{u}\}_s + \theta\{\ddot{u}\}_{s+1} \quad \textbf{(e)}$$

**$\alpha$ and $\gamma(=2\beta)$ are parameters that determine the stability and accuracy of the scheme Equations (c) and (d) can be viewed as Taylor's series expansions of $u$ and $\dot{u}$. The following schemes are special cases of (c) and (d)**

# Time-dependent problems

$$\alpha = \frac{1}{2}, \gamma = 2\beta = \frac{1}{2} \ constant - average \ accleration \ method \ (stable)$$
$$\alpha = \frac{1}{2}, \gamma = 2\beta = \frac{1}{3} \ linear \ accleration \ method \ (\ conditionally \ stable)$$
$$\alpha = \frac{1}{2}, \gamma = 2\beta = 0 \ Central \ difference \ method \ (conditionally \ stable)$$
$$\alpha = \frac{1}{2}, \gamma = 2\beta = \frac{8}{5} \ Galerkin \ method \ (stable)$$
$$\alpha = \frac{1}{2}, \gamma = 2\beta = 2 \ Backward \ difference \ method \ (stable)$$

**For all schemes in which $\gamma < \alpha$ and $\alpha \geq 0.5$, the stability requirement is**

$$\Delta t \leq \Delta t_{cri} = \left[\frac{1}{2}\omega_{max}^2(\alpha - \gamma)\right]^{-1/2}$$

**where wmax is the maximum natural frequency of the system without $[C]$:**

$$\left([M] - \omega^2[K]\right)\{u\} = \{F\}$$

## Fully Discretized Finite Element Equations

**Eliminating $\{\ddot{u}\}_{s+1}$ from Eqs. (c) and (d) and writing the result for $\{\dot{u}\}_{s+1}$, we obtain**

$$\{\dot{u}\}_{s+1} = a_6(\{u\}_{s+1} - \{u\}_s) - a_7\{\dot{u}\}_s - a_8\{\ddot{u}\}_s \qquad \textbf{(f)}$$

$$a_6 = \frac{\alpha}{\beta \Delta t}, a_7 = \frac{\alpha}{\beta} - 1, a_8 = \left(\frac{\alpha}{\gamma} - 1\right)\Delta t$$

**Now premultiplying Eq. (c) with $[M]_{s+1}$**

$$\{u\}_{s+1} = \{u\}_s + \Delta t\{\dot{u}\}_s + \frac{1}{2}(\Delta t)^2\{\ddot{u}\}_{s+\gamma}$$
$$\times [M]_{s+1} \qquad\qquad \times [M]_{s+1}$$

**Substituting for $[M]_{s+1}\{\ddot{u}\}_{s+1}$ from Eq (a):**

$$[K]\{u\} + (C)\{\dot{u}\} + [M]\{\ddot{u}\} = \{F\}$$

# Time-dependent problems

**We obtain**

$$\left([M]_{s+1} + \beta(\Delta t)^2[K]_{s+1}\right)\{u\}_{s+1} = [M]_{s+1}\{b\}_s + \beta(\Delta t)^2[F]_{s+1} - \beta(\Delta t)^2[F]_{s+1}\{\dot{u}\}_{s+1}$$

**where** $$\{b\}_s = \{u\}_s + \Delta t + \frac{1}{2}(1-\gamma)(\Delta t)^2\{\ddot{u}\}_s$$

**Now, multiplying throughout with** $a_3 = 1/[\beta(\Delta t)^2]$

$$(a_3[M]_{s+1} + \{K\}_{s+1})\{u\}_{s+1} = a_3[M]_{s+1}\{b\}_s + [F]_{s+1} - [C]_{s+1}\{\dot{u}\}_{s+1}$$

**Using (f):**

$$\{\ddot{u}\}_{s+1} = a_6(\{u\}_{s+1} - \{u\}_s) - a_7\{\dot{u}\}_s - a_8\{\ddot{u}\}_s \qquad \textbf{for } \{\dot{u}\}_{s+1}$$

