Class-9

NA26018

Finite Element Analysis of Solids and Fluids

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2020年

SUPG/PSPG Method for Incompressible Flow

The finite element (FE) numerical computation of incompressible Navier–Stokes equations (NS) suffers from two main sources of numerical instabilities arising from the associated Galerkin problem:

1. **Equal order finite elements approximation for pressure and velocity, do not satisfy the Babuška–Brezzi condition and leads to instability on the discrete pressure (also called spurious pressure)**

2. **The advection term in the Navier–Stokes equations can produce oscillations in the velocity field (also called spurious velocity)**

Such spurious velocity oscillations become more evident for advection-dominated (i.e., high Reynolds number) flows

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SUPG/PSPG Method for Incompressible Flow

The streamline upwind Petrov–Galerkin pressure-stabilizing Petrov–Galerkin (SUPG/PSPG) formulation for incompressible Navier–Stokes equations can be used for

- **finite element computations of high Reynolds number incompressible flow,**

- **using equal order of finite element space (for velocity and pressure) by introducing additional stabilization terms in the Navier–Stokes Galerkin formulation**

Consider the governing equation of incompressible viscous flows:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f}\right)-\nabla\cdot\boldsymbol{\sigma}=0$$

$$\nabla\cdot\mathbf{u}=0$$

$$\boldsymbol{\sigma}=\boldsymbol{\tau}-\mathrm{p}\mathbf{I}$$

For Newtonian fluid,

$$\boldsymbol{\tau}=2\mu\mathbf{D} \qquad \mathbf{D}=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}})$$

SUPG/PSPG Method for Incompressible Flow

The variational formulations with the SUPG and PSPG stabilization terms will be described

- **Note: These formulations are based on finite element discretization in space only, rather than in both space and time**

$$S_u^h = \left\{ u^h \mid u^h \in \left(\mathrm{H}^{1h}\right)^d, u^h = \hat{u}^h \ on\, \Gamma_u \right\}$$

$$V_u^h = \left\{ w^h \mid u^h \in \left(\mathrm{H}^{1h}\right)^d, w^h = 0 \ \ on\, \Gamma_u \right\}$$

$$S_p^h = V_p^h = \left\{ q^h \mid q^h \in \left(\mathrm{H}^{1h}\right) \right\}$$

Rewrite the momentum equation into tensor format

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho\, f_i$$

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Multiply the velocity test function w_i into tensor format, we obtain

$$\int_\Omega w_i \rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) d\Omega = \int_\Omega w_i \frac{\partial \sigma_{ij}}{\partial x_j} d\Omega + \int_\Omega \rho w_i f_i \, d\Omega$$

The first term of the Rhs for above Eq., can be expanded as

$$\int_\Omega w_i \frac{\partial \sigma_{ij}}{\partial x_j} d\Omega = \int_\Omega \left[\frac{\partial}{\partial x_j}(w_i \sigma_{ij}) - \sigma_{ij} \frac{\partial w_i}{\partial x_j} \right] d\Omega$$

$$= \oint_{\Gamma_u \cup \Gamma_h \cup \Gamma_d} w_i \sigma_{ij} n_j \, d\Gamma - \int_\Omega \sigma_{ij} \frac{\partial w_i}{\partial x_j} d\Omega$$

$$= \oint_{\Gamma_h} w_i \hat{h}_i d\Gamma + \oint_{\Gamma_d^+} w_i (\sigma_{ij} n_j)^+ d\Gamma + \oint_{\Gamma_d^-} w_i (\sigma_{ij} n_j)^- d\Gamma - \int_\Omega \sigma_{ij} \frac{\partial w_i}{\partial x_j} d\Omega$$

