Class-4

NA26018

Finite Element Analysis of Solids and Fluids

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Eigenvalue problems

Introduction

An eigenvalue problem is defined to be one in which we seek the values of the parameter λ such that the equation

$$A(u) = \lambda B(u)$$

is satisfied for nontrivial values of u. Here A and B denote either matrix operators or differential operators, and values of n for which Eq. is satisfied are called eigenvalues. For each value of λ there is a vector u, called an eigenvector or eigenfunction

e.g.

$$-\frac{d^2u}{dx^2} = \lambda u(x), \qquad with\ A = -\frac{d^2}{dx^2}, B = 1$$

which arises in connection with natural axial vibrations of a bar or the transverse vibration of a cable, constitutes an eigenvalue problem. Here λ denotes the square of the frequency of vibration, w

Eigenvalue problems

In general, the determination of the eigenvalues is of engineering as well as mathematical importance

- **In structural problems, the eigenvalues denote either natural frequencies or buckling loads**
- **In fluid mechanics and heat transfer, eigenvalue problems arise in connection with the determination of the homogeneous parts of the transient solution**
- **Eigenvalues often denote amplitudes of Fourier components making up the solution**
- **Eigenvalues are also useful in determining the stability characteristics of temporal schemes**

Formulation of Eigenvalue Problems

Parabolic Equation

Consider the partial differential equation

$$\rho c A \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial t}\right) = q(x,t)$$

which arises in connection with transient heat transfer in one-dimensional systems (e.g a plane wall or a fin). u denotes the temperature, k the thermal conductivity, ρ the density, A the cross-sectional area, c the specific heat, q the heat generation per unit length

- **Equations involving the first-order time derivative are called parabolic equations**

Eigenvalue problems

The homogenous solution (i.e, the solution when $q = 0$) is often sought in the form of a product of a function of x and a function of t (i. e, through the separation-of-variables technique)

$$u^h(x,t) = U(x)T(t)$$

Substitution of this assumed form of solution into the homogeneous form gives

$$\rho c A \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial t}\right) = q(x,t)$$

Separating variables of t and x (assuming that $\rho c A$ and kA are functions of x only), we arrive at

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\rho c A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right)$$

Note that the left-hand side of this equation is a function of t only while the right-hand side is a function of x only

Eigenvalue problems

For two functions of two independent variables to be equal for all values of the independent variables, both functions must be equal to the same constant, say -λ (λ>0):

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\rho c A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{\partial u}{\partial t}\right) = -\lambda$$

$$\frac{dT}{dt} = -\lambda T \qquad \textbf{(p-1)}$$

$$-\frac{d}{dx}\left(kA\frac{\partial u}{\partial t}\right) - \lambda\rho c A U = 0 \qquad \textbf{(p-2)}$$

The negative sign of the constant λ is based on the physical requirement that the solution $U(x)$ be harmonic in x while $T(t)$ decay exponentially with increasing t

The solution of (p-1) is

$$T(t) = Ke^{-\lambda t}$$

Eigenvalue problems

$$T(t) = Ke^{-\lambda t}$$

where k is a constant of integration

- **The values of λ are determined by solving (p-2), which also gives $U(x)$**
- **With $T(t)$ and $U(x)$ known, we have the complete homogeneous solution**
- **The problem of solving (p-2) for λ and $U(x)$ is termed an eigenvalue problem, and λ is called the eigenvalue and $U(x)$ the eigenfunction**

When K, A, ρ, and c are constants, the solution of (p-2) is

$$U(x) = C \sin \alpha x + D \cos \alpha x \, , \alpha^2 = \frac{\rho c}{k} \lambda \qquad \textbf{(p-3)}$$

where C and D are constants of integration. Boundary conditions of the problem are used to find algebraic relations among C and D