**We obtain the final result**

$$[\hat{K}]_{s+1}\{u\}_{s+1} = [\hat{F}]_{s,s+1}$$

# Time-dependent problems

$$\left[\hat{K}\right]_{s+1}\{u\}_{s+1} = \left[\hat{F}\right]_{s,s+1}$$

**where**

$$\left[\hat{K}\right]_{s+1} = [K]_{s+1} + a_3[M]_{s+1} + a_6[C]_{s+1}$$
$$\left\{\hat{F}\right\}_{s,s+1} = [F]_{s+1} + [M]_{s+1}\{A\}_s + [C]_{s+1}\{B\}_s$$
$$\{A\}_s = a_3\{b\}_s = a_3\{u\}_s + a_4\{\dot{u}\}_s + a_5\{\ddot{u}\}_s$$
$$\{B\}_s = a_6\{u\}_s + a_7\{\dot{u}\}_s + a_8\{\ddot{u}\}_s$$
$$a_3 = \frac{1}{\beta(\Delta t)^2}, a_4 = a_3\Delta t, a_5 = \frac{1}{\gamma} - 1$$

**Note:**

- **The calculation of $\left[\hat{K}\right]_{s+1}$ and $\left\{\hat{F}\right\}_{s,s+1}$ requires knowledge of the initial conditions $\{u\}_0$,$\{\dot{u}\}_0$, and $\{\ddot{u}\}_0$. In practice, we do not know $\{\ddot{u}\}_0$**
- **As an approximation, it can be calculated from (a)**

$$[K]\{u\} + (C)\{\dot{u}\} + [M]\{\ddot{u}\} = \{F\}$$

# Time-dependent problems

**we often assume that the applied force is zero at $t = 0$**

$$\{\ddot{u}\}_0 = [M]^{-1}(\{F\}_0 - [K]\{u\}_0 - [C]\{\dot{u}\}_0)$$

**At the end of each time step, the new velocity vector$\{\dot{u}\}_{s+1}$and acceleration vector $\{\ddot{u}\}_{s+1}$ are computed using**

$$\{\ddot{u}\}_{s+1} = a_3(\{u\}_{s+1} - \{u\}_s) - a_4\{\dot{u}\}_s - a_5\{\ddot{u}\}_s$$
$$\{\dot{u}\}_{s+1} = \{\dot{u}\}_s + a_2\{\ddot{u}\}_s + a_1\{\ddot{u}\}_{s+1}$$
$$a_1 = \alpha\Delta t, a_2 = (1-\alpha)\Delta t$$

**The remaining procedure stays the same as in static problems**
**The fully discretized model is based on the assumption that $\gamma \neq 0$.**
**Obviously, for centered difference scheme ($\gamma = 0$), the formulation must be modified**

# Time-dependent problems

## Mass Lumping

**Recall from the time approximation of parabolic equations that use of the forward difference scheme (i.e, $\alpha = 0$) results in the following time marching scheme**

$$[\hat{K}]_{s+1}\{u\}_{s+1} = [\bar{K}]_s\{u\}_s + [\bar{F}]_{s,s+1}$$

$$[\hat{K}]_{s+1} = [M] + a_1[K]_{s+1},$$
$$[\bar{K}]_s = [M] - a_1[K]_s$$
$$[\bar{F}]_{s,s+1} = \Delta t_{s+1}[\alpha\{F\}_{s+1} + (1-\alpha)\{F\}_s]$$
$$a_1 = \alpha\Delta t_{s+1}, a_2 = (1-\alpha)\Delta t_{s+1}$$

$$[M^e]\{u\}_{s+1} = ([M^e] - \Delta t[K^e])\{u\}_s + \Delta t\{F^e\}_s$$

# Time-dependent problems

$$[M^e]\{u\}_{s+1} = ([M^e] - \Delta t[K^e])\{u\}_s + \Delta t\{F^e\}_s$$

**The mass matrix [Me] derived from the weighted-integral formulations of the governing equation is called the consistent mass matrix**

- $[M^e]$ **is symmetric positive-definite and nondiagonal**
- **Solution of the global equations requires inversion of the assembled mass matrix**
- **If the mass matrix is diagonal, then the assembled equations can be solved directly (i.e, without inverting a matrix) and thus saving computational time**

$$(U_I)_{s+I} = M_{II}^{-1}\left[M_{II}(U_I)_s - \Delta t\sum_{j=1}^{N} K_{IJ}\left(U_J\right)_s + \Delta t\left(F_J\right)_s\right]$$

- **The explicit nature of has motivated us to find rational ways of diagonalizing the mass matrix**

# Time-dependent problems

There are several ways of constructing diagonal mass matrices, also known as **lumped matrices**

- Row-sum lumping techniques
- Proportional lumping techniques

## Row-Sum Lumping

The sum of the elements of each row of the consistent mass matrix is used as the diagonal element

$$M_{ii}^e = \sum_{j=1}^{n} \int_{x_a}^{x_b} \rho \psi_i^e \psi_j^e \, dx = \int_{x_a}^{x_b} \rho \psi_i^e \, dx$$

