Class-3

NA26018

Finite Element Analysis of Solids and Fluids

万德成

dcwan@sjtu.edu.cn， http://dcwan.sjtu.edu.cn/

上海交通大学

船舶海洋与建筑工程学院

海洋工程国家重点实验室

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Contents

- ✓ **Finite element formulation of the 1-D fourth-order differential equation that arises in the Euler-Bernoulli beam theory**
- ✓ **Finite element formulation of the pair of 1-D second-order equations associated with the Timoshenko beam theory**
- ✓ **Frame elements that can be used to analyze plane frame structures**

Euler-Bernoulli beam element

Euler-Bernoulli beam theory:

- **It is assumed that plane cross sections perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation**

In this theory, the transverse deflection w of the beam is governed by the fourth-order differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w=q(x)\quad for\quad 0<x<L$$

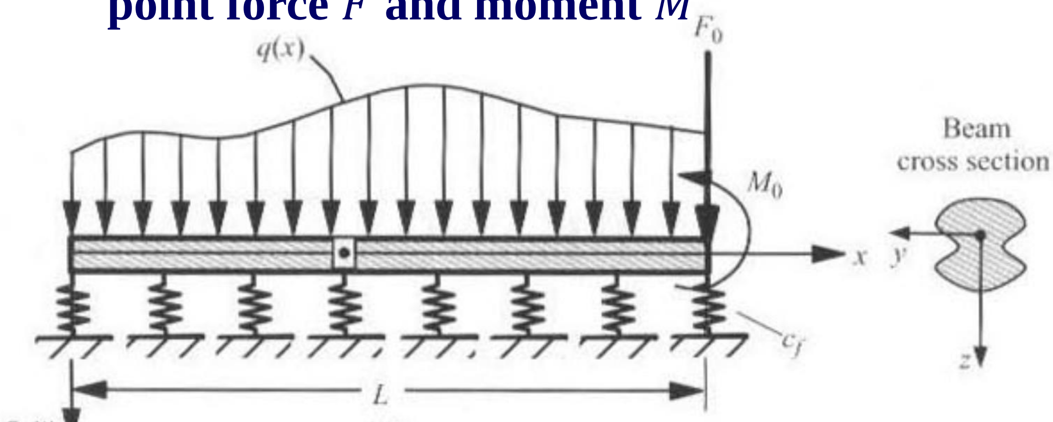
$EI=E(x)I(x)$, $cf=cf(x)$, $q=q(x)$ **are given functions of** x
w**: Dependent variable, transverse deflection of the beam**
E**: Modulus of elasticity**
I**: Scond moment of area about the y axis of the beam**
q**: Distributed transverse load**
c_f **: Elastic foundation modulus**

Euler-Bernoulli beam element

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w=q(x)\quad for\quad 0<x<L$$

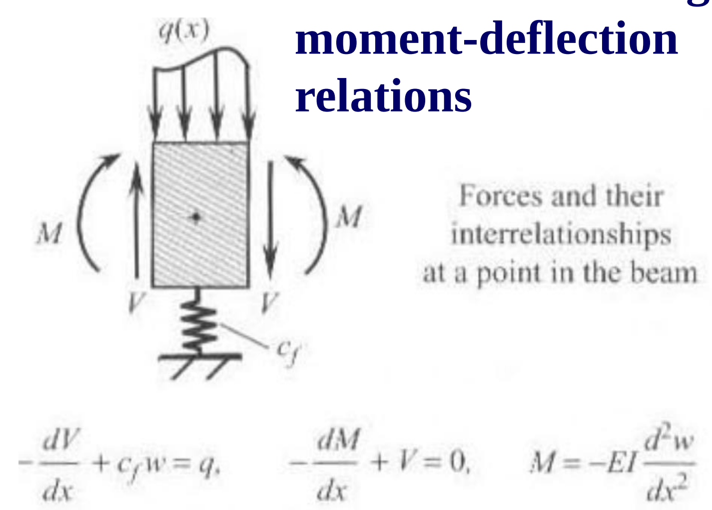
- w must satisfy appropriate boundary conditions, since the equation is of fourth order, four boundary conditions are needed to solve it
- The weak formulation of the equation will provide the form of these four boundary conditions
- A step-by-step procedure for the finite element analysis of DE will be presented

Typical beam with distributed load q and point force F and moment M



(a)

Shear force-bending moment-deflection relations

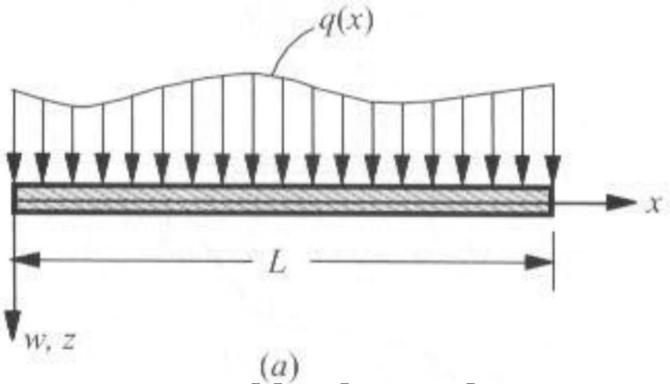


(b)

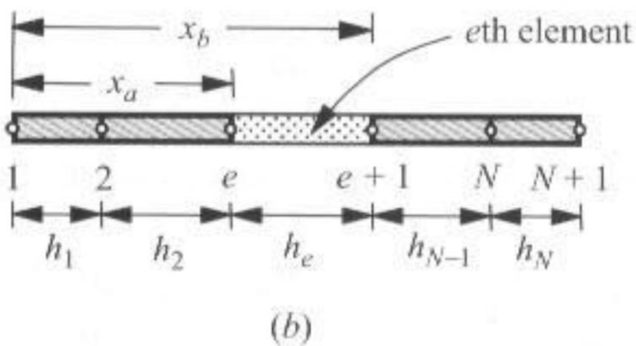
Euler-Bernoulli beam element

The domain of the straight beam is divided into a set of N line elements, each element $\Omega^e=(x_a,x_b)=(x_e,x_{e+1})$ having at least two end nodes

- **The element is geometrically the same as that used for bars, the number and form of the primary and secondary unknowns at each node are dictated by the variational formulation of the differential equation**
- **In most practical problems, the discretization of a given structure into a minimum number of elements is often dictated by the geometry, loading, and material properties**



Geometry and loads on a beam

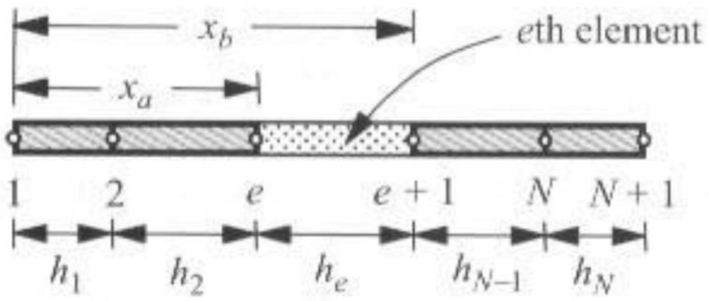


Finite element discretization

Derivation of Element Equations

- **Variational formulation (provides the primary and secondary variables of the problem)**
- **Suitable approximations, interpolation functions for the primary variables**
- **Element equations**