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$$= \oint_{\Gamma_h} w_i \hat{h}_i d\Gamma + \oint_{\Gamma_d^+} w_i (\sigma_{ij} n_j)^+ d\Gamma + \oint_{\Gamma_d^-} w_i (\sigma_{ij} n_j)^- d\Gamma - \int_\Omega \sigma_{ij} \frac{\partial w_i}{\partial x_j} d\Omega$$

$$= \oint_{\Gamma_h} w_i \hat{h}_i d\Gamma + \oint_{\Gamma_d} w_i \hat{n}_j (\sigma_{ij}^+ + \sigma_{ij}^-) d\Gamma - \int_\Omega \sigma_{ij} \frac{\partial w_i}{\partial x_j} d\Omega$$

(a)

$$= \oint_{\Gamma_h} w_i \hat{h}_i d\Gamma + \oint_{\Gamma_d} \gamma \kappa w_i \hat{n}_j d\Gamma - \int_\Omega \sigma_{ij} \frac{\partial w_i}{\partial x_j} d\Omega$$

Note: if surface tension is considered(for multiphase flow). Otherwise, this term is zero

Multiply the pressure test function p_i into tensor format, we obtain

$$\int_\Omega q_i \left(\frac{\partial u_j}{\partial x_j} \right) d\Omega = 0 \qquad \textbf{(b)}$$

Combing (a) and (b), we obtain:

$$\int_{\Omega} w_i \rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) d\Omega + \int_{\Omega} \sigma_{ij} \frac{\partial w_i}{\partial x_j} d\Omega - \oint_{\Gamma_h} w_i \hat{h}_i d\Gamma$$

$$+ \int_{\Omega} q_i \left(\frac{\partial u_j}{\partial x_j} \right) d\Omega = \int_{\Omega} \rho w_i f_i \; d\Omega + \oint_{\Gamma_d} \gamma \kappa w_i \hat{n}_j d\Gamma$$

Written into vector form, we obtain:

$$\int_{\Omega} \mathbf{w}^{\mathrm{h}} \rho \left(\frac{\partial \mathbf{u}^{\mathrm{h}}}{\partial \mathrm{t}} + \mathbf{u}^{\mathrm{h}} \cdot \nabla \mathbf{u}^{\mathrm{h}} \right) \mathrm{d}\Omega + \int_{\Omega} \varepsilon(\mathbf{w}^{\mathrm{h}}) : \sigma(\mathbf{u}^{\mathrm{h}}, \mathrm{p}^{\mathrm{h}}) \; \mathrm{d}\Omega$$

$$- \oint_{\Gamma_{\mathrm{h}}} \mathbf{w}^{\mathrm{h}} \hat{\mathrm{h}} \mathrm{d}\Gamma + \int_{\Omega} \mathrm{q}^{\mathrm{h}} \nabla \cdot \mathbf{u}^{\mathrm{h}} \mathrm{d}\Omega = \int_{\Omega} \rho \mathbf{w}^{\mathrm{h}} \cdot \mathbf{f} \; \mathrm{d}\Omega + \oint_{\Gamma_{\mathrm{d}}} \gamma \kappa \mathbf{w}^{\mathrm{h}} \cdot \hat{\mathrm{n}}_{\mathrm{j}} \mathrm{d}\Gamma$$

SUPG/PSPG Method for Incompressible Flow

Use the SUPG/PSPG by Tezduyar and Osawa, above Eq. can be given by

$$\int_{\Omega} \mathbf{w}^{h} \cdot \rho\left(\frac{\partial \mathbf{u}^{h}}{\partial t} + \mathbf{u}^{h} \cdot \nabla \mathbf{u}^{h} - \mathbf{f}\right) \mathrm{d}\Omega + \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{w}^{h}) : \boldsymbol{\sigma}(\mathrm{p}^{h}, \mathbf{u}^{h}) \mathrm{d}\Omega - \int_{\Gamma} \mathbf{w}^{h} \cdot \hat{\mathbf{h}} \mathrm{d}\Gamma + \int_{\Omega} \mathrm{q}^{h} \nabla \cdot \mathbf{u}^{h} \mathrm{d}\Omega$$