$\sum_{j=1}^{n} \psi_j^e = 1$ of the interpolation functions is used

When $\rho$ is constant, $[M^e]_L = \frac{\rho h_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (linear element)

# Time-dependent problems

$$[M^e]_L = \frac{\rho h_e}{6}\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ (quadratic element)}$$

**Compare these lumped mass matrices with the consistent mass matrices**

$$[M^e]_C = \frac{\rho h_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \text{ (linear element)}$$

$$[M^e]_C = \frac{\rho h_e}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix} \text{ (quadratic element)}$$

**Here subscripts $L$ and $C$ refer to lumped and consistent mass matrices, respectively**

## Proportional Lumping

Here the diagonal elements of the lumped mass matrix are computed to be proportional to the diagonal elements of the consistent mass matrix while conserving the total mass of the element :

$$M_{ii}^e = \alpha \int_{x_a}^{x_b} \rho \psi_i^e \psi_i^e \, dx, \alpha = \int_{x_a}^{x_b} \rho \, dx \Big/ \sum_{i=1}^{n} \int_{x_a}^{x_b} \rho \psi_i^e \psi_i^e \, dx$$

**NOTE:**

For constant $\rho$, proportional lumping gives the same lumped mass matrices as those obtained in the row-sum technique for the Lagrange linear and quadratic elements

# Time-dependent problems

**The use of a lumped mass matrix in transient analyses can save computational time in two ways.**

1. **For forward difference schemes, lumped mass matrices result in explicit algebraic equations not requiring matrix inversions**

2. **The critical time step required for conditionally stable schemes is larger, and hence less computational time is required when lumped mass matrices are used**

**To see this, consider the stability criterion:**

$$\Delta t \leq \Delta t_{cri} = \left[\frac{1}{2} w_{max}^2 (\alpha - \gamma)\right]^{-1/2}$$

**for the case** $\alpha = 1/2$ **and** $\beta = 0$**.**

# Time-dependent problems

**For a one linear element model of a uniform bar of stiffness $EA$ and mass $\rho A$, fixed at the left end, the eigenvalue problem with a consistent mass matrix is**

$$\left(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \omega^2\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} = \begin{Bmatrix}Q_1^1\\ Q_2^1\end{Bmatrix}$$

**Since $U_1 = 0$ and $Q_2^1 = 0$, we have**

$$\omega^2 = \frac{EA}{h}\Big/\frac{\rho Ah}{3} = \frac{3E}{\rho h^2}$$

**Substituting this into the critical time step relation, we have**

$$(\Delta t_{cri})_C = 2/\omega_{max} = h(4\rho/3E)^{1/2}$$

**If we use the lumped matrix, w is given by**

$$w = (2E/\rho)^{1/2}/h$$

**and the critical time step is**

$$(\Delta t_{cri})_C = h(2\rho/E)^{1/2} > (\Delta t_{cri})_C$$

**Thus, explicit schemes require larger time steps than implicit schemes**

# Time-dependent problems

## Example 1 Transient heat transfer(parabolic equation)

**Consider the transient heat conduction problem described by the differential equation**

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } 0 < x < 1$$

**with boundary conditions**

$$u(0,t) = 0, \frac{\partial u}{\partial t}(1,t) = 0$$

**and initial condition**

$$u(x,0) = 1.0$$

**where $u$ is the nondimensionalized temperature**

# Time-dependent problems

**The problem at hand is a special case of**

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial x}+c_2\frac{\partial^2 u}{\partial x^2}=f(x,t)$$

**with** $a=1, b=0, c_0=0, c_1=1, c_2=0$ **and** $f=0$

$$[K]\{u\}+[M^1]\{\dot{u}\}+[M^2]\{\ddot{u}\}=\{F\}$$

$$[K]=[K^1]+[K^2]+[M^0]$$

$$M_{ij}^0=\int_{x_a}^{x_b} c_0\,\psi_i\psi_j dx, M_{ij}^1=\int_{x_a}^{x_b} c_1\,\psi_i\psi_j dx$$

$$M_{ij}^2=\int_{x_a}^{x_b} c_2\,\psi_i\psi_j dx, K_{ij}^1=\int_{x_a}^{x_b} a\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^2=\int_{x_a}^{x_b} b\frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}dx, F_i=\int_{x_a}^{x_b}\psi_i f\,dx+\hat{Q}_i$$