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w = q(x) \quad for \quad 0<x<L$$



Weak Form

Construct the weak form over the element

$$0=\int_{x_e}^{x_{e+1}} v[\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-q]dx$$

$$=\int_{x_e}^{x_{e+1}}\left[\frac{dv}{dx}\frac{d}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w-vq\right]dx+\left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e}^{x_{e+1}}$$

$$=\int_{x_e}^{x_{e+1}}\left[EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}+c_f vw-vq\right]dx+\left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)-\frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}}$$

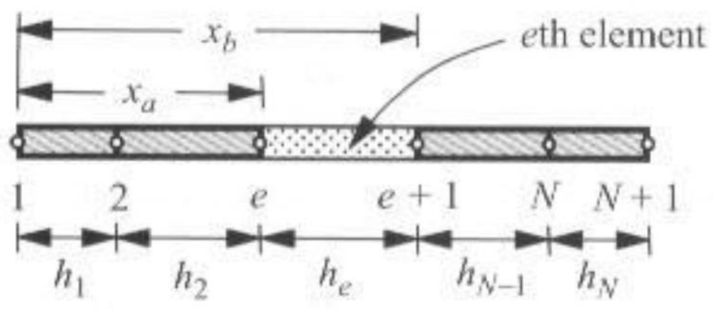
$v(x)$ is a weight function that is twice differentiable with respect to x

- **The first term of the equation is integrated twice by parts, to yield two differentiations to the weight function v while retaining two derivatives of the dependent variable w**
- **Now the differentiation is distributed equally between the weight function u and the dependent variable w**
- **Because of the two integration by parts, there appear two boundary expressions, which are to be evaluated at the two boundary points**

Euler-Bernoulli beam element

$$0 = \int_{x_e}^{x_{e+1}} \left[EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w - v q \right] dx + \left[v \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right) - \frac{dv}{dx} EI \frac{d^2 w}{dx^2} \right]_{x_e}^{x_{e+1}}$$

- **Examination of the boundary terms indicates that the essential boundary conditions involve the specification of the deflection w and slope dw/dx**
- **The natural boundary conditions involve the specification of the bending moment ($EI \frac{d^2 w}{dx^2}$) and shear force ($\frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right)$) at the endpoints of the element**



Euler-Bernoulli beam element

$$0 = \int_{x_e}^{x_{e+1}} \left[EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w - v q \right] dx + \left[v \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right) - \frac{dv}{dx} EI \frac{d^2 w}{dx^2} \right]_{x_e}^{x_{e+1}}$$

For this case:

- **There are two essential boundary conditions and two natural boundary conditions**
- **we must identify w and dw/dx as the primary variables at each node (so that the essential boundary conditions are included in the interpolation)**
- **The natural boundary conditions always remain in the weak form and end up on the right-hand side of the equation**

Euler-Bernoulli beam element

Introduce the following notation for the secondary variables

Q_1^e, Q_3^e: shear force
Q_2^e, Q_4^e: bending moment

Generalized forces:

$$Q_1^e \equiv \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e} = -V(x_e)$$

$$Q_2^e \equiv \left(EI\frac{d^2w}{dx^2}\right)|_{x_e} = -M(x_e)$$

$$Q_3^e \equiv -\left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_{e+1}} = V(x_{e+1})$$

$$Q_4^e \equiv -\left(EI\frac{d^2w}{dx^2}\right)|_{x_{e+1}} = M(x_{e+1})$$

corresponding displacements and rotations are called the generalized displacements

$$0 = \int_{x_e}^{x_{e+1}} \left(EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw - vq\right)dx + \left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) - \frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}}$$

Euler-Bernoulli beam element

$$0 = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w - v q \right) dx$$

$$-v(x_e) Q_1^e - \left(-\frac{dv}{dx} \right) |_{x_e} Q_2^e - v(x_{e+1}) Q_3^e - \left(-\frac{dv}{dx} \right) |_{x_{e+1}} Q_4^e$$

$$\equiv B(v, w) - l(v)$$

Bilinear and linear forms of this problem:

$$B(v, w) = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w \right) dx$$

$$l(v) = \int_{x_e}^{x_{e+1}} v q dx + v(x_e) Q_1^e + \left(-\frac{dv}{dx} \right) |_{x_e} Q_2^e$$

$$+v(x_{e+1}) Q_3^e + \left(-\frac{dv}{dx} \right) |_{x_{e+1}} Q_4^e$$

A statement of the principle of virtual displacements (u denotes virtual displacement) for the Euler-Bernoulli beam theory

Euler-Bernoulli beam element

The quadratic functional (total potential energy) of the isolated beam element, is given by

$$\prod_e(w) = \int_{x_e}^{x_{e+1}} \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 + \frac{1}{2}c_f w^2 - wq\right] dx - w(x_e)Q_1^e - w(x_{e+1})Q_3^e$$

$$-\left(-\frac{dw}{dx}\right)|_{x_e} Q_2^e - w(x_{e+1})Q_3^e - \left(-\frac{dw}{dx}\right)|_{x_{e+1}} Q_4^e$$

- **First term in the square brackets represents the elastic strain energy due to bending**
- **Second is the strain energy stored in the elastic foundation, Third is the work done by the distributed load**
- **Remaining terms account for the work done by the generalized forces Q_i^e in moving through the respective generalized displacements of the element**

We may go from the total potential energy functional to the weak form by using the principle of minimum potential energy,

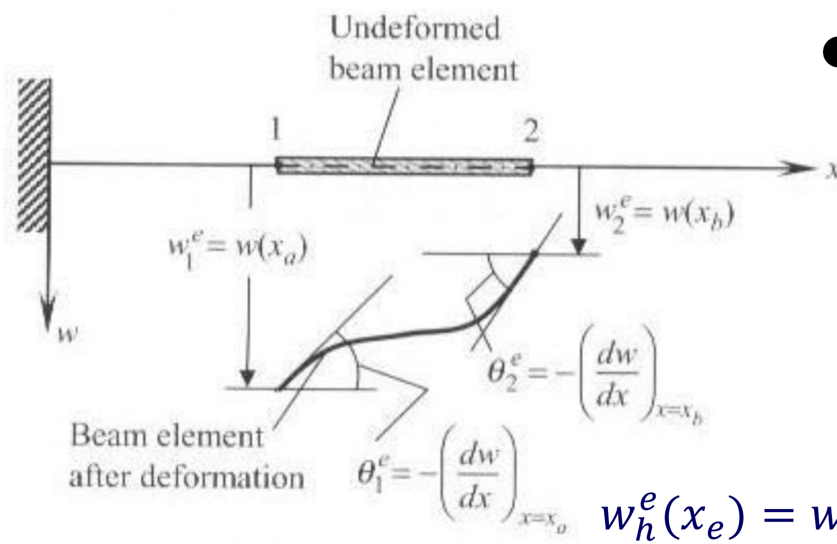
$\delta\Pi = 0$

Interpolation Functions

$$0 = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w - v q \right) dx$$

$$- v(x_e) Q_1^e - \left(-\frac{dv}{dx} \right) |_{x_e} Q_2^e - v(x_{e+1}) Q_3^e - \left(-\frac{dv}{dx} \right) |_{x_{e+1}} Q_4^e$$

