Class-4

NA26018

# Finite Element Analysis of Solids and Fluids


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# Time-dependent problems

## Introduction

**In this section,**

- **we develop the finite element models of one-dimensional time-dependent problems and**
- **describe time approximation schemes to convert ordinary differential equations in time to algebraic equations.**

**We consider finite element models of the time-dependent version of the differential equations studied previously. These include**

- **The second-order（in space）parabolic equations**

  **(i.e, first time derivative)**

- **The second-order hyperbolice quations**

  **(i.e, second time derivative)**

- **The fourth-order hyperbolic equations**

**arising in connection with the bending of beams**

# Time-dependent problems

Finite element models of time-dependent problems can be developed in two alternative ways:

(a) coupled formulation in which the time $t$ is treated as an additional coordinate along with the spatial coordinate $x$

(b) decoupled formulation where time and spatial variations are assumed to be separable

Thus, the approximation in the two formulations takes the form

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} \hat{u}_j^e \hat{\psi}_j^e(x,t) \quad \text{(coupled formulation)}$$

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} u_j^e(t) \psi_j^e(x) \quad \text{(coupled formulation)}$$

- $\hat{\psi}_j^e(x,t)$are time-space (two-dimensional) interpolation functions
- $\hat{u}_j^e$are the nodal values that are independent of $x$ and $t$,
- $\psi_j^e(x)$) are the usual one-dimensional interpolation functions in spatial coordinate $x$ only
- $u_j^e(t)$ are functions of time $t$ only

# Time-dependent problems

Space-time coupled finite element formulations are not common, and they have not been adequately studied. In this section, we consider the space-time **decoupled formulation** only

The space-time decoupled finite element formulation of time-dependent problems involves **2 steps** :

## 1. Spatial approximation,

- The solution $u$ of the equation under consideration is approximate by decoupled form, and the spatial finite element model of the equation is developed using the procedures of static or steady-state problems while carrying all time-dependent terms in the formulation.
- This step results in a set of ordinary differential equations （i.e, a semidiscrete system of equations） in time for the nodal variables $u_j^e(t)$ of the element. Decoupled Eq. represents the spatial approximation of $u$ for any time $t$
- When the solution is separable into functions of time only and space only, $u(x,t) = T(t)X(x)$, the approximation is clearly justified

# Time-dependent problems

- **Even when the solution is not separable, decoupled Eq. can represent a good approximation of the actual solution provided a sufficiently small time step is used**

## 2. Temporal approximation

- **The system of ordinary differential equations are further approximated in time, often using finite difference formulae for the time derivatives**
- **This step allows conversion of the system of ordinary differential equations into a set of algebraic equations among $u$ at time $t_{s+1} = (s+1)\Delta t$, where $\Delta t$ is the time increment and $s$ is a nonnegative integer**