Eigenvalue problems

To fix the ideas, consider Eq. (p-2) subject to the boundary conditions: (e. g, a fin with specified temperature at $x = 0$ and insulated at $x = L$)

$$U(0) = 0, \left[kA\frac{dU}{dx}\right]_{x=L} = 0$$

Using the above boundary conditions in (p-2),we obtain

$$0 = C \cdot 0 + D \cdot 1, 0 = \alpha(C\cos\alpha L - D\sin\alpha L)$$

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \alpha\begin{bmatrix} 0 & 0 \\ \cos\alpha L & \sin\alpha L \end{bmatrix}\right)\begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \textbf{(p-4)}$$

For nontrivial solution （i. e, not both C and D are equal to zero）, we set the determinant of the coefficient matrix in (p-4) to zero and obtain（since α cannot be zero)

$$\cos\alpha L = 0 \longrightarrow \alpha_n L = \frac{(2n-1)\pi}{2}$$

Eigenvalue problems

Hence, the homogeneous solution becomes

$$u^h(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \sin \alpha_n x \,, \lambda_n = \alpha_n^2 \left(\frac{k}{\rho c}\right), \alpha_n = \frac{(2n-1)\pi}{2L}$$

The constants C_n are determined using the initial condition of the problem, $u(x,0) = u_0(x)$

$$u^h(x,0) = \sum_{n=1}^{\infty} C_n \sin \alpha_n x = u_0(x)$$

Multiplying both sides with $\sin \alpha_m x$, integrating over the interval $(0, L)$, and making use of the orthogonality condition

$$\int_0^L sin\, \alpha_n x\, sin\, \alpha_m x\, dx = \begin{cases} 0, \text{if } m \neq n \\ \dfrac{L}{2}, \text{if } m = n \end{cases} \qquad C_n = \frac{2}{L}\int_0^L u_0(x) \sin \alpha_n x\, dx$$

The complete solution of the Parabolic equation is given by the sum of the homogeneous solution and the particular solution

$$u(x,t) = u^h(x,t) + u^p(x,t)$$

Eigenvalue problems

Hyperbolic Equation

The axial motion of a bar, for example, is described by the equation

$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = f(x,t)$$

where u denotes the axial displacement, E the modulus of elasticity, A the cross-sectional area, ρ the density, and f the axial force per unit length

- **The solution consists of two parts: homogeneous solution u^h (i.e, when $f = 0$) and particular solution up. The homogeneous part is determined by the separation-of-variables technique, as we discussed for the parabolic equation**

The homogenous solution is also assumed to be of the form:

$$u^h(x,t) = U(x)T(t)$$

Substitution into the homogeneous form of hyperbolic eq., gives

Eigenvalue problems

$$\rho A \frac{d^2T}{dt^2} - \frac{d}{dx}\left(EA\frac{dU}{dx}\right)T = 0$$

Assuming that ρA and EA are functions of x only, we arrive at

$$\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{\rho A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) = -\alpha^2$$

or

$$\frac{d^2T}{dt^2} + \alpha^2 T = 0 \qquad \textbf{(h-1)}$$

$$\frac{d}{dx}\left(EA\frac{dU}{dx}\right) - \alpha^2 \rho A U = 0 \qquad \textbf{(h-2)}$$

The solution of (h-1) is

$$T(t) = Ke^{-i\alpha t} = K_1 \cos\alpha t + K_2 \sin\alpha t$$

The solution of (h-2) is

$$U(x) = C\sin\overline{\alpha}x + D\cos\overline{\alpha}x\,,\overline{\alpha}^2 = \frac{\rho}{E}\alpha^2$$

the constants C and D are determined using the boundary conditions of the problem

Eigenvalue problems

Once again, we are required to solve an eigenvalue problem（the steps are analogous to those described for a parabolic equation）

