上海交通大学
SHANGHAI JIAO TONG UNIVERSITY

SJTU COMPUTATIONAL MARINE HYDRODYNAMICS LAB
上海交大船舶与海洋工程计算水动力学研究中心

Class-2

NA26018

Finite Element Analysis of Solids and Fluids

万德成

dcwan@sjtu.edu.cn , http://dcwan.sjtu.edu.cn/

上海交通大学

船舶海洋与建筑工程学院
海洋工程国家重点实验室

2020年

Contents

- ✓ **Basic steps** of finite element analysis for **1-D problems**
  - Division of whole domain (into subdomains)
  - Derivation of approximation functions (over each element)
  - Assembly of elements (based on continuity of the solution and balance of internal fluxes)
- ✓ **A few remarks** on deriving model equation

Background

The traditional variational methods (e.g, the Ritz, Galerkin, and least-squares) cease to be effective because of the dificulty in constructing the approximation functions

- The approximation functions are arbitrary (satisfying continuity, linear independence, completeness, and essential boundary conditions)

- The given domain is geometrically complex

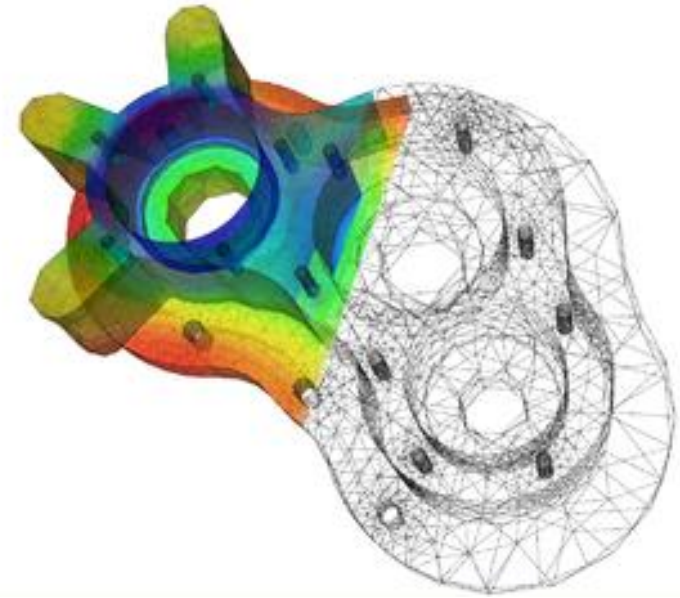
Finite element method overcomes the shortcomings of the traditional variational methods by providing a systematic way of constructing the approximation functions

Background

An effective computational method should have the following features:

- It should have a sound mathematical as well as physical basis
- It should not have limitations with regard to the geometry, the physical composition of the domain, or the nature of the "loading"
- The formulative procedure should be independent of the shape of the domain and the specific form of the boundary conditions
- It should be flexible enough to allow different degrees of approximation without reformulating the entire problem
- It should involve a systematic procedure that can be automated for use on digital computers

The Finite Element Method:



- ✓ Given domain is represented as a collection of simple domains (finite elements)
- ✓ Approximation functions are constructed in a variational or weighted-residual manner over element

Background

3 Features of FEM:

- Division of whole domain enables a systematic derivation of the approximation functions as well as representation of complex domains
- Approximation functions are constructed over each element. The approximation functions need not be polynomials (as in meshless form of FEM)
- The assemblage of elements results in a numerical analog of the mathematical model of the problem being analyzed (assembly ofelements is based on continuity of the solution and balance of internal fluxes)

Discretization of Domain

3 Basic Steps of FEM:

- Generate elements used to represent the domain of a problem such that the approximation functions can be uniquely derived
- Approximation functions depend not only on the geometry but also on the number and location of points(nodes), in the element and the quantities to be interpolated
- Once the approximation functions have been derived, the procedure to obtain algebraic relations among the unknown coefficients is exactly the same as that used in the Ritz and weighted-residual methods