# Time-dependent problems

**All time approximation schemes seek to find $u_j$ at time $t_s + 1$ using the values of $u_j$ from previous times :**

$$\text{compute } \{u\}_{s+1} \text{ using} \{u\}_s, \{u\}_{s-1}, \ldots$$

**Thus, at the end of the two-stage approximation, we have a continuous spatial solution at discrete intervals of time**

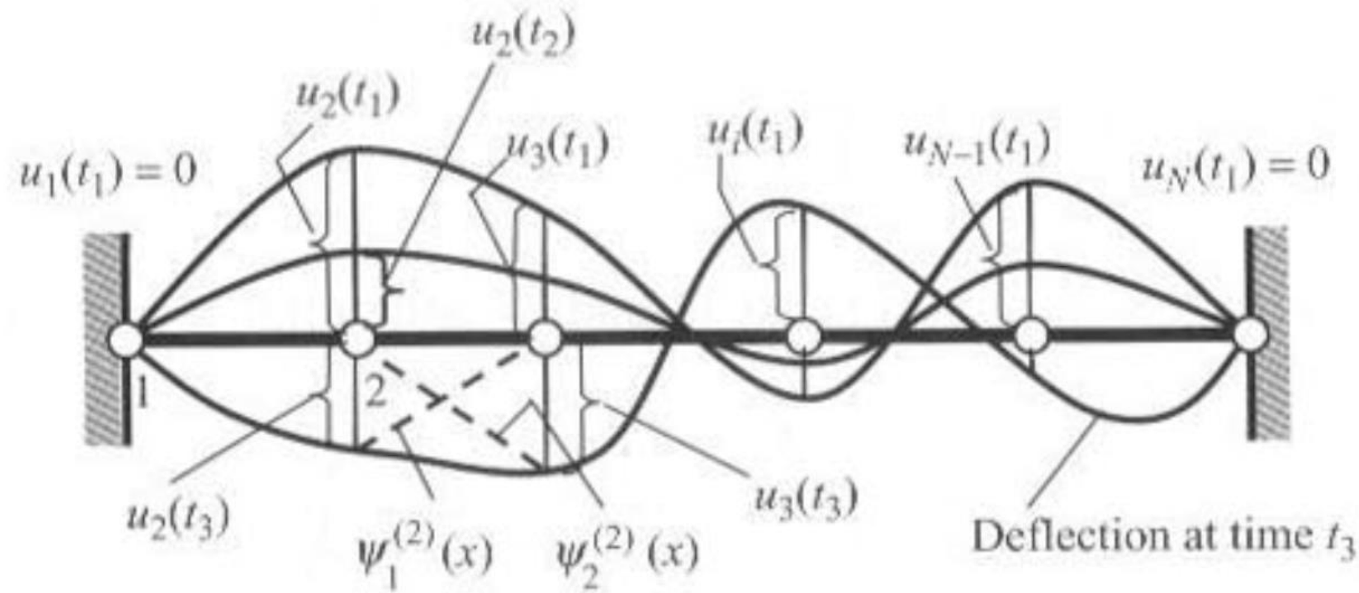
$$u(x, t_s) \approx u_h^e(x, t_s) = \sum_{j=1}^{n} u_j^e(t_s)\psi_j^e(t_s) \; (s = 0,1, \ldots)$$

**Note: the approximate solution has the same form as that in the separation-of variables technique used to solve boundary value and initial value problems**

# Time-dependent problems

$$u(x,t_s) \approx u_h^e(x,t_s) = \sum_{j=1}^{n} u_j^e(t_s)\psi_j^e(t_s)\ (s = 0,1,\dots)$$

**Note: By taking nodal values to be functions of time, we see that the spatial points in an element take on different values for different times**



# Time-dependent problems

**We study the details of the two steps by considering a model differential equation that contains both**

- **second-and fourth-order spatial derivatives**
- **first-and second-order time derivatives**

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}=f(x,t)$$

**The above equation is subject to appropriate boundary and initial conditions. The boundary conditions are of the form**

$$specify\ \ u(x,t)\ \ or\ -a\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)$$

$$specify\ \ \frac{\partial u}{\partial x}\ \ or\ \ b\frac{\partial^2 u}{\partial x^2}$$

**at $x=0,L$, and the initial conditions involve specifying**

$$c_2 u(x,0)\ \ and\ \ c_2\dot{u}(x,0)+c_1 u(x,0) \qquad \dot{u}\equiv\partial u/\partial t$$

# Time-dependent problems

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}=f(x,t)$$

**Above equation describes following physical roblems**

**a. Heat transfer and fluid flow:**

$\boldsymbol{c_2 = 0\ and\ b = 0}$

**b. Transverse motion of a cable:**

$\boldsymbol{a = T, c_0 = 0, b = 0, c_1 = \rho, c_2 = 0}$

**c. The longitudinal motion of a rod:**

$\boldsymbol{a = EA, b = 0}$**; if damping is not considered,**$\boldsymbol{c_1 = 0, c_2 = \rho A}$

**d. The transverse motion of an Euler-Bernoulli beam:**

$\boldsymbol{a = 0, b = EI, c_0 = k, c_1 = 0, c_2 = \rho A}$

# Semidiscrete Finite Element Models

**The semidiscrete formulation involves approximation of the spatial variation of the dependent variable. The formulation follows essentially the same steps as described in previous**

- **The first step involves the construction of the weak form of the equation over a typical element**
- **In the second step, we develop the finite element model by seeking approximation of the decoupled form**