- **Interpolation functions of an element be continuous with nonzero derivatives up to order two**



- **The approximation $w_h^e(x)$ over a finite element should be twice differentiable and satisfies the interpolation properties (i.e., satisfies the essential boundary conditions of the element)**

$$w_h^e(x_e) = w_1^e, w_h^e(x_{e+1}) = w_2^e, \theta_h^e(x_e) = \theta_1^e, \theta_h^e(x_{e+1}) = \theta_2^e$$

Euler-Bernoulli beam element

$$w_h^e(x_e) = w_1^e, w_h^e(x_{e+1}) = w_2^e, \theta_h^e(x_e) = \theta_1^e, \theta_h^e(x_{e+1}) = \theta_2^e$$

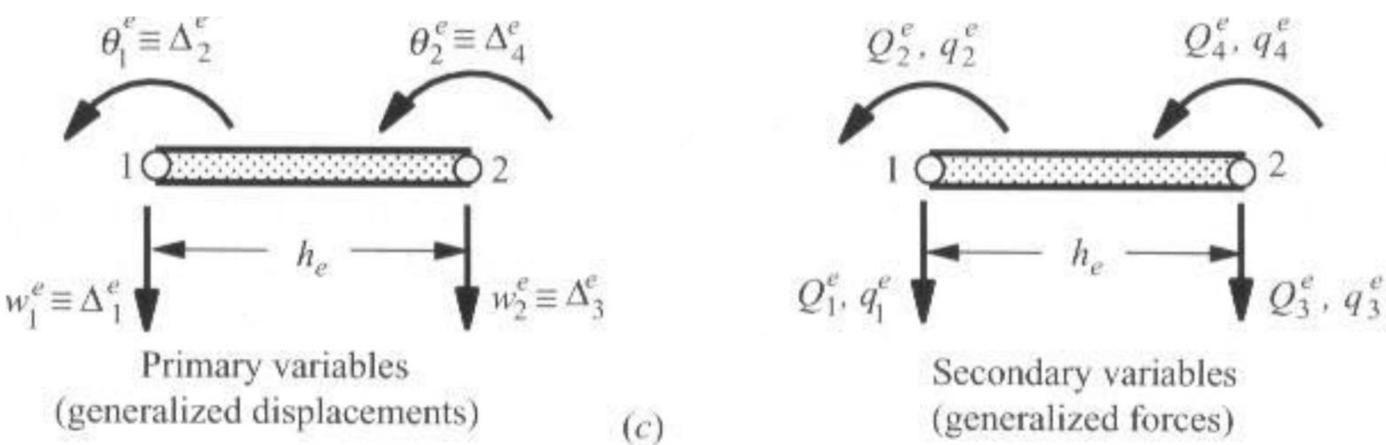
$$\boxed{\theta = -\, d\,w/d\,x}$$

There are a total of 4 conditions in an element (two per node), a 4-parameter polynomial must be selected for w

$$w(x) \approx w_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 + c_4^e x^3$$

expressing cej in terms of the primary nodal variables

$$\Delta_1^e \equiv w_h^e(x_e), \Delta_2^e \equiv -\frac{dw_h^e}{dx}|_{x=x_e}, \Delta_3^e \equiv w_h^e(x_{e+1}), \Delta_4^e \equiv -\frac{dw_h^e}{dx}|_{x=x_{e+1}}$$



Euler-Bernoulli beam element

$$\Delta_1^e \equiv w_h^e(x_e) \qquad = c_1^e + c_2^e x_e + c_3^e x_e^2 + c_4^e x_e^3$$

$$\Delta_2^e \equiv -\frac{dw_h^e}{dx}\Big|_{x=x_e} = -c_2^e - 2c_3^e x_e - 3c_4^e x_e^2$$

$$\Delta_3^e \equiv w_h^e(x_{e+1}) \qquad = c_1^e + c_2^e x_{e+1} + c_3^e x_{e+1}^2 + c_4^e x_{e+1}^3$$

$$\Delta_4^e \equiv -\frac{dw_h^e}{dx}\Big|_{x=x_{e+1}} = -c_2^e - 2c_3^e x_{e+1} - 3c_4^e x_{e+1}$$

$$\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{bmatrix} 1 & x_e & x_e^2 & x_e^3 \\ 0 & -1 & -2x_e & -3x_e^2 \\ 1 & x_{e+1} & x_{e+1}^2 & x_{e+1}^3 \\ 0 & -1 & -2x_{e+1} & -3x_{e+1}^2 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \\ c_4^e \end{Bmatrix}$$

Inverting this matrix equation to express cej in terms of Δ_1^e, Δ_2^e, Δ_3^e and Δ_4^e, and substituting the result to $w_h^e(x)$

Euler-Bernoulli beam element

$$w_h^e(x_e) = \Delta_1^e \phi_1^e + \Delta_2^e \phi_2^e + \Delta_3^e \phi_3^e + \Delta_4^e \phi_4^e = \sum_{j=1}^{4} \Delta_j^e \phi_j^e$$

Hermite cubic (or cubic spline) interpolation functions

$$x_{e+1} = x_e + h_e$$

$$\phi_1^e = 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_2^e = -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3$$

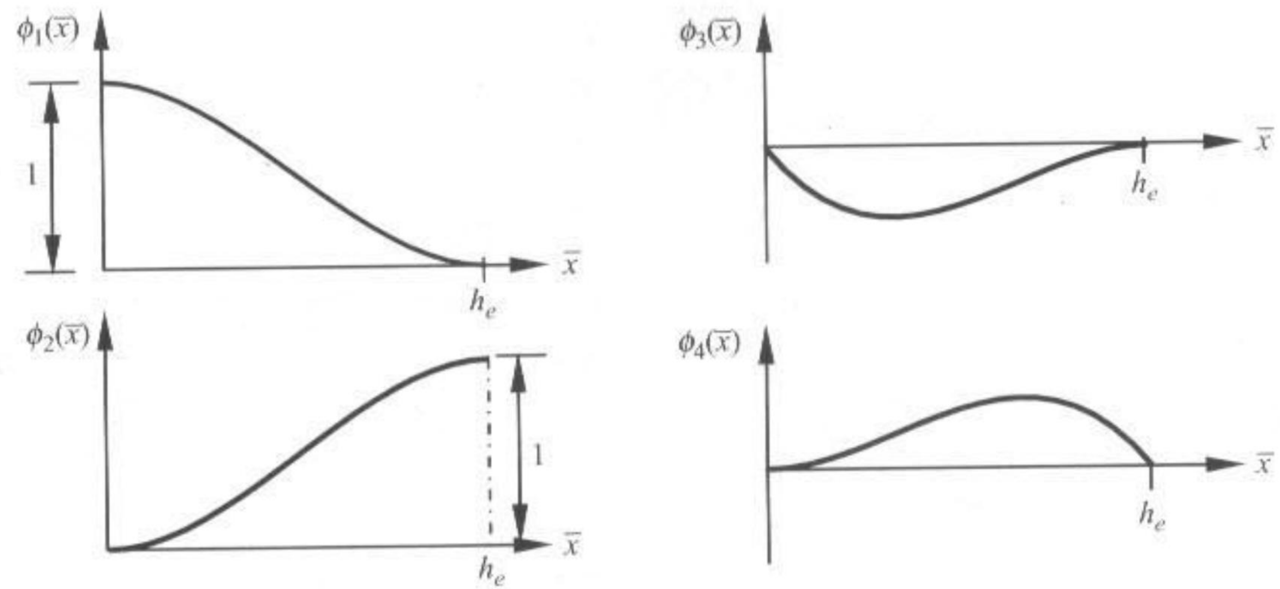
$$\phi_4^e = -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right]$$