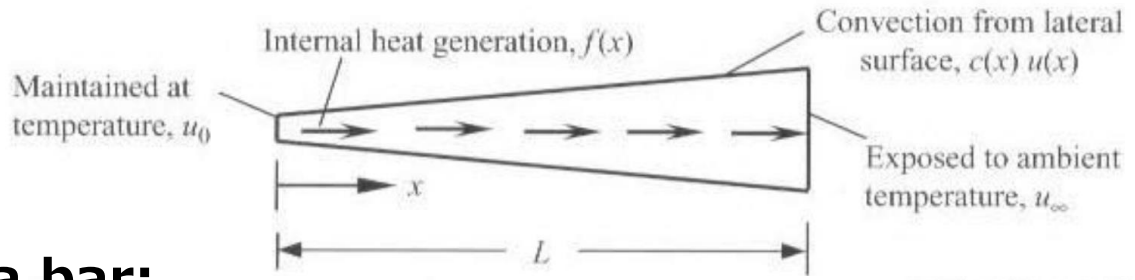
Model Boundary Value Problem

Consider the problem of finding the function $u(x)$ that satisfies the differential equation, with B. C.

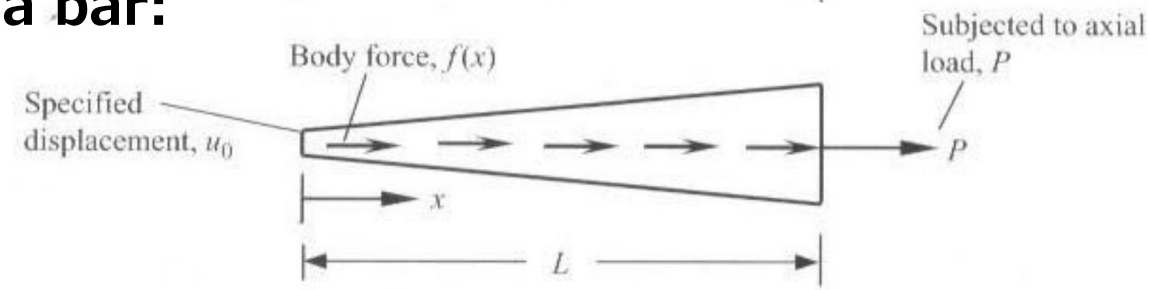
$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for} \quad 0<x<L$$

$$u(0)=u_0, \qquad \left(a\frac{du}{dx}\right)\Big|_{x=L}=Q_0$$

Heat transfer in a fin:



Axial deformation of a bar:



Basic Steps of FEM

A step-by-step procedure for the FE formulation and solution of the second-order equation in 1-D

Step 1: Discretization（or representation） of the given domain into a collection of preselected finite elements

（a）Construct the finite element mesh of preselected elements

（b）Number the nodes and the elements

（c）Generate the geometric properties （e. g, coordinates and cross-sectional areas）needed for the problem

Basic Steps of FEM

A step-by-step procedure for the FE formulation and solution of the second-order equation in 1-D

Step 2: Derivation of element equations for all typical elements in the mesh

(a) Construct the variational formulation of the given differential equation over the typical element

(b) Assume that a typical dependent variable u is of the form

$$u = \sum_{i=1}^{n} u_i \psi_i$$

and substitute it into Step 2a to obtain element equations in the form

$$[K^e]\{u^e\} = \{F^e\}$$

Basic Steps of FEM

A step-by-step procedure for the FE formulation and solution of the second-order equation in 1-D

Step 3: Assembly of element equations to obtain the equations of the whole problem

(a) Identify the interelement continuity conditions among the primary variables (relationship between the local degrees of freedom and the global degrees of freedom-connectivity of elements by relating element nodes to global nodes

(b) Identify the "equilibrium" conditions among the secondary variables (relationship between the local source or force components and the globally specified source components)

(c) Assemble element equations using Steps 3a and 3b

Basic Steps of FEM

A step-by-step procedure for the FE formulation and solution of the second-order equation in 1-D

Step 4: Imposition of the boundary conditions of the problem

（a） Identify the specified global primary degrees of freedom

（b） Identify the specified global secondary degrees of freedom（if not already done in Step 3b）

Step 5. Solution of the assembled equations

Step 6. Postprocessing of the results.