# Time-dependent problems

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}=f(x,t)$$

$$\begin{aligned}
0 &= \int_{x_a}^{x_b} w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}-f\right]dx \\
&= \int_{x_a}^{x_b}\left[\frac{\partial w}{\partial x}a\frac{\partial u}{\partial x}+\frac{\partial^2 w}{\partial x^2}b\frac{\partial^2 u}{\partial x^2}+c_0 wu+c_1 w\frac{\partial u}{\partial t}+c_2 w\frac{\partial^2 u}{\partial t^2}-wf\right]dx \\
&\quad +\left[w\left[\left(-a\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]+\frac{\partial w}{\partial x}\left(-b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_a}^{x_b} \\
&= \int_{x_a}^{x_b}\left[a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+b\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 u}{\partial x^2}+c_0 wu+c_1 w\frac{\partial u}{\partial t}+c_2 w\frac{\partial^2 u}{\partial t^2}-wf\right]dx \\
&\quad -\hat{Q}_1 w(x_a)-\hat{Q}_3 w(x_b)-\hat{Q}_2\left(-\frac{\partial w}{\partial x}\right)\Big|_{x_a}-\hat{Q}_4\left(-\frac{\partial w}{\partial x}\right)\Big|_{x_b}
\end{aligned}$$

# Time-dependent problems

$$\hat{Q}_1 = \left[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_a}, \hat{Q}_2 = \left[b\frac{\partial^2 u}{\partial x^2}\right]_{x_a}$$

$$\hat{Q}_3 = -\left[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_b}, \hat{Q}_4 = -\left[b\frac{\partial^2 u}{\partial x^2}\right]_{x_b}$$

**Next, we assume that u is interpolated by an expression of the decoupled form:**

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} u_j^e(t)\psi_j^e(x) \text{ (decoupled formulation)}$$

**This equation implies that, at any arbitrarily fixed time $t > 0$, the function $u$ can be approximated by a linear combination of the $\psi_j^e$ and $u_j^e(t)$, with $u_j^e(t)$ being the value of $u$ at time $t$ at the jth node of the element Omegae. In other words, the time and spatial variations of $u$ are separable.**

# Time-dependent problems

- **This assumption is not valid, in general, because it may not be possible to write the solution $u(x,t)$ as the product of a function of time only and a function of space only**

- **However, with sufficiently small time steps, it is possible to obtain accurate solutions to even those problems for which the solution is not separable in time and space**

- **The finite element solution that we obtain at the end of the analysis is continuous in space but not in time**

**We only obtain the finite element solution in the form**

$$u(x,t_s) = \sum_{j=1}^{n} u_j^e(t_s)\psi_j^e(t_s) = \sum_{j=1}^{n} \left(u_j^s\right)^e \psi_j^e(x) \quad (s = 0,1,\dots)$$

**Where$\left(u_j^s\right)^e$is the value of $u(x,t)$ at time $t = t_s$and node $j$ of the element Omegae**

# Time-dependent problems

**Substituting** $w = \psi_j^e(x)$ **(to obtain the ith equation of the system) and substitute decoupled approximation into weak form, we obtain**

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} u_j^e(t)\psi_j^e(x) \ \text{(decoupled formulation)}$$

$$0 = \int_{x_a}^{x_b} \left[ a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + b\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + c_0 wu + c_1 w\frac{\partial u}{\partial t} + c_2 w\frac{\partial^2 u}{\partial t^2} - wf \right] dx$$
$$-\hat{Q}_1 w(x_a) - \hat{Q}_3 w(x_b) - \hat{Q}_2 \left(-\frac{\partial w}{\partial x}\right)\Bigg|_{x_a} - \hat{Q}_4 \left(-\frac{\partial w}{\partial x}\right)\Bigg|_{x_b}$$

# Time-dependent problems

$$0 = \int_{x_a}^{x_b} \left[ a\frac{d\psi_i}{dx}\left(\sum_{j=1}^{n} u_j \frac{d\psi_j}{dx}\right) + b\frac{d^2\psi_i}{dx^2}\left(\sum_{j=1}^{n} u_j \frac{d^2\psi_j}{dx^2}\right) + c_0\psi_i\left(\sum_{j=1}^{n} u_j\psi_j\right) \right.$$

$$\left. + c_1\psi_i\left(\sum_{j=1}^{n}\frac{du_j}{dt}\psi_j\right) + c_2\psi_i\left(\sum_{j=1}^{n}\frac{d^2u_j}{dt^2}\psi_j\right) - \psi_i f \right] dx$$

$$-\hat{Q}_1\psi_i(x_a) - \hat{Q}_3\psi_i(x_b) - \hat{Q}_2\left(-\frac{d\psi_j}{dx}\right)\bigg|_{x_a} - \hat{Q}_4\left(-\frac{d\psi_j}{dx}\right)\bigg|_{x_b}$$

$$= \sum_{i=1}^{n}\left[(K_{ij}^1 + K_{ij}^2)u_j + M_{ij}^1\frac{du_j}{dt} + M_{ij}^2\frac{d^2u_j}{dt^2}\right] - F_i$$