# Time-dependent problems

**The finite element model is**

$$[M^e]\{\dot{u}\} + [K^e]\{u\} = \{Q^e\}$$

$$M_{ij}^e = \int_{x_a}^{x_b} \psi_i^e \psi_j^e dx,\ K_{ij}^e = \int_{x_a}^{x_b} \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} dx$$

**For the choice of linear interpolation functions, the semidiscrete equations of a typical element are**

$$\frac{h_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\dot{u}_1^e\\ \dot{u}_2^e\end{Bmatrix} + \frac{1}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1^e\\ u_2^e\end{Bmatrix} = \begin{Bmatrix}Q_1^e\\ Q_2^e\end{Bmatrix}$$

**where he is the length of the element. Use of the $\alpha$-family of approximation results in the equation**

$$([M^e] + \Delta t\alpha[K^e])\{u^e\}_{s+1} = ([M^e] - \Delta t(1-\alpha)\,[K^e])\{u^e\}_s + \Delta t(\alpha[Q^e]_{s+1} + (1-\alpha)[Q^e]_s)$$

# Time-dependent problems

**First consider a one-element mesh, we have**

$$\begin{bmatrix} \frac{1}{3}h+\alpha\frac{\Delta t}{h} & \frac{1}{6}h-\alpha\frac{\Delta t}{h} \\ \frac{1}{6}h-\alpha\frac{\Delta t}{h} & \frac{1}{3}h+\alpha\frac{\Delta t}{h} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}_{s+1} = \begin{bmatrix} \frac{1}{3}h-(1-\alpha)\frac{\Delta t}{h} & \frac{1}{6}h+(1-\alpha)\frac{\Delta t}{h} \\ \frac{1}{6}h+(1-\alpha)\frac{\Delta t}{h} & \frac{1}{3}h-(1-\alpha) \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}_{s+1} + \Delta t\begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix}$$

**where** $\bar{Q}_i = \alpha\left(Q_i^1\right)_{s+1} + (1-\alpha)\left(Q_i^1\right)_s$

**b.c.** $u(0,t)=0,\quad \frac{\partial u}{\partial x}(1,t)=0$ **and i.c.** $u(x,0)=1.0$

**The boundary conditions of the problem require**

$$(U_1)_s = 0, \left(Q_2^1\right)_s = 0 \ \ for\ all\ s>0\ (i.e., t>0)$$

**However, the initial condition requires**

$$U_1(0)\psi_1(x) + U_2(0)\psi_2(x) = 1$$

$$U_1(0) = \mathbf{0}$$
$$U_2(0) = \mathbf{0}$$

# Time-dependent problems

**Using the boundary conditions, we can write for the one-element model**

$$\left(\frac{1}{3h}+\alpha\frac{\Delta t}{h}\right)(U_2)_{s+1}=\left[\frac{1}{3}h-(1-\alpha)\frac{\Delta t}{h}\right](U_2)_s$$

**which can be solved repeatedly for $U_2$ at different times, $s=0,1,\ldots$**

**NOTE:**

**Repeated use of above Eq. can cause the temporal approximation error to grow with time depending on the value of $\alpha$. As noted earlier, the forward difference scheme $(\alpha=0)$ is a conditionally stable scheme**

# Time-dependent problems

**To determine the critical time step for the one-element mesh. We can first calculate the maximum eigenvalue of the associated system**

- **By using the stability condition, for the forward difference scheme, the time step should be smaller than $\Delta t_{cri} = 0.6667$; otherwise, the solution will be unstable**

$$\lambda = 3/h^2 = 3 \qquad \Delta t < \Delta t_{cri} \equiv \frac{2}{(1-2\alpha)\lambda}$$

**For a two-element mesh, we have ($h_1$=$h_2$=$h$=0.5): the condensed equations of the timeemarehing scheme are given by**

$$\begin{bmatrix} \frac{2}{3}h + 2\alpha\frac{\Delta t}{h} & \frac{1}{6}h - \alpha\frac{\Delta t}{h} \\ \frac{1}{6}h - \alpha\frac{\Delta t}{h} & \frac{1}{3}h + \alpha\frac{\Delta t}{h} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_{s+1} = \begin{bmatrix} \frac{2}{3}h - 2(1-\alpha)\frac{\Delta t}{h} & \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} \\ \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} & \frac{1}{3}h - (1-\alpha) \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_s$$

# Time-dependent problems

**With $(U_2)_0$ and $(U_3)_0$, the forward difference scheme yield**

$$\frac{h}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}_{s+1} = \frac{h}{6}\begin{bmatrix}4-2\mu & 1+\mu\\ 1+\mu & 2-\mu\end{bmatrix}\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}_{s}, \mu = \frac{6\Delta t}{h^2}$$