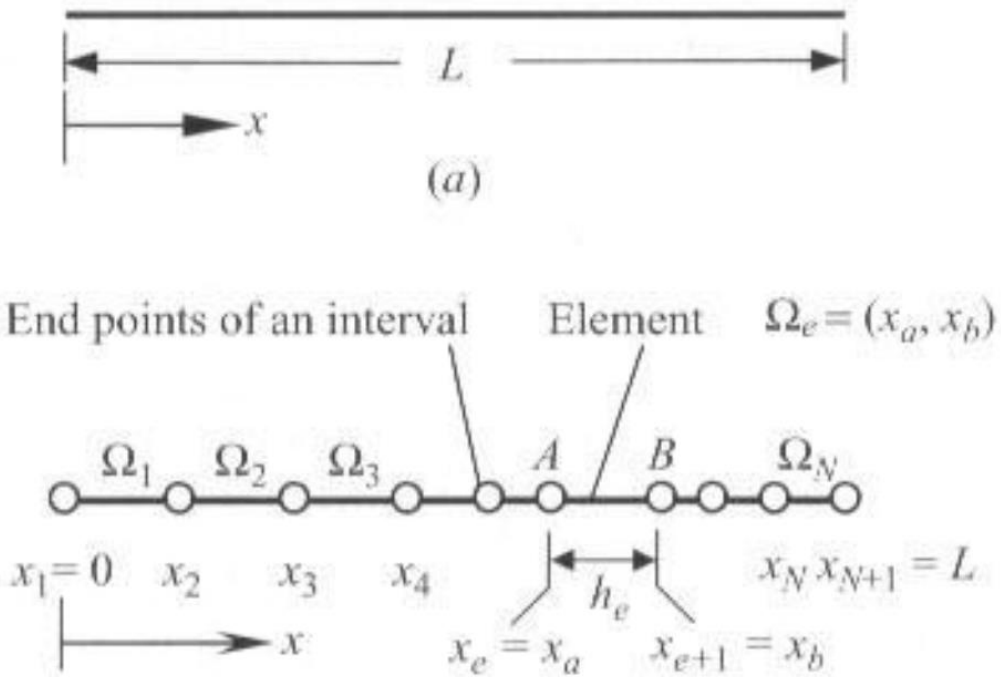
（a） Compute the gradient of the solution or other desired quantities from the primary degrees of freedom computed in Step 5

（b） Represent the results in tabular and/or graphical form

Basic Steps of FEM

Discretization of the domain

- In the FE method, the domain of the problem is divided into a set of subintervals (finite elements)
- The collection of finite elements in a domain is called the FE mesh of the domain



Basic Steps of FEM

The reason for dividing a domain into subdomains:

- **Domains of most systems, by design, are a composite of geometrically and/or materially different parts, and the solution on these subdomains is represented by different functions that are continuous at the interfaces of these subdomains. Therefore, it is appropriate to seek approximation of the solution over each subdomain**

- **Approximation of the solution over each element of the mesh is simpler than its approximation over the entire domain**

Basic Steps of FEM

How to Make a Suitable Mesh?

The number of elements in which the total domain is divided depends mainly on the geometry of the domain and on the desired accuracy of the solution

- **Choose a starting mesh based on the qualitative behavior of the solution**
- **Investigate the convergence of the finite element solution by gradually refining the mesh and comparing the solution with those obtained by higher-order elements (The order of an element refers to the degree of polynomial used to represent the solution over the element)**

Basic Steps of FEM

Develop the algebraic equations among the unknown parameters, as we did in the Ritz and Galerkin methods

Derivation of Element Equations

(not for the domain)

- Here we work with a finite element as opposed to the total domain
- This step results in a matrix equation of the form $K_e u_e = f_e$ (finite element model of the original equation), matrix equation can not be put together simply
- Since the element is physically connected to its neighbors, the resulting algebraic equations will contain more unknowns than the number of algebraic equations. Then it becomes necessary to **assemble the elements together** to **eliminate the extra unknowns**