**In matrix form, we have**

$$[K]\{u\} + [M^1]\{\dot{u}\} + [M^2]\{\ddot{u}\} = \{F\}$$

# Time-dependent problems

$$[K]\{u\} + [M^1]\{\dot{u}\} + [M^2]\{\ddot{u}\} = \{F\} \qquad \textbf{(a)}$$

**where**

$$[K] = [K^1] + [K^2] + [M^0]$$

$$M_{ij}^0 = \int_{x_a}^{x_b} c_0\, \psi_i \psi_j dx, M_{ij}^1 = \int_{x_a}^{x_b} c_1\, \psi_i \psi_j dx$$

$$M_{ij}^2 = \int_{x_a}^{x_b} c_2\, \psi_i \psi_j dx, K_{ij}^1 = \int_{x_a}^{x_b} a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx$$

$$K_{ij}^2 = \int_{x_a}^{x_b} b \frac{d^2\psi_i}{dx^2} \frac{d^2\psi_j}{dx^2} dx, F_i = \int_{x_a}^{x_b} \psi_i f\, dx + \hat{Q}_i$$

**Equation (a) is a hyperbolic equation, and it contains the parabolic equation as a special case (set $[M] = [0]$). The time approximation of（$a$） for these two cases will be considered separately**

## Parabolic Equations-Time Approximation

**The time approximation is discussed with the help of a single first-order differential equation**

- **Suppose that we wish to determine $u(t)$ for $t > 0$ such that $u(t)$ satisfies**

$$a\frac{du}{dt} + bu = f(t), \qquad 0 < t < T \ \ and\ u(0) = u_0$$

**where $a \neq 0, \ \ b,$ and $u_0$ are constants, and $f$ is a function of time $t$.**

- **The exact solution of the problem consists of two parts: the homogeneous and particular solutions. The homogeneous solution is**

$$u^h(t) = Ae^{-kt}, k = \frac{b}{a}$$

**The particular solution is**

$$u^p(t) = \frac{1}{a}e^{-kt}\left(\int_0^t e^{k\tau} f(\tau)\right) d\tau$$

# Time-dependent problems

The complete solution is given by

$$u(t) = e^{-kt}\left(A + \frac{1}{a}\int_0^t e^{k\tau} f(\tau)\right) d\tau$$

- In the finite difference solution of parabolic Eq., we replace the derivatives with their finite difference approximation