$$+ \sum_{\mathrm{e}=1}^{\mathrm{n_{e1}}} \int_{\Omega^{\mathrm{e}}} \frac{1}{\rho}\left[\tau_{SUPG}\, \rho \mathbf{u}^{h} \cdot \nabla \mathbf{w}^{h} + \tau_{PSPG} \nabla \mathrm{q}^{h}\right] \cdot \left[\rho\left(\frac{\partial \mathbf{u}^{h}}{\partial t} + \mathbf{u}^{h} \cdot \nabla \mathbf{u}^{h}\right) - \nabla \cdot \boldsymbol{\sigma}(\mathrm{p}^{h}, \mathbf{u}^{h}) - \rho \mathbf{f}\right] \mathrm{d}\Omega^{\mathrm{e}} = 0 \qquad \textbf{(c)}$$

w^h and q^h are test function for velocity and pressure.

The preceding 4 terms are from weak form of Galerkin method, others are from stabilizing form introduced by the SUPG/PSPG

τ_{SUPG} τ_{PSPG} **are parameters for stablizing**

$$\tau_{SUPG} = \tau_{PSPG} = \left[\left(\frac{2\|\mathrm{u}^{h}\|}{h}\right)^{2} + 9\left(\frac{4\mu}{h^{2}}\right)^{2}\right]^{-\frac{1}{2}}$$

Interpolation for velocity and pressure can be given by

$$\mathbf{u}^{\mathrm{h}} = \overline{\mathbf{N}}\mathbf{v} \qquad \mathrm{p}^{\mathrm{h}} = \mathbf{N}\mathrm{p}$$

$$\mathbf{N} = [N_1, N_2, \ldots, N_{\mathrm{n}}]$$

Eq. (c) leads to the following non-linear ordinary differential equations

$$\begin{aligned}
&(\mathbf{M}+\mathbf{M}_{\boldsymbol{\delta}})\mathbf{a}+\mathbf{C}(\mathbf{v})+\mathbf{C}_{\boldsymbol{\delta}}(\mathbf{v})+(\mathbf{K}+\mathbf{K}_{\boldsymbol{\delta}})\mathbf{v}-(\mathbf{G}+\mathbf{G}_{\boldsymbol{\delta}})\mathbf{p}=(\mathbf{F}+\mathbf{F}_{\boldsymbol{\delta}}) \\
&\mathbf{G}^{T}\mathbf{v}+\mathbf{M}_{\boldsymbol{\varepsilon}}\mathbf{a}+\mathbf{C}_{\boldsymbol{\varepsilon}}(\mathbf{v})+\mathbf{K}_{\boldsymbol{\varepsilon}}\mathbf{v}+\mathbf{G}_{\boldsymbol{\varepsilon}}\mathbf{p}=\mathbf{E}+\mathbf{E}_{\boldsymbol{\varepsilon}}
\end{aligned} \qquad \textbf{(d)}$$

$$\mathbf{M} = \int_{\Omega} \rho \overline{\mathrm{N}}^{\mathrm{T}} \overline{\mathrm{N}} dV$$

$$\mathbf{M}_{\boldsymbol{\delta}} = \int_{\Omega} \tau_{SUPG} \rho \mathbf{u} \nabla \overline{\mathrm{N}}^{\mathrm{T}} \overline{\mathrm{N}} dV$$

Acceleration term

$$\mathbf{M}_{\boldsymbol{\varepsilon}} = \int_{\Omega} \tau_{PSPG} \nabla \overline{\mathrm{N}}^{\mathrm{T}} \overline{\mathrm{N}} dV$$

SUPG/PSPG Method for Incompressible Flow

$$\mathbf{N}(\mathbf{u}) = \int_{\Omega} \rho \overline{\mathrm{N}}^{\mathrm{T}} \mathbf{u} \cdot \nabla \overline{\mathrm{N}} dV$$

$$\mathbf{N}_{\delta}(\mathbf{u}) = \int_{\Omega} \tau_{SUPG} \rho \mathbf{u} \nabla \overline{\mathrm{N}}^{\mathrm{T}} \mathbf{u} \cdot \nabla \overline{\mathrm{N}} dV$$

$$\mathbf{N}_{\boldsymbol{\varepsilon}}(\mathbf{u}) = \int \tau_{PSPG} \nabla \overline{\mathrm{N}}^{\mathrm{T}} \mathbf{u} \cdot \nabla \overline{\mathrm{N}} dV$$

