上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

CMHL SJTU COMPUTATIONAL MARINE HYDRODYNAMICS LAB
上海交大船舶与海洋工程计算水动力学研究中心

Class-9

NA26018

Finite Element Analysis of Solids and Fluids

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Contents

- **Introduction of the XFEM**
- **XFEM in computational solid mechanics**
- **XFEM in computational fluid mechanics**

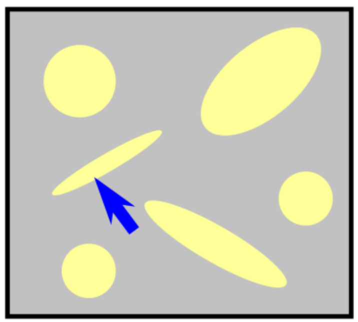
Introduction of the XFEM

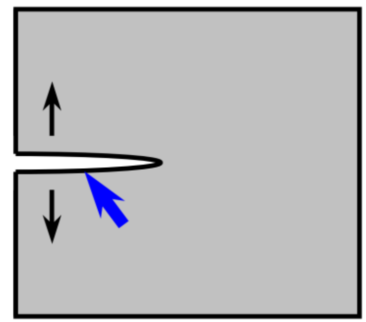
Motivation

- **Field quantities change discontinuously across boundaries and interfaces**

Examples:

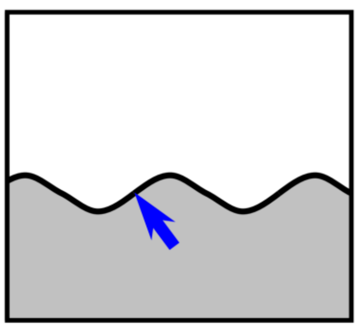
Material interfaces

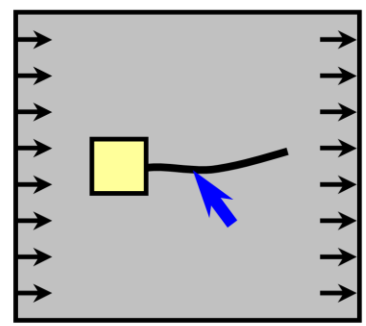




Cracks

Two-phase flow





Fluid-structure interaction

Introduction of the XFEM

Motivation

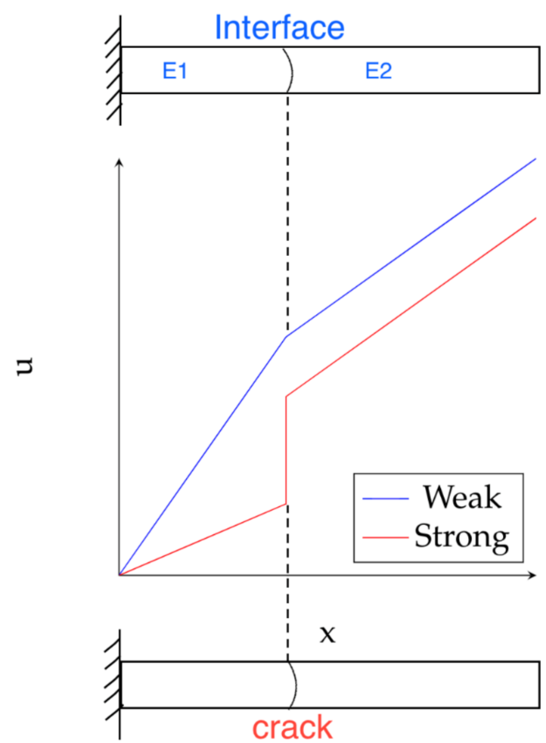
Discontinuities :

- **Large gradient or variation of the physical field occurs in limited region**

e.g., displacement field, temperature field, potential field

- **Discontinuities can be classified onto strong and weak discontinuities**

e.g., strong discontinuities: displacement field around crack; weak discontinuities: displacement field around material inclusion