Alternatively,

Eigenvalue problems associated with parabolic equations are obtained from corresponding equations of motion by assuming solution of the form:

$$u(x,t) = U(x)e^{-\alpha t}, \lambda = \alpha$$

Eigenvalue problems associated with hyperbolic equations are obtained by assuming solution of the form:

$$u(x,t) = U(x)e^{-iwt}, \lambda = w^2$$

λ denotes the eigenvalue

Also (p-2), (h-2) will be derived

Finite Element Formulation

Comparison of parabolic and hyperbolic eqs. with the previous model equation reveals that the equations governing eigenvalue problems are special cases of the model equations studied

Parabolic eq.

$$\frac{dT}{dt} = -\lambda T \qquad \textbf{(p-1)}$$

$$-\frac{d}{dx}\left(kA\frac{dU}{dx}\right) - \lambda\rho c A U = 0 \qquad \textbf{(p-2)}$$

Hyperbolic eq.

$$\frac{d^2T}{dt^2} + \alpha^2 T = 0 \qquad \textbf{(h-1)}$$

$$-\frac{d}{dx}\left(EA\frac{dU}{dx}\right) - \alpha^2\rho A U = 0 \qquad \textbf{(h-2)}$$

Second order differential eq. in Class-3

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f = 0 \ \text{ for } \ o < x < L$$

Eigenvalue problems

Here we will summarize the steps in the finite element formulation of eigenvalue problems for the sake of completeness

- **We will consider eigenvalue problems described by**
  1. **A single equation in a single unknown**
     (e. g, heat transfer, bar, and Euler-Bernoulli beam problems)
  2. **A pair of equations in two variables**
     (e.g, Timoshenko beam theory)

Eigenvalue problems

Heat Transfer and Bar-Like Problems

Consider the problem of solving the equation

$$-\frac{d}{dx}\left[a(x)\frac{dU}{dx}\right]+c(x)U(x)=\lambda c_0(x)U(x)$$

for λ and $U(x)$. Here a, c, and c_0 are known quantities that depend on the physical problem, λ is eigenvalue, and U is eigenfunction. Special cases of above Eq. are given below

$$\text{Heat transfer:}\quad a=kA, c=P\beta, c_0=\rho cA$$
$$\text{Bars:}\quad a=EA, c=0, c_0=\rho A$$

Over typical element Ω_e, we seek a finite element approximation of U in the form

$$U_h^e(x)=\sum_{j=1}^{n}u_j^e\psi_j^e(x)$$

Eigenvalue problems

The weak form of governing equation is

$$0 = \int_{xa}^{x_b} \left(a \frac{dw}{dx}\frac{dU}{dx} + cwU(x) - \lambda c_0 wU \right) dx - Q_1^e w(x_a) - Q_n^e w(x_b)$$

where w **is the weight function, and** Q_1^e **and** Q_n^e **are the secondary variables at node** 1 **and node** n**, respectively**

$$Q_1^e = -\left[a\frac{dU}{dx} \right]_{x_a}, Q_n^e = -\left[a\frac{dU}{dx} \right]_{x_b}$$

Substitution of the finite element approximation into the weak form gives the finite element model of the eigenvalue equation

$$[K^e]\{u^e\} - \lambda[M^e]\{u^e\} = \{Q^e\}$$

$$K_{i,j}^e = \int_{x_a}^{x_b} \left[a(x)\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c(x)\psi_i^e\psi_j^e \right] dx, M_{i,j}^e = \int_{x_a}^{x_b} c_0(x)\psi_i^e\psi_j^e dx$$

Above equation contains the finite element models of the eigenvalue equations（p-2）and（h-2）as special cases

Eigenvalue problems

Example

Consider a plane wall, initially at a uniform temperature T_0, which has both surfaces suddenly exposed to a fluid at temperature T_∞. The governing differential equation is

$$k\frac{\partial^2}{\partial x^2} = \rho c_0 \frac{\partial T}{\partial t}$$

and the initial condition is

$$T(x,0) = T_0$$

where k is the thermal conductivity, p the density, and c_0 the specitic heat at constant pressure, Equation is also known as the difusion equation with diffusion coefficient $a = k/pc_0$