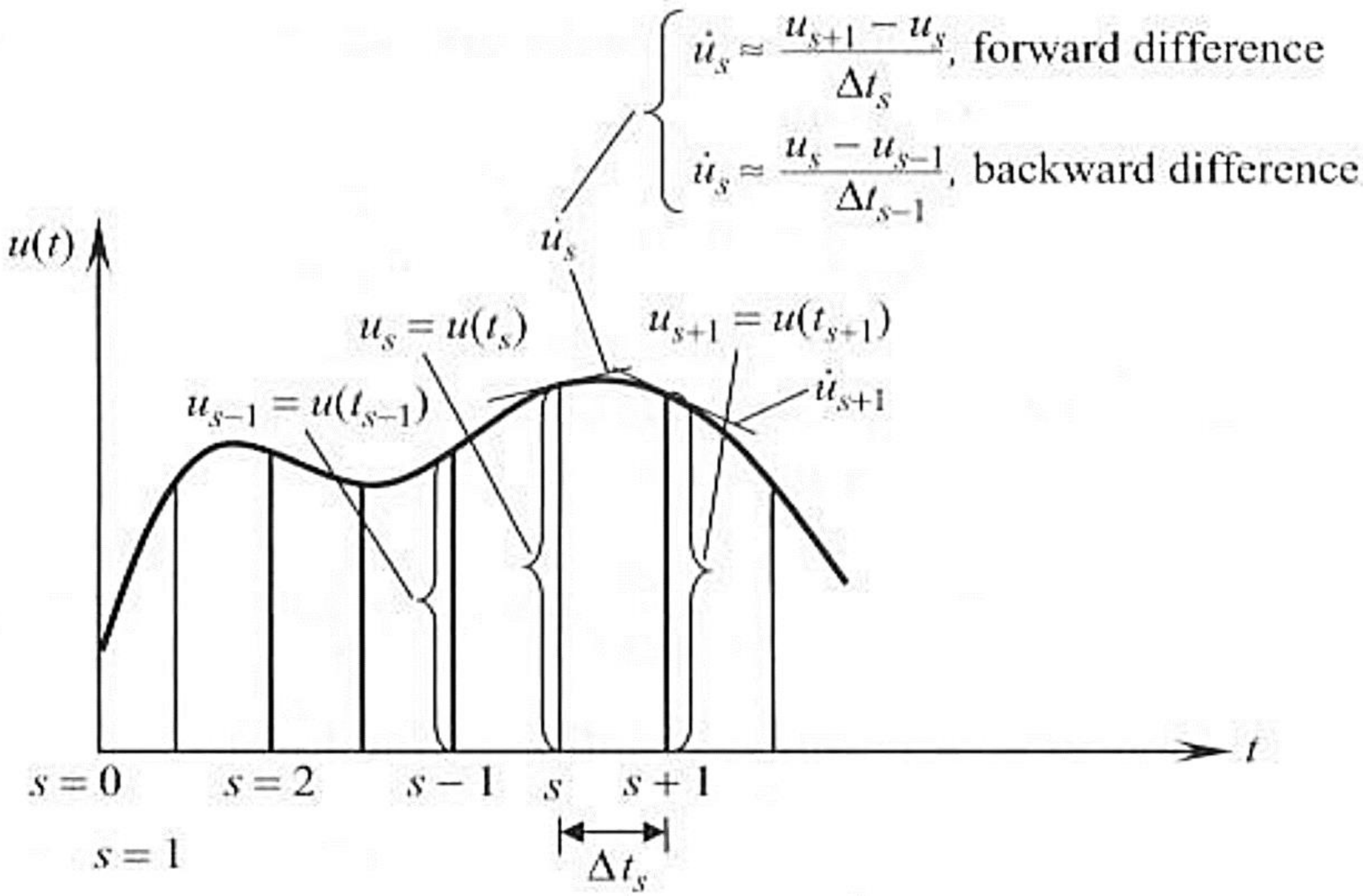
- The most commonly used scheme is the a family of approximation in which a weighted average of the time derivatives at two consecutive time steps is approximated by linear interpolation of the values of the variable at the two steps (as in Fig. next)

$$(1-\alpha)\dot{u}_s + \alpha \dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \text{ for } 0 \le \alpha \le 1$$

us, denotes the value of $u(t)$ at time $t = t_s = \sum_{i=1}^{s} \Delta t_i$

$\Delta t_s = t_s - t_{s-1}$ is the sth time step

# Time-dependent problems



# Time-dependent problems

**If the total time $[0, T]$ is divided into equal time steps, then $t_s = s\Delta t$, and**

$$(1-\alpha)\dot{u}_s + \alpha\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \text{ for } 0 \le \alpha \le 1 \qquad \textbf{(a)}$$

$$u_{s+1} = u_s + \Delta t\dot{u}_{s+\alpha}$$
$$\dot{u}_{s+\alpha} = (1-\alpha)\dot{u}_s + \alpha\dot{u}_{s+1} \text{ for } 0 \le \alpha \le 1 \qquad \textbf{(b)}$$

**When $\alpha = 0$, Eq.(a) gives**

$$\dot{u}_s = \frac{u_{s+1} - u_s}{\Delta t_{s+1}}$$

**This is the slope of the function $u(t)$ at time $t = t_s$ based on the values of the function at time ts and$t_s + 1$**

- **Since the value of the function from a step in front is used, it is termed a forward difference approximation**

# Time-dependent problems

$$(1-\alpha)\dot{u}_s + \alpha\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \text{ for } 0 \leq \alpha \leq 1 \qquad \textbf{(a)}$$

**When $\alpha = 1$, we obtain**

$$\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \longrightarrow \dot{u}_s = \frac{u_s - u_{s-1}}{\Delta t_s}$$