Cubic interpolation functions are derived by interpolating w as well as its derivative dw/dx at the nodes

Euler-Bernoulli beam element

- **Recall that Lagrange cubic interpolation functions are derived to interpolate a function, but not its derivatives, at the nodes**
- **Hence, a Lagrange cubic element will have 4 nodes, with the dependent variable, not its derivative, as the nodal degree of freedom**
- **Since the slope (or derivative) of the dependent variable is also required by the weak form to be continuous at the nodes for the Euler-Bernoulli beam theory, the Lagrange cubic interpolation of w, although it meets the continuity requirement for w, is not admissible in the finite element approximation of the Euler-Bernoulli beam theory**

Euler-Bernoulli beam element



Hermite cubic interpolations functions used in the Euler-Bernoulli beam element

Euler-Bernoulli beam element

$$\phi_1^e = 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_2^e = -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_4^e = -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right]$$

$$\bar{x} = x - x_e$$

$$\phi_1^e = 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3, \phi_2^e = -\bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3, \phi_4^e = -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \frac{\bar{x}}{h_e}\right]$$

Euler-Bernoulli beam element

The first, second and third derivatives of ϕ_i^e with respect to x are

$$\frac{d\phi_1^e}{d\bar{x}} = -\frac{6}{h_e}\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right) \frac{d\phi_2^e}{d\bar{x}} = -\left[1+3\left(\frac{\bar{x}}{h_e}\right)^2 - 4\frac{\bar{x}}{h_e}\right]$$

$$\frac{d\phi_3^e}{d\bar{x}} = -\frac{d\phi_1^e}{d\bar{x}} = \frac{6}{h_e}\frac{\bar{x}}{h_e}\left(1-\frac{\bar{x}}{h_e}\right) \frac{d\phi_4^e}{d\bar{x}} = -\frac{\bar{x}}{h_e}\left(3\frac{\bar{x}}{h_e}-2\right)$$

$$\frac{d^2\phi_1^e}{d\bar{x}^2} = -\frac{6}{h_e^2}\left(1-2\frac{\bar{x}}{h_e}\right) \frac{d^2\phi_2^e}{d\bar{x}^2} = -\frac{2}{h_e}\left(3\frac{\bar{x}}{h_e}-2\right)$$

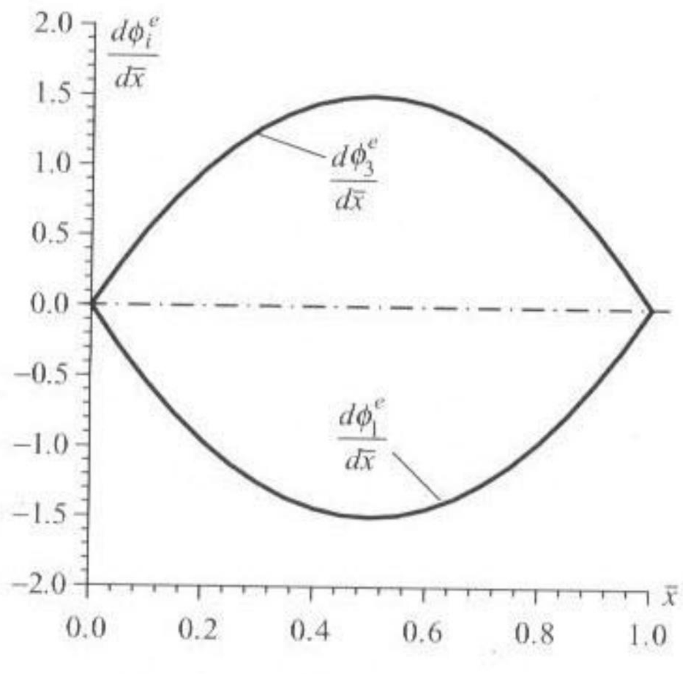
$$\frac{d^2\phi_3^e}{d\bar{x}^2} = -\frac{d^2\phi_1^e}{d\bar{x}^2} = \frac{6}{h_e^2}\left(1-2\frac{\bar{x}}{h_e}\right) \frac{d^2\phi_4^e}{d\bar{x}^2} = -\frac{2}{h_e}\left(3\frac{\bar{x}}{h_e}-1\right)$$

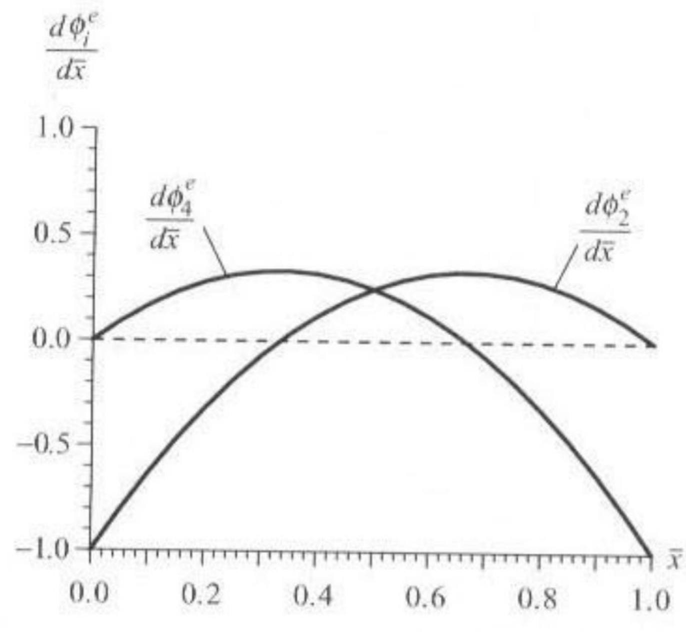
$$\frac{d^3\phi_1^e}{d\bar{x}^3} = \frac{12}{h_e^3}, \frac{d^3\phi_2^e}{d\bar{x}^3} = -\frac{6}{h_e^2}$$

$$\frac{d^3\phi_3^e}{d\bar{x}^3} = \frac{12}{h_e^3}, \frac{d^3\phi_4^e}{d\bar{x}^3} = -\frac{6}{h_e^2}$$