- **We consider two sets of boundary conditions, each being representative of a different scenario for $x = L$. It amounts to solving for two different sets of boundary conditions**

Eigenvalue problems

BC Set 1: If the heat transfer coefficient at the surfaces ot the wall is assumed to be infinite the boundary conditions can be expressed as

$$T(0,t) = T_\infty, T(L,t) = T_\infty \text{ for } t > 0$$

BC Set 2: If we assume that the wall at $x = L$ is subjected to ambient temperature, we have

$$T(0,t) = T_\infty, \left[k\frac{\partial T}{\partial x} + \beta(T - T_\infty) \right]\Bigg|_{x=L} = 0$$

Equation can be normalized to make the boundary conditions homogeneous. Let

$$\alpha = \frac{k}{\rho c_0}, \hat{x} = \frac{x}{L}, \hat{t} = \frac{\alpha t}{L^2}, u = \frac{T - T_\infty}{T_0 - T_\infty}$$

The differential equation, initial condition, and boundary conditions become

Eigenvalue problems

$$-\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}=0$$

$$u(0,t)=0, u(1,t)=0, u(x,0)=1 \qquad BC \text{ set 1}$$

$$u(0,t)=0, \left(\frac{\partial u}{\partial x}+Hu\right)\Bigg|_{x=1}=0, H=\frac{\beta L}{k} \qquad BC \text{ set 2}$$

where the bars over x and u are omitted in the interest of brevity

By separation-of variables technique (or substitute $u=Ue^{-\lambda t}$ leads to the solution of the eigenvalue problem:

$$-\frac{d^2 U}{dx^2}-\lambda U=0, U(0)=0, U(1)=0 \qquad BC \text{ set 1}$$

$$-\frac{d^2 U}{dx^2}-\lambda U=0, U(0)=0, \left(\frac{dU}{dx}+HU\right)\Bigg|_{x=1}=0 \qquad BC \text{ set 2}$$

Eigenvalue problems

$$-\frac{d^2U}{dx^2}-\lambda U=0$$

Recall in Heat Transfer and Bar-Like Problems, we solve

$$-\frac{d}{dx}\left[a(x)\frac{dU}{dx}\right]+c(x)U(x)=\lambda c_0(x)U(x)$$

This differential equation is a special case with $a=1, c=0$, and $c_0=1$. For a linearelement, the element equations have the explicit form:

$$\left(\frac{1}{h_c}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}-\lambda\frac{h_c}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}u_1^e\\ u_2^e\end{Bmatrix}=\begin{Bmatrix}Q_1^e\\ Q_2^e\end{Bmatrix}$$

For a quadratic element, we have

$$\left(\frac{1}{3h_c}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}-\lambda\frac{h_e}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}\right)\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\end{Bmatrix}=\begin{Bmatrix}Q_1^e\\ Q_2^e\\ Q_3^e\end{Bmatrix}$$

Eigenvalue problems

Solution for Set 1

For a mesh of two linear elements (the minimum number needed for Set 1 boundary conditions), with $h_1 = h_2 = 0.5$, the assembled equations are

$$\left(2\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \lambda\frac{1}{12}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_1\\ U_2\\ U_3\end{Bmatrix} = \begin{Bmatrix}Q_1^1\\ Q_2^1 + Q_1^2\\ Q_2^2\end{Bmatrix}$$

The boundary conditions $U(0) = 0,\ \ Q_2^1 + Q_1^2 = 0, and\ U(1) = 0$ require $U_1 = U_3 = 0$. Hence, the eigenvalue problem reduces to the single equation

$$\left(4 - \lambda\frac{4}{12}\right)U_2 = 0, \text{or } \lambda_1 = 12.0, U_2 \neq 0$$

The mode shape is given （within an arbitrary constant take $U_2 = 1$） by

$$U(x) = U_2\Phi_2(x) = \begin{cases} U_2\psi_2^1(x) = x/h = 2x & 0 \le x \le 0.5\\ U_2\psi_2^1(x) = (2h - x)/h = 2(1-1) & 0.5 \le x \le 1.0\end{cases}$$