Basic Steps of FEM

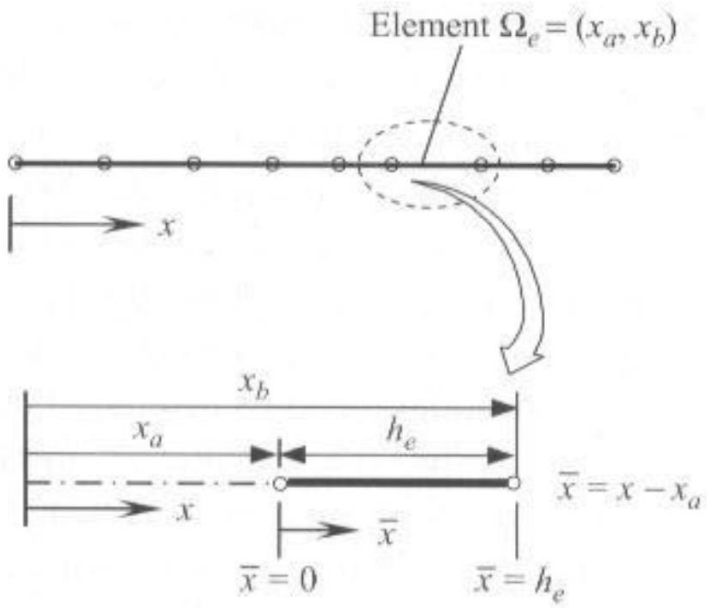
The derivation of FE equations involves the following 3 steps :

1. Construct the weighted-residual or weak form of the differential equation

2. Assume the form of the approximate solution over a typical finite element

3. Derive the finite element equations by substituting the approximate solution into the weighted-residual or weak form

Basic Steps of FEM



- A typical element with endpoints $x = x_a$, x_b, is isolated from the mesh. We seek an approximate solution to the governing differential equation over the element $[x_a, x_b]$
- We develop the algebraic equations using the Ritz method

Basic Steps of FEM

Step 1. Weak Form and Minimum of a Quadratic Functional

The polynomial approximation of the solution within a typical finite element $[\Omega_e]$ is assumed to be of the form:

$$u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e(x)$$

uej: solution at the nodes of element Omegae
Ψ: approximation functions over the element

Note that:

- Writing the approximation in terms of the nodal values of the solution is necessitated by the fact that the continuity of $u(x)$ between elements can be imposed automatically

Basic Steps of FEM

The coefficients u are determined such that original PDE is satisfied in a weighted-integral sense

$$0 = \int_{x_a}^{x_b} w \left[-\frac{d}{dx}\left(a \frac{du}{dx} \right) + cu - f \right] dx$$

w: weight function

Ω_e=[x_a, x_b]: the domain of a typical element

- **For $u = u_h^e$ and each independent choice of w, we obtain an independent algebraic equation relating all u of the element**
- **A total of n independent equations are required to solve for the parameters u_j^e $(j = 1, 2, \ldots n)$**
- **When $w = \Psi$, weighted-integral form is used to obtain the ith equation of the required n equations, this is termed the Galerkin finite element model.**
- **Weighted-integral form contains the second derivative of u, the approximation functions Ψ_j^e must be twice differentiable**

Basic Steps of FEM

To weaken the continuity required of Ψ, we trade the differentiation from u to w such that both u and w are differentiated equally, The resulting integral form is termed the weak form of PDE This form is not only equivalent to PDE but it also contains the NBC of the problem.

Recall the 3-step procedure of constructing the weak form

1: To multiply the PDE with a weight function w and integrate over a typical element

2: To trade differentiation from u to w using integration by parts

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right) dx - \left[wa \frac{du}{dx} \right]_{x_a}^{x_b}$$

Basic Steps of FEM

Identify the primary and secondary variables of the weak form

- **Requires to classify the boundary conditions of each differential equation into EBC/NBC, by examining the boundary term appearing in the weak form**

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right) dx - \left[wa \frac{du}{dx} \right]_{x_a}^{x_b}$$