Euler-Bernoulli beam element

First derivatives dw/dx of the Hermite cubic interpolations functions





Euler-Bernoulli beam element

Hermite cubic interpolation functions satisfy the following interpolations properties:

$$\phi_1^e(x_e) = 1, \quad \phi_i^e(x_e) = 0 \qquad (i \neq 1)$$

$$\phi_3^e(x_{e+1}) = 1, \quad \phi_i^e(x_{e+1}) = 0 \qquad (i \neq 3)$$

$$\left(-\frac{d\phi_2^e}{dx}\right)|_{x_e} = 1, \left(-\frac{d\phi_i^e}{dx}\right)|_{x_e} = 0 \qquad (i \neq 2)$$

$$\left(-\frac{d\phi_4^e}{dx}\right)|_{x_{e+1}} = 1, \left(-\frac{d\phi_i^e}{dx}\right)|_{x_{e+1}} = 0 \qquad (i \neq 4)$$

Can be stated in a compact form:

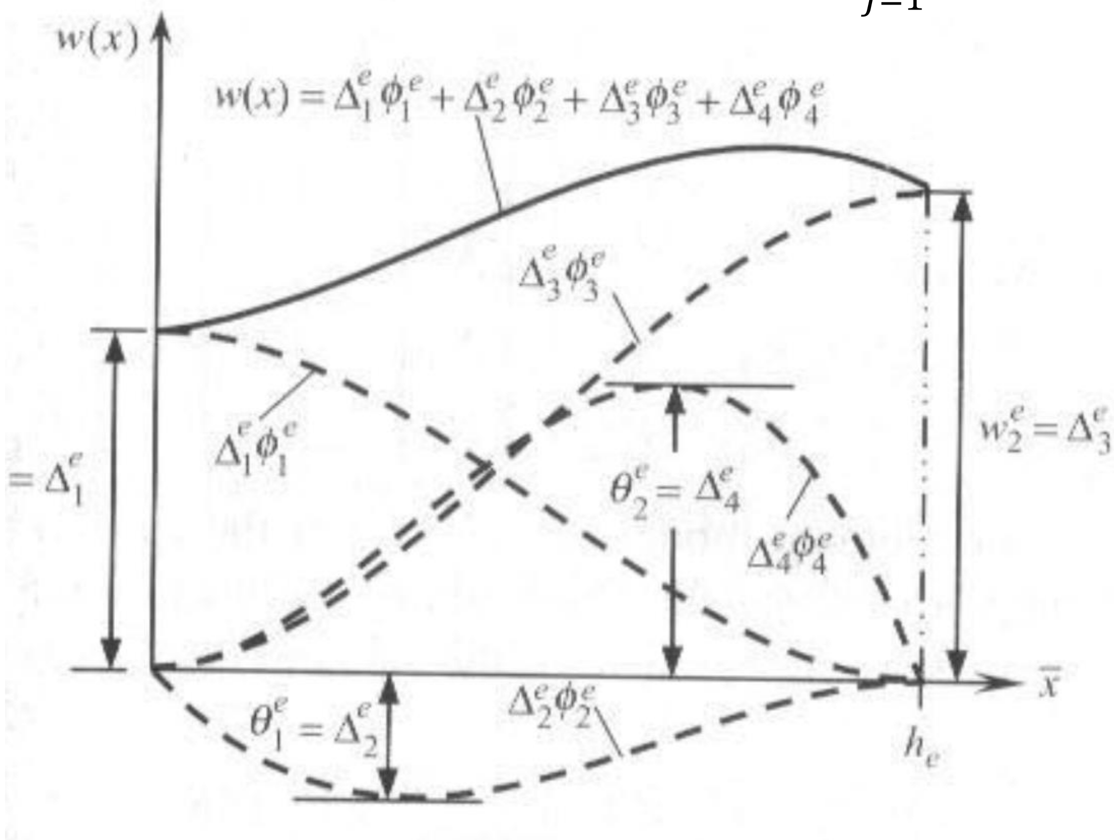
$$\phi_{2i-1}^e(\bar{x}_j) = \delta_{ij}, \phi_{2i}^e(\bar{x}_j) = 0, \qquad \sum_{i=1}^{2} \phi_{2i-1}^e = 1$$

$$\frac{d\phi_{2i-1}^e}{dx}|_{\bar{x}_j} = 0, \left(-\frac{d\phi_{2i-1}^e}{dx}\right)|_{\bar{x}_j} = \delta_{ij}$$

$\bar{x}_1 = 0$ **and** $\bar{x}_1 = h_e$ **are the local coordinates of nodes 1 and 2 of the element** Ω^e $[x_e, x_{e+1}]$

Euler-Bernoulli beam element

$$w_h^e(x_e) = \Delta_1^e\phi_1^e + \Delta_2^e\phi_2^e + \Delta_3^e\phi_3^e + \Delta_4^e\phi_4^e = \sum_{j=1}^{4} \Delta_j^e\phi_j^e$$



Finite element solution over an element

Euler-Bernoulli beam element

$$w_h^e(x_e) = \Delta_1^e \phi_1^e + \Delta_2^e \phi_2^e + \Delta_3^e \phi_3^e + \Delta_4^e \phi_4^e = \sum_{j=1}^{4} \Delta_j^e \phi_j^e$$

- **The order of the interpolation functions derived above is the minimum required for the variational formulation**
- **If a higher-order (i. e, higher than cubic) approximation of w is desired, we must either identify additional primary unknowns at each of the two nodes or add additional nodes with the two degrees of freedom (w, $-dw/dx$)**

For example, if we add d^2w/dx^2 as the primary unknown at each of the two nodes or add a third node with $(w, -dw/dx)$ at each node, there will be a total of six conditions, and a fifth-order polynomial is required to interpolate the end conditions. Interelement continuity of d^2w/dx^2 is not required by the weak form

Finite Element Model

The finite element model of the Euler-Bernoulli beam is obtained by substituting the the finite element interpolation for w and the ϕ_j^e for the weight function v into the weak form

- **Since there are 4 nodal variables Δ_i^e, 4 different choices are used for v: $v = \phi_1^e, \phi_2^e, \phi_3^e, \phi_4^e$ allowing us to obtain a set of 4 algebraic equations**

The ith algebraic equation of the finite element model is ($v = \phi_i^e$)

$$w_h^e(x_e) = \Delta_1^e \phi_1^e + \Delta_2^e \phi_2^e + \Delta_3^e \phi_3^e + \Delta_4^e \phi_4^e = \sum_{j=1}^{4} \Delta_j^e \phi_j^e \qquad v = \phi_i^e$$

$$0 = \int_{x_e}^{x_{e+1}} \left[EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w - v q \right] dx$$

$$-v(x_e) Q_1^e - \left(-\frac{dv}{dx} \right) |_{x_e} Q_2^e - v(x_{e+1}) Q_3^e - \left(-\frac{dv}{dx} \right) |_{x_{e+1}} Q_4^e$$