Convection term

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mu (\mathrm{I}_0 - \frac{2}{3} \mathrm{mm}^{\mathrm{T}}) \mathbf{B} dV$$

Viscous term

$$\mathbf{K}_{\delta} = \int_{\Omega} \tau_{SUPG} \nabla \overline{\mathrm{N}}^{\mathrm{T}} \mathbf{u} \mathbf{B}^{\mathrm{T}} \mu (\mathrm{I}_0 - \frac{2}{3} \mathrm{mm}^{\mathrm{T}}) \mathbf{B} dV$$

$$\mathbf{G} = \int_{\Omega} (\nabla \overline{\mathrm{N}})^{\mathrm{T}} \overline{\mathrm{N}} dV$$

$$\mathbf{G}_{\boldsymbol{\delta}} = \int_{\Omega} \tau_{SUPG} \rho \mathbf{u} (\nabla \overline{\mathrm{N}})^{\mathrm{T}} \nabla \overline{\mathrm{N}} dV$$

Pressure term

$$\mathbf{G}_{\boldsymbol{\varepsilon}} = \int_{\Omega} \tau_{PSPG} \frac{1}{\rho} (\nabla \overline{\mathrm{N}})^{\mathrm{T}} \nabla \overline{\mathrm{N}} dV$$

SUPG/PSPG Method for Incompressible Flow

$$\mathbf{F} = \int_{\Omega} \overline{\mathrm{N}}^{\mathrm{T}} \rho \mathbf{g} dV + \int_{\Gamma} \overline{\mathrm{N}}^{\mathrm{T}} \mathbf{t}_s dS \qquad \mathbf{E} = \int_{\Omega} (\nabla \overline{\mathrm{N}})^{\mathrm{T}} \mathbf{g} dV$$

Force and boundary term

$$\mathbf{F}_{\boldsymbol{\delta}} = \int_{\Omega} \tau_{SUPG} \mathbf{u} \nabla \overline{\mathrm{N}}^{\mathrm{T}} \rho \mathbf{g} dV \qquad \mathbf{E}_{\boldsymbol{\varepsilon}} = \int_{\Omega} \tau_{PSPG} (\nabla \overline{\mathrm{N}})^{\mathrm{T}} \mathbf{g} dV$$

where

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \dots \quad \mathbf{B}_n]^T \qquad \mathbf{B}_I = \begin{bmatrix} \frac{\partial \mathrm{N}_{\mathrm{I}}}{\partial \mathrm{x}} & 0 \\ 0 & \frac{\partial \mathrm{N}_{\mathrm{I}}}{\partial \mathrm{y}} \\ \frac{\partial \mathrm{N}_{\mathrm{I}}}{\partial \mathrm{x}} & \frac{\partial \mathrm{N}_{\mathrm{I}}}{\partial \mathrm{y}} \end{bmatrix} \qquad \mathrm{m} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad I_0 = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Solving procedure

Suppose the acceleration, velocity and pressure at $t+\Delta t$ have the following relations

$$\mathbf{a}^{t+\Delta t} = \mathbf{a}^{t} + \Delta\mathbf{a}, \qquad \frac{\mathbf{v}^{t+\Delta t} - \mathbf{v}^{t}}{\Delta t} = \alpha \cdot \mathbf{a}^{t+\Delta t} + (1-\alpha)\cdot \mathbf{a}^{t}, \qquad \mathbf{p}^{t+\Delta t} = \mathbf{p}^{t} + \Delta\mathbf{p}$$