**Similarly, we can calculate the maximum eigenvalue of the associated 2-mesh system,**

$$\bar{\lambda} = \frac{\lambda h^2}{6}$$

**By using the stability condition, we obtain the critical time step**

$\Delta t_{cri} = 0.0631$

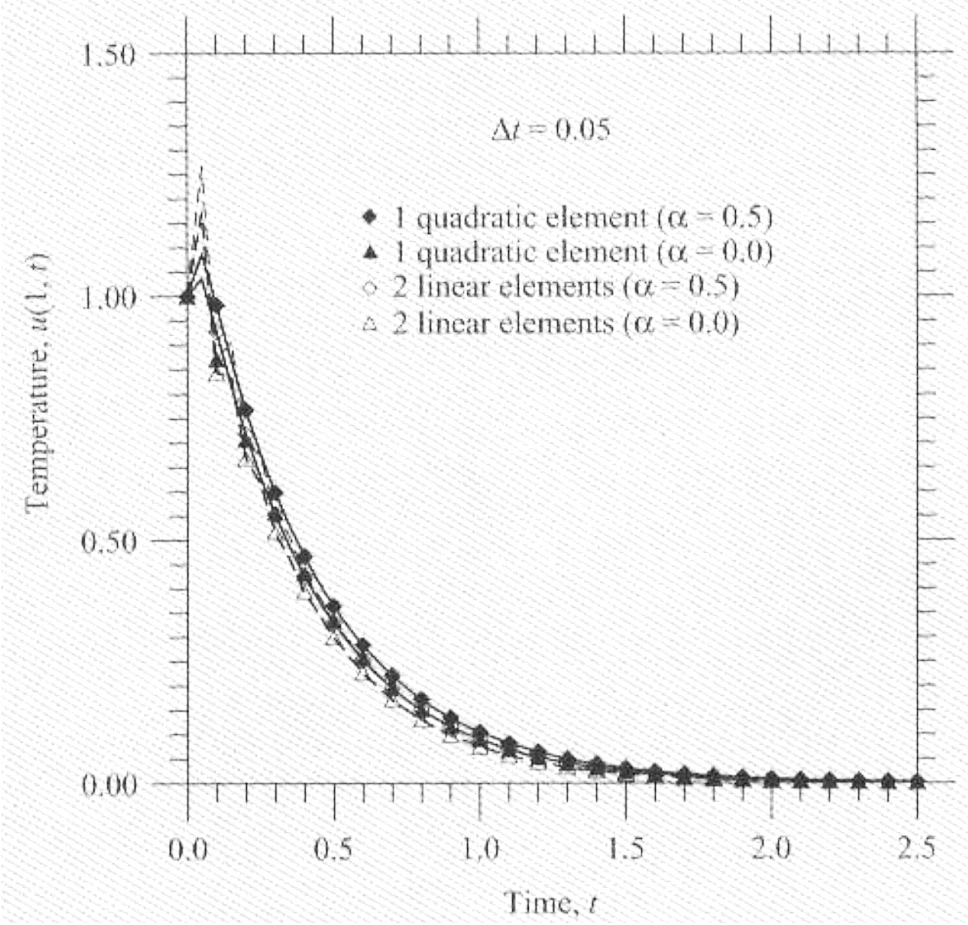
# Time-dependent problems

**Discussion:**

- **For (unconditionally) stable schemes ($\alpha \geq 0.5$), there is no restriction on the time step (e.g, Crank-Nicolson method)**
- **However, to obtain a sufficiently accurate solution, the time step must be taken as a fraction of $\Delta t_{cri}$**
- **The accuracy of the solution also depends on the mesh size. As this is decreased (i.e. the number of elements is increased). $\Delta t_{cri}$ decreased**

# Time-dependent problems

**Solutions predicted by meshes of one, two, or four linear or quadratic elements are compared**




**The convergence of the solution with increasing number of elements is clear**

# 谢 谢！

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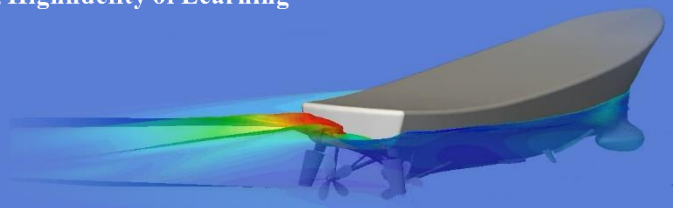

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