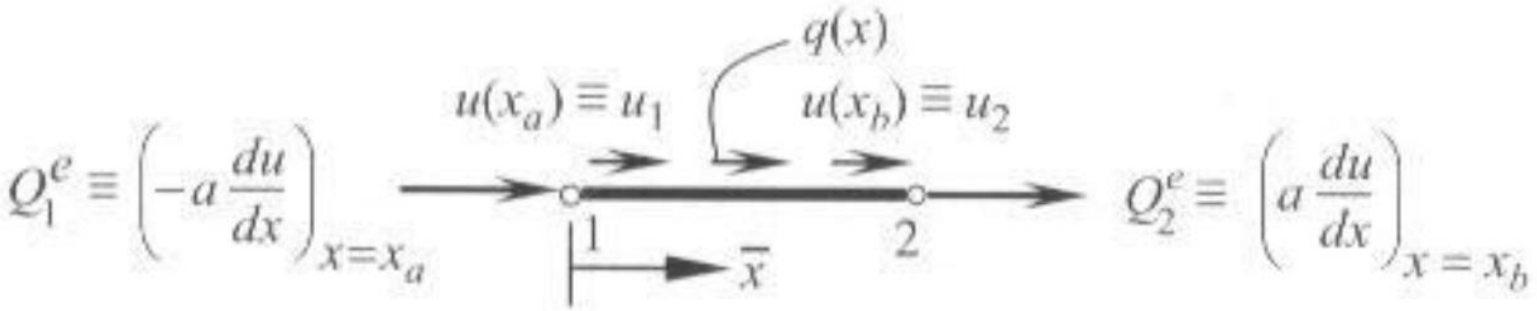
Recall:

- **The coefficient of the weight function w in the boundary expression is called a secondary variable, its specification constitutes the natural or Neumann bound condition**
- **The dependent unknown u in the same form as the weight function w appearing in the boundary expression is termed a primary variable, its specification constitutes the essential /Dirichlet boundary condition**

Basic Steps of FEM

The primary and secondary variables are

$$u \quad \text{and} \quad a\frac{du}{dx} \equiv Q$$



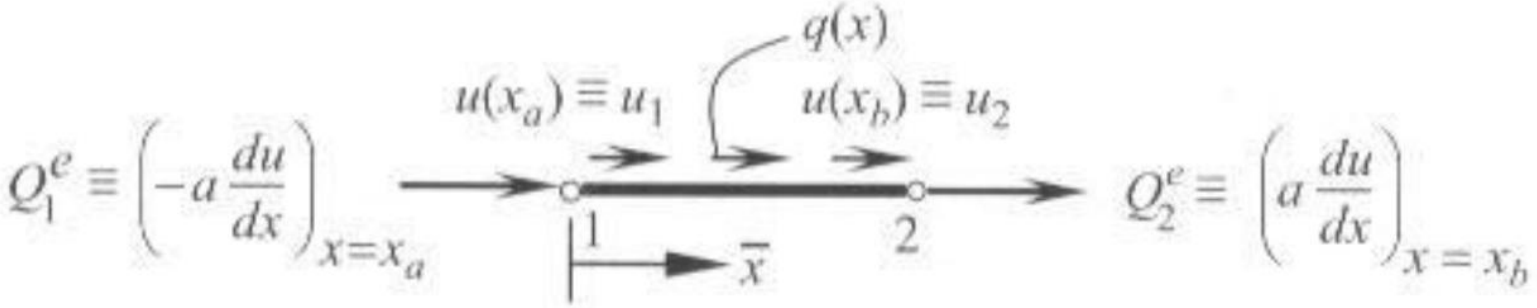
In writing the final form of the weighted-integral statement, we must address the fate of the boundary terms. For a line element end points （boundary points ） 1 and 2, we have the following 4 conditions:

$$u_h^e(x_a) = u_1^e, \left(-a\frac{du}{dx}\right)_{x=x_a} = Q_1^e, u_h^e(x_b) = u_2^e, \left(a\frac{du}{dx}\right)_{x=x_a} = Q_2^e$$

Basic Steps of FEM

Physical Meanings:

Free-body diagram of a typical element



- For **axial deformation of bars**, u denotes displacement, du/dx is the strain ε, $E\varepsilon$ is the stress σ, and $A\sigma$ denotes the force, where E is Young's modulus and A is the area of cross section of the bar
- $Q = EA(du/dx) = a(du/dx)$ has the meaning of force. Q_1^e and Q_2^e are the reaction forces at the left and right ends of the member (Q_1^e is a compressive Q_2^e is a tensile force)

Basic Steps of FEM

$$u_h^e(x_a) = u_1^e, \left(-a\frac{du}{dx}\right)_{x=x_a} = Q_1^e, u_h^e(x_b) = u_2^e, \left(a\frac{du}{dx}\right)_{x=x_a} = Q_2^e$$