Eq.(d) can be written in increment form

$$\mathbf{M}^{*}\Delta\mathbf{a} - \mathbf{G}^{*}\Delta\mathbf{p} = \mathbf{R}$$

$$(\mathbf{G}^{T})^{*}\Delta\mathbf{a} - \mathbf{G}_{\varepsilon}^{*}\Delta\mathbf{p} = \mathbf{Q}$$

R, Q and other terms is given as

Solving procedure

$$\mathbf{R} = \mathbf{F} + \mathbf{F}_\delta - [(\mathbf{M} + \mathbf{M}_\delta)\mathbf{a} + \mathbf{C}(\mathbf{v}) + \mathbf{C}_\delta(\mathbf{v}) + (\mathbf{K} + \mathbf{K}_\delta)\mathbf{v} - (\mathbf{G} + \mathbf{G}_\delta)\mathrm{p}]$$

$$\mathbf{Q} = \mathbf{E} + \mathbf{E}_\varepsilon - [\mathbf{G}^T\mathbf{v} + \mathbf{M}_\varepsilon\mathbf{a} + \mathbf{C}_\varepsilon(\mathbf{v}) + \mathbf{K}_\varepsilon\mathbf{v} - \mathbf{G}_\varepsilon\mathrm{p}]$$

$$\mathbf{M}^* = \mathbf{M} + \mathbf{M}_\delta + \alpha\Delta t\left(\frac{\partial \mathbf{C}(\mathbf{v})}{\partial \mathbf{v}} + \frac{\partial \mathbf{C}_\delta(\mathbf{v})}{\partial \mathbf{v}} + \mathbf{K} + \mathbf{K}_\delta\right)$$

$$\mathbf{G}^* = \mathbf{G} + \mathbf{G}_\delta$$

$$(\mathbf{G}^{\mathrm{T}})^* = \mathbf{M}_\varepsilon + \alpha\Delta t\left(\frac{\partial \mathbf{C}(\mathbf{v})}{\partial \mathbf{v}} + \mathbf{K}_\varepsilon + \mathbf{G}^{\mathrm{T}}\right)$$

The increment Eqs. can be solved with initial guess

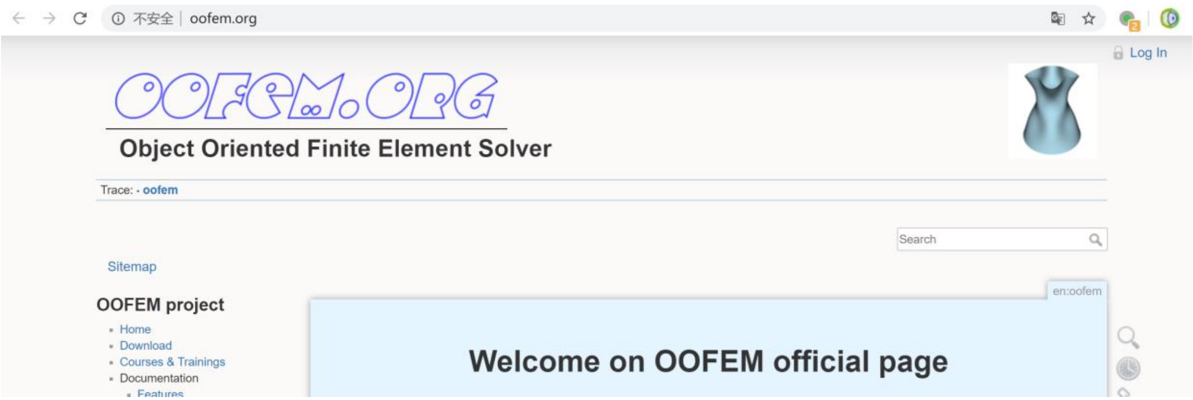
$$\mathbf{v}_0^{t+\Delta t} = \mathbf{v}^t + \Delta t\mathbf{a}^t, \qquad \mathbf{a}_0^{t+\Delta t} = \mathbf{a}^t, \qquad \mathbf{p}_0^{t+\Delta t} = \mathbf{p}^t$$