**which is termed, for obvious reason, the backward difference approximation**

# Time-dependent problems

**Recall the parabolic Eq.:**

$$a\frac{du}{dt} + bu = f(t), 0 < t < T \; and \;\; u(0) = u_0$$

**Note that it is valid for all times** $0 < t < T$**. In particular, it is valid at times** $t = t_s$**, and** $t = t_s + 1$**. Hence,**

$$\dot{u}_s = \frac{1}{a}(f_s - bu_s), \dot{u}_{s+1} = \frac{1}{a}(f_{s+1} - bu_{s+1})$$

**Substituting the above expressions into finite difference approximation (a):**

$$(1-\alpha)\dot{u}_s + \alpha\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \text{ for } 0 \le \alpha \le 1 \qquad \textbf{(a)}$$

$$(1-\alpha)(f_s - bu_s) + \alpha(f_{s+1} - bu_{s+1}) = a\frac{u_{s+1} - u_s}{\Delta t_{s+1}}$$

**Solving for** $u_s + 1$**,we obtain**

# Time-dependent problems

$$[a+\alpha\Delta t_{s+1}b]u_{s+1}=[a-(1-\alpha)\Delta t_{s+1}b]\,u_s+\Delta t_{s+1}[\alpha f_{s+1}+(1-\alpha)f_s]$$

$$u_{s+1}=\frac{a-(1-\alpha)\Delta t_{s+1}b}{a+\alpha\Delta t_{s+1}b}u_s+\Delta t_{s+1}\frac{[\alpha f_{s+1}+(1-\alpha)f_s]}{a+\alpha\Delta t_{s+1}b}$$

**Thus, above Eq. can be used repeatedly to march in time and obtain the solution at times** $t=t_s+1,\ t_s+2,\ \ldots,\ t_N$**, Ntime is the number of time steps required to reach the final** $T$

**At the very beginning, i. e** $s=0$**, the solution u is calculated using the initial value** $u_0$**:**

$$u_1=\frac{a-(1-\alpha)\Delta t_1 b}{a+\alpha\Delta t_1 b}u_0+\Delta t_1\frac{[\alpha f_1+(1-\alpha)f_0]}{a+\alpha\Delta t_1 b}$$

# Time-dependent problems

**We may also develop a time approximation scheme using the finite element method**

- **To this end, we consider the same parabolic problem**

$$a\frac{du}{dt} + bu = f(t), 0 < t < T \ \ and \ \ u(0) = u_0$$

- **We wish to determine $u_{s+1}$ in terms of $u_s$**

**The weighted-integral form of the parabolic over the time interval $(t_s, t_{s+1})$ is**

$$0 = \int_{t_s}^{t_{s+1}} v(t)\left(a\frac{du}{dt} + bu - f\right)dt$$

**where $u$ is the weight function. Assuming a solution of the form**

$$u(t) \approx \sum_{j=1}^{n} u_j \psi_j(t)$$

**where $\psi_j(t)$ are interpolation functions of order $(n-1)$. The Galerkin finite element model is obtained by substituting the above approximation for $u$ and $v = \psi i$. We obtain**

# Time-dependent problems

$$[A]\{u\} = \{F\}$$

$$A_{ij} = \int_{t_s}^{t_{s+1}} \psi_i(t)\left(a\frac{d\psi_j}{dt} + b\psi_j\right)dt\ , F_i = \int_{t_s}^{t_{s+1}} \psi_i(t)f(t)dt$$

- **Equation is valid with the time interval ($t_s$, $t_{s+1}$), and it represents a relationship be tween the values $u_1, u_2, \dots, u_n$, which are the values of $u$ at time** $t_s, t_s + \Delta t/(n-1), t_s + 2\Delta t/(n-1), \dots, t_{s+1}$, **respectively**

**This would yield a multistep approximation scheme**

**To obtain a single-step approximation scheme, i.e., write $u_s + 1$ in terms of $u_s$ only, we assume linear approximation (i.e. $n = 2$)**

$$u(t) = u_s\psi_1(t) + u_{s+1}\psi_2(t)$$

$$\psi_1(t) = \frac{t_{s+1} - t_s}{\Delta t}\ and\ \psi_2(t) = \frac{t - t_s}{\Delta t}$$

**For this choice of approximation, the Matrix form can be written as**

# Time-dependent problems

$$\left(\frac{a}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}+\frac{b\Delta t}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}u_s\\ u_{s+1}\end{Bmatrix}=\frac{\Delta t}{6}\begin{Bmatrix}f_s\\ 2f_{s+1}\end{Bmatrix}$$