Euler-Bernoulli beam element

$$0 = \sum_{j=1}^{4} \left[\int_{x_e}^{x_{e+1}} \left(EI \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_i^e}{dx^2} + c_f \phi_i^e \phi_j^e \right) dx \right] u_j^e - \int_{x_e}^{x_{e+1}} \phi_i^e q dx - Q_i^e$$

$$\sum_{j=1}^{4} K_{ij}^e \Delta_j^e - F_i^e = 0 \;\; or \; [K^e]\{\Delta^e\} = \{F^e\}$$

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_j^e}{dx^2} + c_f \phi_i^e \phi_j^e \right) dx$$

$$F_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e q dx + Q_i^e$$

Coefficients K are symmetric: $K_{ij}^e = K_{ji}^e$

Euler-Bernoulli beam element

$$\sum_{j=1}^{4} K_{ij}^{e}\Delta_j^{e} - F_i^{e} = 0 \;\; or \;\; [K^e]\{\Delta^e\} = \{F^e\}$$

$$[K^e] \quad = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

For the case in which EI and q are constant over an element, the element stiffness matrix K^e have the following specific forms

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2} + c_f \phi_i^e \phi_j^e \right) dx$$

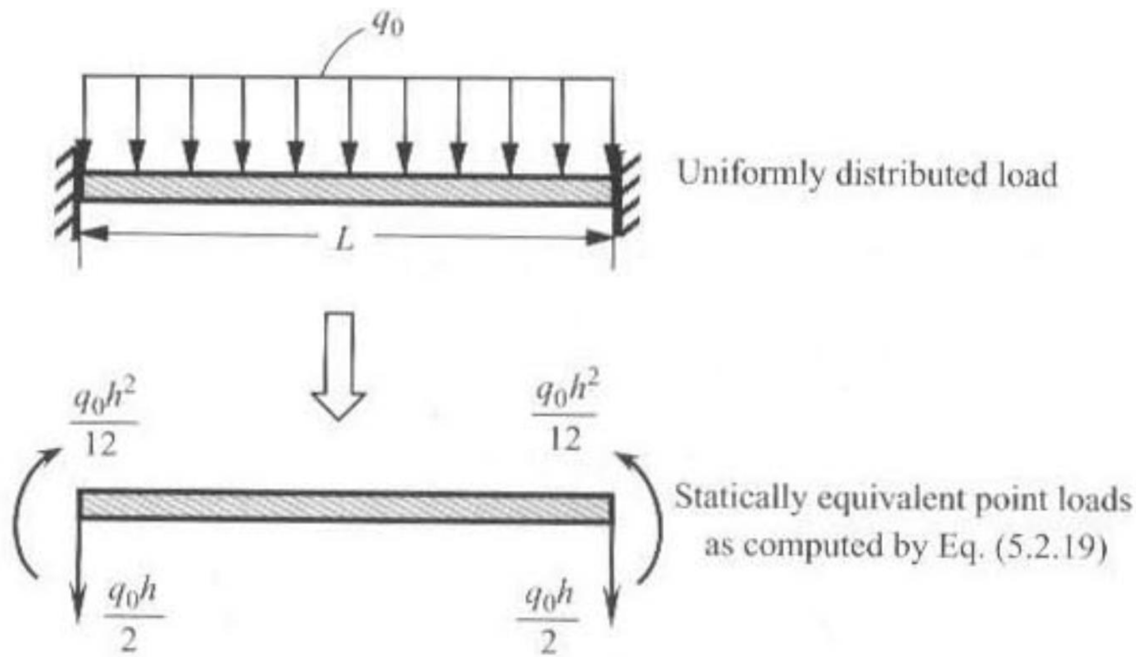
$$[K^e] \;= \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} + \frac{c_f^e h_e}{420} \begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & -22h_e \\ 13h_e & -3h_e^2 & -22h_e & 4h_e^2 \end{bmatrix}$$

Euler-Bernoulli beam element

Force vector Fe

$$F_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e q dx + Q_i^e$$

$$\{F_i^e\} = \frac{q_e h_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

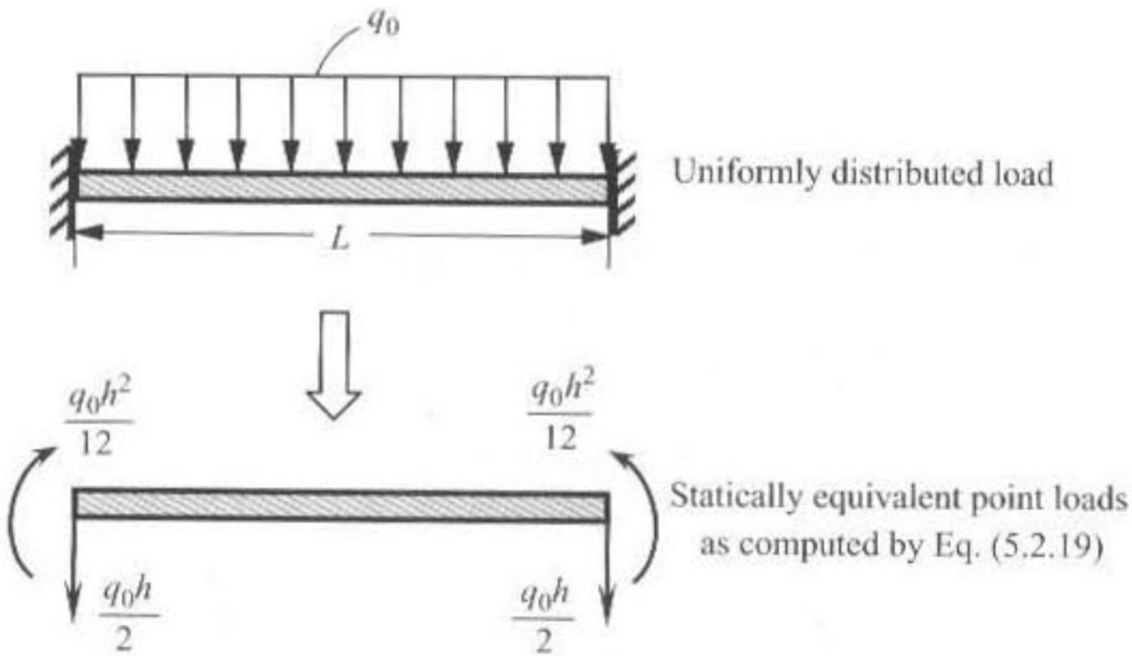


- **It can be verified that the generalized force vector represents the "staticall equivalent" forces and moments at nodes 1 and 2 due to the uniformly distributed load of intensity qe over the element**
- **For given function $q(x)$, provides a straightforward way of computing the components of the generalized force vector qe**

Euler-Bernoulli beam element

Recall Remark 5:

When a transverse point force F_0^e is applied at a point x_0 inside the element, it is distributed to the element nodes by the relation



$$q_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e(x) F_0^e \delta(x - x_0) dx = F_0^e \phi_i^e(x_0), \qquad x_e \le x_0 \le x_{e+1}$$