If we select ueh such that it automatically satisfies the end conditions $u_h^e(x_a) = u_1^e$ and $u_h^e(x_b) = u_2^e$, then only the 2 conditions remained:

$$Q_1 = \left(-a\frac{du}{dx}\right)\Big|_{x_a}, \qquad Q_2 = \left(a\frac{du}{dx}\right)\Big|_{x_b}$$

$$0 = \int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right)dx - \left[wa\frac{du}{dx}\right]_{x_a}^{x_b}$$

$$0 = \int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right)dx - w(x_a)Q_1 - w(x_b)\,Q_2$$

The FE model based on above weak form is the weak form Galerkin FE model

- **The weak form admits approximation functions that are lower order than the weighted-residual statement**

Basic Steps of FEM

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right) dx - w(x_a) Q_1 - w(x_b) Q_2$$

The weak form contains two types of expressions: bilinear form (those containing both w and u) and linear form (those containing only w)

$$B^e(w,u) = \int_{x_a}^{x_b} \left(a \frac{dw}{dx} \frac{du}{dx} + cwu \right) dx$$

$$l^e(w) = \int_{x_a}^{x_b} w\, f\, dx + w(x_a)\, Q_1 + w(x_b) Q_2$$

The weak form can be expressed as

$$B^e(w,u) = l^e(w)$$

variational problem associated with original DE

Basic Steps of FEM

$$B^e(w,u) = l^e(w)$$

- **Bilinear form results directly in the element coefficient matrix of the FE equations**
- **Linear form leads to the right-hand-side column vector of the FE equations**
- **Derivation of the variational problem of the type is possible for all problems described by DE. However, the bilinear form $B(w,u)$ may not be linear in u, and it may not be symmetric in its arguments w and u**

Recall:
When $B(w,u)$ is symmetric ($B(w,u) = B(u,w)$) in u, w, and $l(w)$ is linear in w, its variational problem is identical with the quadratic functional $\delta I = 0$

Basic Steps of FEM

$$I^e(u) = \frac{1}{2}B^e(u,u) - l^e(u)$$
$$= \frac{1}{2}\int_{x_a}^{x_a}\left[a\left(\frac{du}{dx}\right)^2 + cu^2\right]dx$$
$$- \int_{x_a}^{x_b} u\, f\, dx - u(x_a)Q_1 - u(x_b)Q_2$$

Thus, the relationship between the weak form and the minimum of quadratic functional is

$$0 = \delta I^e = B^e(w,u) - l^e(w), w = \delta u$$

Basic Steps of FEM

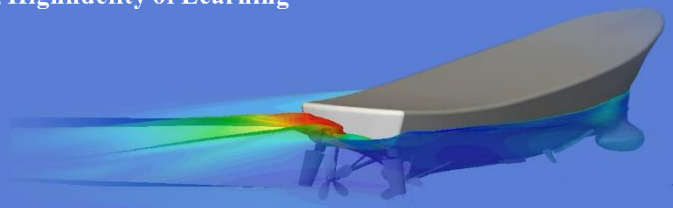
Physical Meanings of $\delta I^e = 0$:

- $\delta I^e = 0$ **in solid and structural mechanics is also known as the princ of minimum total potential energy**
- **When original DE describes the axial deformation of a bar, $0.5B^e(u,u)$ represents the elastic strain energy stored in the bar element, $I^e(u)$ represents the work done by applied forces, and $I^e(u)$ is the total potential energy of the bar element**
- **Thus, the finite element model can be developed using either the statement of the principle of minimum total potential energy of an element or the weak form of the governing equations of an element**
- **We can always construct a weak form of any set of differential equations(linear or not of order 2 and higher). Finite element formulations do not require the existence of the functional $I^e(u)$ they only need weighted-integral statements or weak forms**

谢 谢！

CMHL SJTU COMPUTATIONAL MARINE HYDRODYNAMICS LAB
上海交大船舶与海洋工程计算水动力学研究中心

创新创智·求真求实
Creation of Mind, Highfidelity of Learning



http://dcwan.sjtu.edu.cn