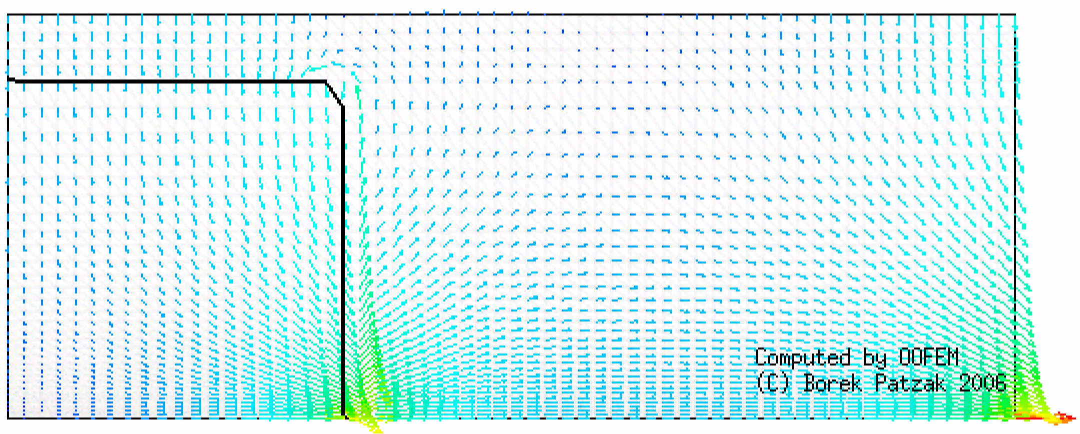
Solutions can be updated iteratively by

$$\mathbf{a}_{i+1}^{t+\Delta t} = \mathbf{a}_i^{t+\Delta t} + \Delta\mathbf{a}, \qquad \mathbf{v}_{i+1}^{t+\Delta t} = \mathbf{v}_i^{t+\Delta t} + \Delta t\alpha\Delta\mathbf{a}, \qquad \mathrm{p}_{i+1}^{t+\Delta t} = \mathrm{p}_i^{t+\Delta t} + \Delta\mathrm{p}$$

For α in the incremental Eqs, if α≥0.5 : unconditional stable

Open source solvers by SUPG/PSPG: OOFEM





dam breaking problem - incompressible, free surface flow

谢 谢！

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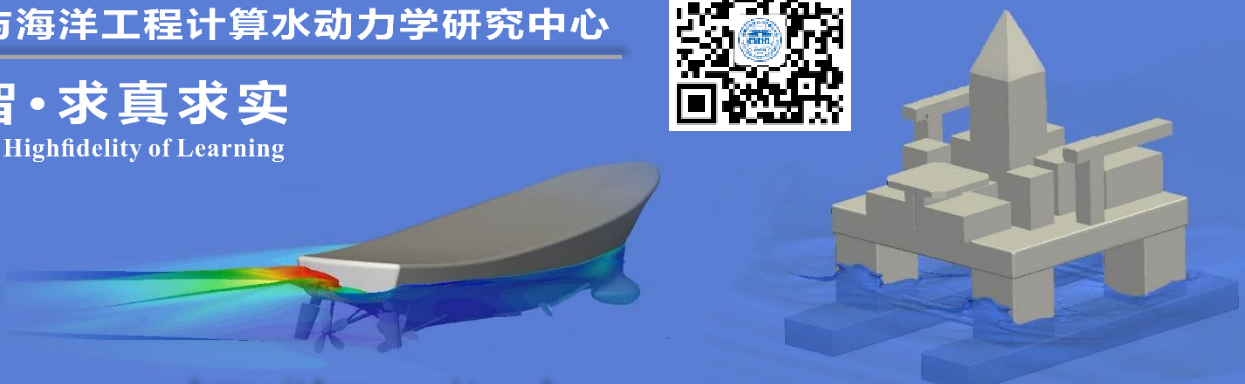

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