Assembly of Element Equations

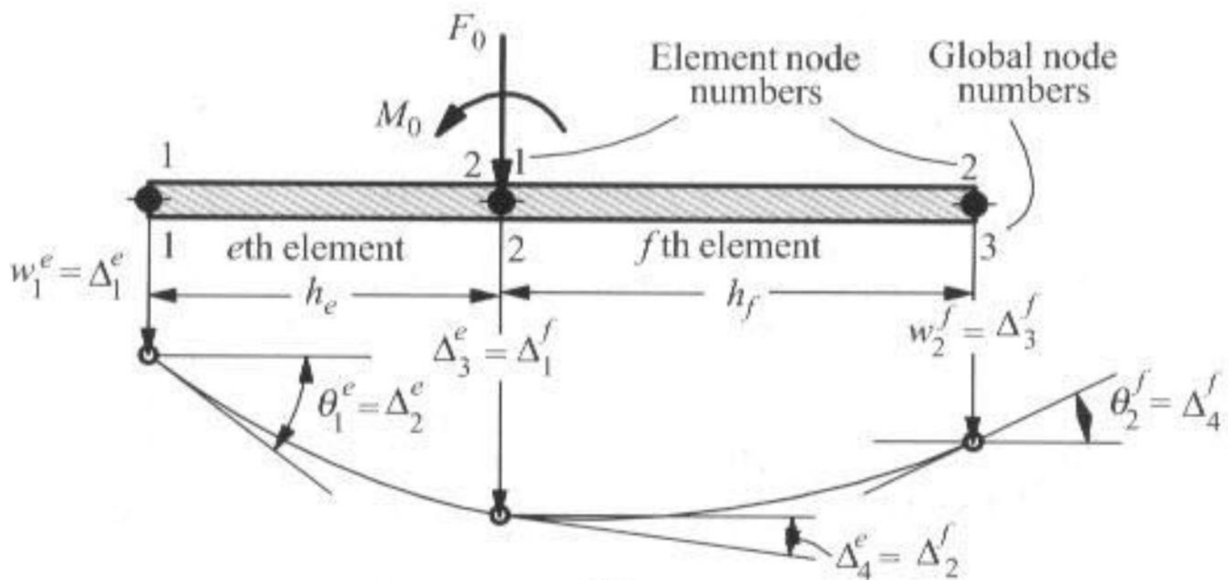
The assembly procedure for beam elements is the same as that used for bar elements except that we must take into account the two degrees of freedom at each node

- **Interelement continuity of the primary variables (deflection and slope)**
- **Interelement equilibrium of the secondary variable (shear force and bending moment) at the nodes common to elements.**

Euler-Bernoulli beam element

To demonstrate the assembly procedure, we select a 2-element model

- **There are 3 global nodes and a total of 6 global generalized displacements and 6 generalized forces in the problem**

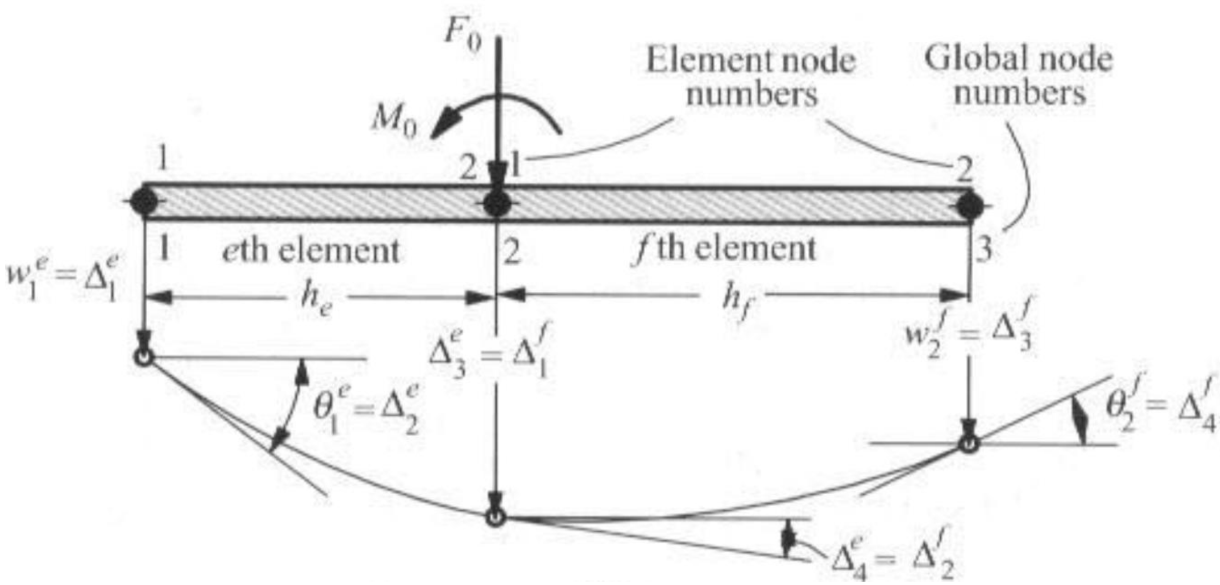


Euler-Bernoulli beam element

The **continuity of the primary variables** **implies the following relation between the element degrees of freedom Δ_i^e and the global degrees of freedom U_i**

$$\Delta_1^1 = U_1,\quad \Delta_3^1 = U_2,\quad \Delta_3^1 = \Delta_1^2 = U_3$$

$$\Delta_4^1 = \Delta_2^2 = U_4,\qquad \Delta_3^2 = U_5 \quad \Delta_4^2 = U_6$$



Euler-Bernoulli beam element

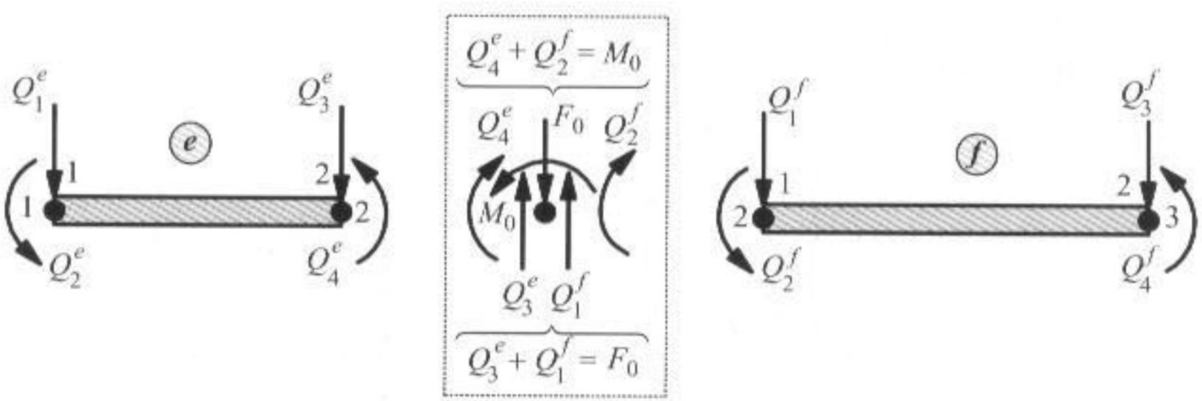
In general, the equilibrium of the generalized forces at a node between two connecting elements Ω_e and Ω_f requires that

$$Q_3^e + Q_1^f = \text{applied external point force}$$

$$Q_4^e + Q_2^f = \text{applied external bending moment}$$

- **If no external applied forces are given, the sum should be equated to zero**
- **In equating the sums to the applied generalized forces (force or moment) the sign convention for the element force degrees of freedom should be followed:**