Eigenvalue problems

For a mesh of one quadratic element, we have ($h = 1.0$)

$$\frac{16}{3} - \lambda\frac{16}{30} = 0, or\ \lambda_1 = 10.0, U_2 \neq 0$$

The corresponding eigenfunction is

$$U(x) = U_2\Phi_2(x) = U_2\psi_2^1 = 4\frac{x}{h}\left(1 - \frac{x}{h}\right), 0 \leq x \leq 1.0$$

The exact eigenvalues: $\lambda_n = (n\pi)^2\ and\ \lambda_1 = (\pi)^2 = 9.8696$

By comparison, one quadratic element gives more accurate solution than two linear elements

Natural Vibration of Beams

Euler-Bernoulli Beam

For the Euler-Bernoulli beam theory, the equation of motion is of the form:

$$\rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial t^2}\left(EI \frac{\partial^2 w}{\partial t^2} \right) = q(x,t)$$

where ρ denotes the mass density per unit length, A the area of cross section, E the modulus and I the second moment of area. The expression involving ρI is called rotary inertia term

- **Equation can be formulated as an eigenvalue problem in the interest of finding the frequency of natural vibration by assuming periodic motion**

$$w(x,t) = W(x)e^{-iwt}$$

where w is the frequency of natural transverse motion and $W(x)$ is the mode shape of the transverse motion. Substitution into Euler-Bernoulli beam equation, yields

Eigenvalue problems

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\lambda\left(\rho AW-\rho I\frac{d^2w}{dx^2}\right)=0 \qquad \lambda=w^2$$

The weak form of above Eq. is given by

$$0=\int_{x_a}^{x_b}\left(EI\frac{d^2v}{dx^2}\frac{d^2W}{dx^2}-\lambda\rho A_vW-\lambda\rho I\frac{dv}{dx}\frac{dW}{dx}\right)dx$$
$$+\left\{v\left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right)+\lambda\rho I\frac{dW}{dx}\right]\right\}_{x_a}^{x_b}+\left[\left(-\frac{dv}{dx}\right)EI\frac{d^2W}{dx^2}\right]_{x_a}^{x_b}$$

where v is the weight function

Note: The rotary inertia term contributes to the shear force term, giving rise to an effective shear force that must be known at a boundary point when the deflection is unknown at the point

To obtain the finite element model of weak form, assume finite element approximation of the form

$$W(x)=\sum_{j=1}^{4}\Delta_j^e\phi_j^e(x)$$

where ϕ_j^e are the Hermite cubic polynomials

Eigenvalue problems

We obtain the finite element model

$$\left([K^e] - w^2[M^e]\{\Delta^e\} = \{Q^e\}\right)$$

$$K_{ij}^e = \int_{x_a}^{x_b} EI \frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}dx,\ M_{ij}^e = \int_{x_a}^{x_b}\left(\rho A\phi_i^e\phi_j^e + \rho I\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\right)dx$$

$$Q_1^e = \left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\Bigg|_{x_a},Q_2^e = \left(EI\frac{d^2W}{dx^2}\right)\Bigg|_{x_a}$$

$$Q_3^e = -\left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\Bigg|_{x_b},Q_4^e = -\left(EI\frac{d^2W}{dx^2}\right)\Bigg|_{x_b}$$

For constant values of EI and ρA, the stiffness matrix $[Ke]$ and mass matrix $[Me]$ are

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

Eigenvalue problems

$$[M^e] = \frac{\rho^e A_e h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}$$

$$+\frac{\rho^e I_e}{30h_e}\begin{bmatrix} 36 & -3h_e & -36 & -3h_e \\ -3h & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix}$$

When rotary inertia is neglected, we omit the second part of the mass matrix in $[M^e]$