**Assuming that $u_s$ is known, we solve for $u_{s+1}$ from the second equation**

$$\left(a+\frac{2b\Delta t}{3}\right)u_{s+1}=\left(a-\frac{b\Delta t}{3}\right)u_s+\Delta t\left(\frac{f_s}{3}+\frac{2f_{s+1}}{3}\right)$$

**Recall, using Finite difference method, we get**

$$[a+\alpha\Delta t_{s+1}b]u_{s+1}=[a-(1-\alpha)\Delta t_{s+1}b]\,u_s+\Delta t_{s+1}[\alpha f_{s+1}+(1-\alpha)f_s]$$

**By comparison,**
**we find that the Galerkin scheme is a specia case of the a finite difference family of approximation, with** $\alpha=2/3$

## Parabolic Equations- Stable and Conditionally Stable Schemes

$$u_{s+1} = \frac{a-(1-\alpha)\Delta t_{s+1} b}{a+\alpha \Delta t_{s+1} b} u_s + \Delta t_{s+1} \frac{[\alpha f_{s+1} + (1-\alpha) f_s]}{a+\alpha \Delta t_{s+1} b}$$

**can be written in the form:**

$$u_{s+1} = A(u_s) + F_{s,s+1}, A = \frac{a-(1-\alpha)\Delta t_{s+1} b}{a+\alpha \Delta t_{s+1} b}$$

$$F_{s,s+1} = \Delta t_{s+1} \frac{[\alpha f_{s+1} + (1-\alpha) f_s]}{a+\alpha \Delta t_{s+1} b}$$

**The operator $A$ is known as the amplification operator. Since us is an approximate solution, the error $E_s = u_a(t_s) - u_s$at time $t_s$ (where $u_a$ is the exact solution) will influence the solution at $t_{s+1}$**

# Time-dependent problems

$$u_{s+1} = A(u_s) + F_{s,s+1}, A = \frac{a - (1-\alpha)\Delta t_{s+1} b}{a + \alpha \Delta t_{s+1} b}$$

$$F_{s,s+1} = \Delta t_{s+1} \frac{[\alpha f_{s+1} + (1-\alpha) f_s]}{a + \alpha \Delta t_{s+1} b}$$

- **The error will grow (i.e, Es will be amplified) as we march in time if the magnitude of the operator is greater than 1,** $|A| > 1$
- **When the error grows without bound, the computational scheme becomes unstable (i.e, solution** $u_{s+1}$ **becomes unbounded with time)**
- **Therefore, in order for the scheme to be stable, it is necessary that** $|A| \leq 1$ **:**

$$\left| \frac{a - (1-\alpha)\Delta t_{s+1} b}{a + \alpha \Delta t_{s+1} b} \right| \leq 1$$

# Time-dependent problems

**The above equation places a restriction on the magnitude of the time step for certain values of $\alpha$**

- **When the error remains bounded for any time step (i.e., condition is trivially satisfied for any value of $\Delta t$, the scheme is stable**
- **If the error remains bounded only when the time step $\Delta t$ remains below certain value, the scheme is said to be conditionally stable**

**For different values of $\alpha$, the time approximation scheme yields a different scheme. The following well-known time-approximation schemes along with their order of accuracy and stability should be noted**

$$\alpha = \begin{cases} 0, & \text{The forward difference (or Euler) scheme (conditionally stable); order of accuracy} = O(\Delta t) \\ \frac{1}{2}, & \text{The Crank-Nicolson scheme (stable);} O(\Delta t)^2 \\ \frac{2}{3}, & \text{The Galerkin method (stable);} O(\Delta t)^2 \\ 1, & \text{The backward difference scheme (stable);} O(\Delta t) \end{cases}$$

# 谢 谢！

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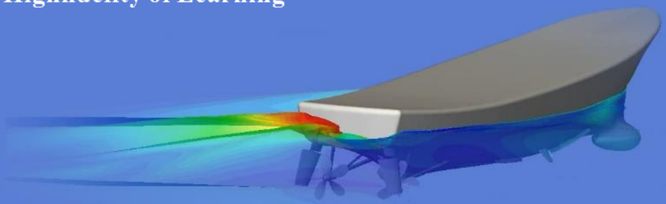

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