Forces are taken as positive when they act in the direction of positive z-axis, and moments are taken as positive when they follow the right-hand screw rule (i.e, when the thumb is along the positive y-axis, the four fingers show the direction of the moment)



Forces acting downward are positive and counterclockwise moments are positive

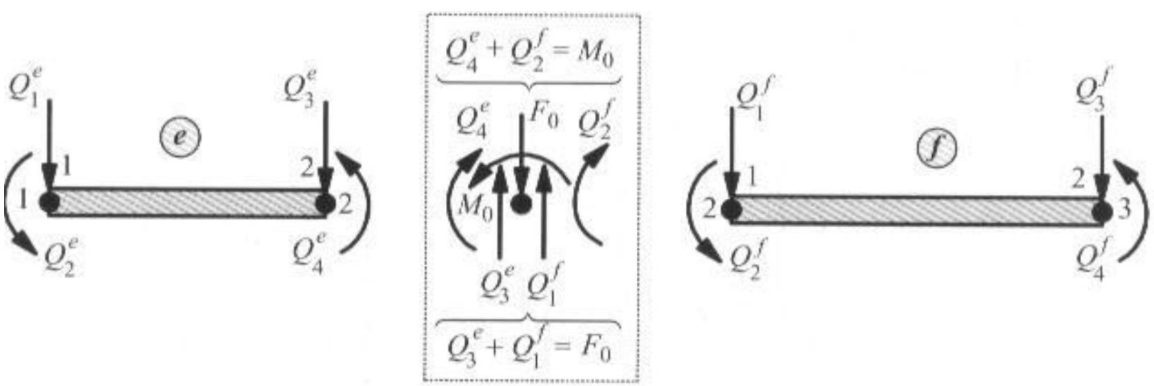
Euler-Bernoulli beam element

In general, the equilibrium of the generalized forces at a node between two connecting elements Ω_e and Ω_f requires that

$$Q_3^e + Q_1^f = \text{applied external point force}$$

$$Q_4^e + Q_2^f = \text{applied external bending moment}$$

- **To impose the equiliibrium of forces, it is necessary to add the third and fourth equations (corresponding to the second node) of element Ωe to the first and second equations (corresponding to the first node) of element Ωf**
- **Global stiffness parameters K_{33}, K_{34}, K_{43} and K_{44} associated with global node 2 are the superposition of the element stiffnesses**



$$K_{33} = K_{33}^1 + K_{11}^2,\ K_{34} = K_{34}^1 + K_{12}^2$$
$$K_{43} = K_{43}^1 + K_{21}^2,\ K_{44} = K_{44}^1 + K_{22}^2$$

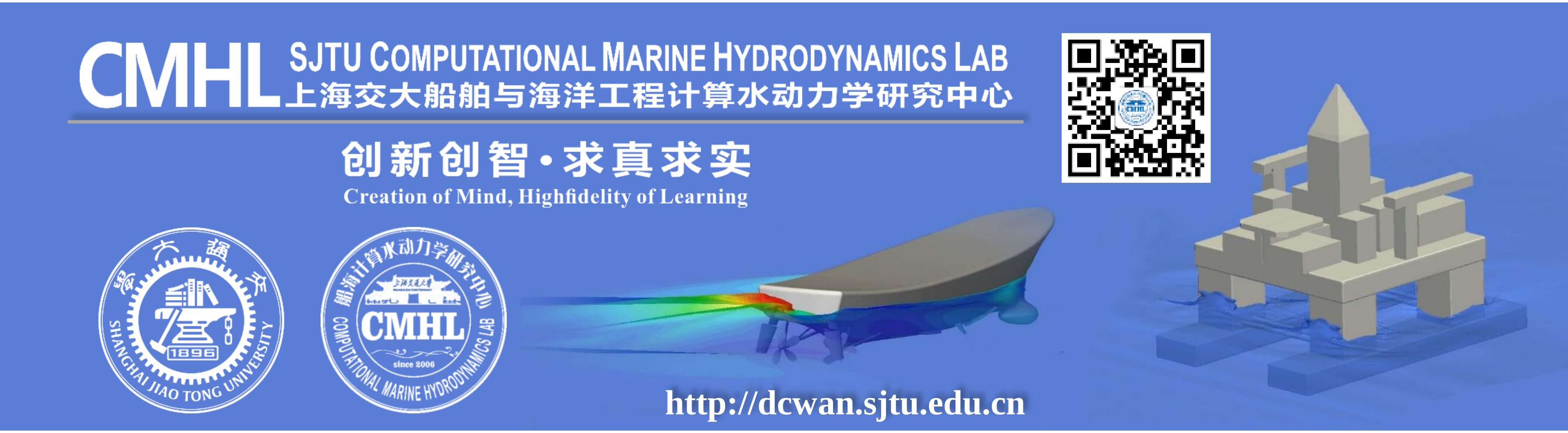
Euler-Bernoulli beam element

In general, the assembled stiffness matrix and force vector for beam elements connected in series have the forms given in

$$
[\mathrm{K}] = \begin{array}{c}
\begin{matrix} \overbrace{\hspace{6em}}^{\text{Global node 1}} & \overbrace{\hspace{14em}}^{\text{Global node 2}} & \overbrace{\hspace{6em}}^{\text{Global node 3}} \end{matrix} \\
\begin{bmatrix}
K_{11}^{1} & K_{12}^{1} & K_{13}^{1} & K_{14}^{1} & & \\
K_{21}^{1} & K_{22}^{1} & K_{23}^{1} & K_{24}^{1} & & \\
K_{31}^{1} & K_{23}^{1} & K_{33}^{1} + K_{11}^{2} & K_{33}^{1} + K_{12}^{2} & K_{13}^{2} & K_{14}^{2} \\
K_{41}^{1} & K_{24}^{1} & K_{43}^{1} + K_{21}^{2} & K_{44}^{1} + K_{22}^{2} & K_{23}^{2} & K_{24}^{2} \\
 & & K_{31}^{2} & K_{32}^{2} & K_{33}^{2} & K_{34}^{2} \\
 & & K_{41}^{2} & K_{42}^{2} & K_{43}^{2} & K_{44}^{2}
\end{bmatrix}
\end{array}
\begin{matrix}
\left.\vphantom{\begin{matrix}K\\K\end{matrix}}\right\} 1 \\
\left.\vphantom{\begin{matrix}K\\K\end{matrix}}\right\} 2 \\
\left.\vphantom{\begin{matrix}K\\K\end{matrix}}\right\} 3
\end{matrix}
$$

$$
\{F\} = \begin{Bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 + q_1^2 \\ q_4^1 + q_2^2 \\ q_3^2 \\ q_4^2 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^2 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}
$$

谢 谢！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