Eigenvalue problems

Timoshenko Beam

For the Timoshenko beam theory, the equation of motion is of the form:

$$\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[GAK_s\left(\frac{\partial w}{\partial x} + \Psi\right)\right] = 0$$

$$\rho I \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial}{\partial x}\left(EI\frac{\partial \Psi}{\partial x}\right) + GAK_s\left(\frac{\partial w}{\partial x} + \Psi\right) = 0$$

where G is the shear modulus $(G = E/2(1+v))$ and K_s is the shear correction factor $(K_s = 5/6)$. Note that above Eq. contains the rotary inertia term. Once again, we assume periodic motion and write

$$w(x,t) = W(x)e^{-iwt}, \Psi(x,t) = S(x)e^{-iwt}$$

and obtain the eigenvalue problem

$$-\frac{d}{dx}\left[GAK_s\left(\frac{\partial w}{\partial x} + s\right)\right] - w^2\rho AW = 0$$

$$-\frac{d}{dx}\left(EI\frac{ds}{dx}\right) + GAK_s\left(\frac{\partial w}{\partial x} + s\right) - w^2\rho IS = 0$$

Eigenvalue problems

For equal interpolation of $W(x)$ and $S(x)$,

$$W(x) = \sum_{j=1}^{n} W_j^e \psi_j^e(x)\,, S(x) = \sum_{j=1}^{n} S_j^e \psi_j^e(x)$$

where ψ_j^e are the $(n-1)$ order Lagrange polynomials, the finite element model is given by

$$\left(\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} - w^2 \begin{bmatrix} [M^{11}] & 0 \\ 0 & [M^{22}] \end{bmatrix} \right) \begin{Bmatrix} \{W\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$[K^e]$ is the stiffness matrix and $[M^e]$ is the mass matrix

$$K_{ij}^{11} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx} \psi_j^e dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(EI \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + GAK_s \psi_i^e \psi_j^e \right) dx,$$

$$M_{ij}^{11} = \int_{x_a}^{x_b} \rho A \psi_i^e \psi_j^e dx, M_{ij}^{22} = \int_{x_a}^{x_b} \rho AI \psi_i^e \psi_j^e dx,$$

Eigenvalue problems

$$F_j^1 = Q_{2i-1}, F_j^2 = Q_{2i}$$

$$Q_1^e \equiv -\left[GAK_s\left(s + \frac{dW}{dx}\right)\right]\Bigg|_{x_a}, Q_2^e \equiv -\left(EI\frac{dS}{dx}\right)\Bigg|_{x_a}$$

$$Q_3^e \equiv \left[GAK_s\left(s + \frac{dW}{dx}\right)\right]\Bigg|_{x_b}, Q_4^e \equiv -\left(EI\frac{dS}{dx}\right)\Bigg|_{x_b}$$

For the choice of linear interpolation functions, we have

$$[K^e] = \left(\frac{2E_eI_e}{\mu_0 h_e^3}\right)\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5+6\Lambda_e) & 3h_e & h_e^2(1.5-6\Lambda_e) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5-6\Lambda_e) & 3h_e & h_e^2(1.5+6\Lambda_e) \end{bmatrix}$$

$$[M^e] = \frac{\rho^e A_e}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2r_e & 0 & r_e \\ 1 & 0 & 2 & 0 \\ 0 & r_e & 0 & 2r_e \end{bmatrix}, r_e = \frac{I_e}{A_e}$$

$$\Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2}, \mu_0 = 12\Lambda_e$$

谢　谢！

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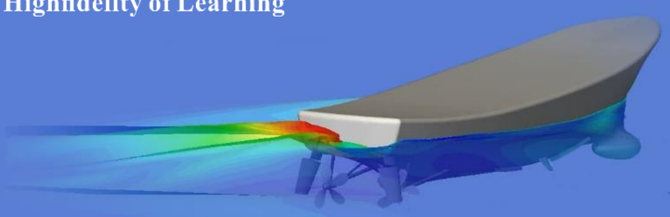

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