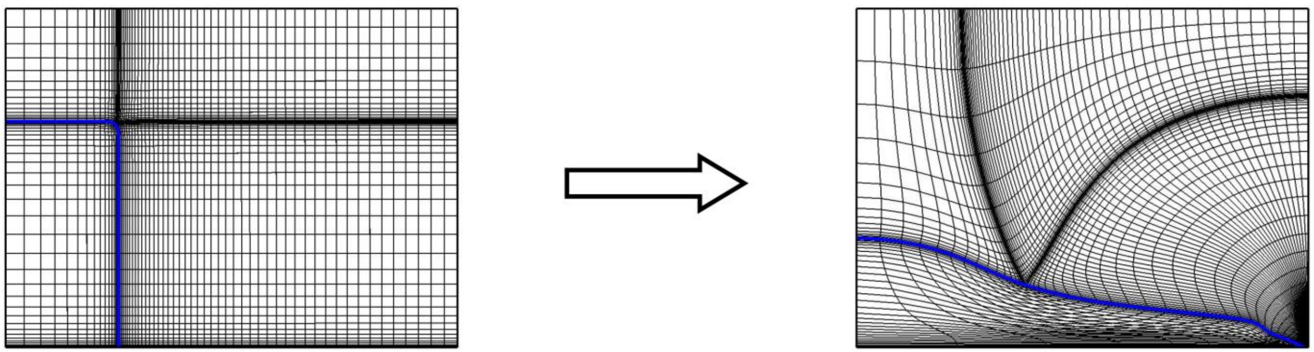


inclusion

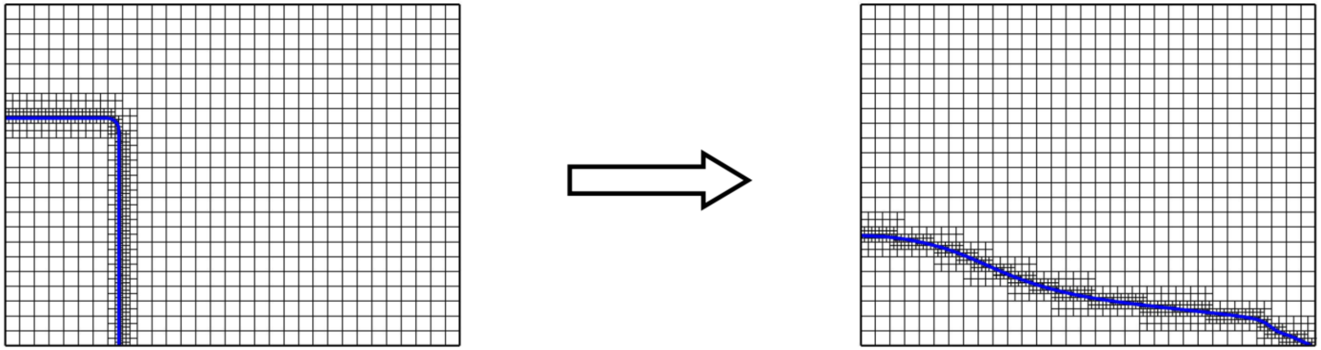
crack

Introduction of the XFEM

- How to handle these situation with classical FEM?
- Interface tracking: topology must not change



- Interface capturing:refinement needed



Introduction of the XFEM

- **In classical FEM, mesh construction and maintenance are crucial for the success**
- **The eXtended Finite Element Method (XFEM) avoids mesh manipulations and adjusts the approximation space**

- **XFEM can be regards as a typical Generalized Finite Element Method（广义有限元方法）**

Introduction of the XFEM

History

- In 1995, the idea of enrichment based on Partition of Unity method (PUM) was mentioned by J. M. Melenk in his Dr. thesis
- In 1996, XFEM theory was proposed and published formally by J. M. Melenk and his supervisor I. Babuška. At the same time, Dr. C. A. Duarte proposed similar approach named as hp cloud FE method
- In 1996-1999, the General Finite Element Method (GFEM) was extended in fracture mechanics problems by T. Belytschko and his colleges. Finally, it was proved that the XFEM and GFEM are mathematically identical
- In 2000-2010, T. Fries proposed and developed the intrinsic XFEM, the corrected XFEM. XFEM shows potential in the application of pressure jump and two-phase flow problems

Introduction of the XFEM

What is different in an XFEM-code ? (compared to standard FEM)

- Enrichment functions are evaluated
- Special integration rules are used for cut elements
- The number of DOFs per node varies (element matrices overall system of equations)
- Possibly the post-processing

Special considerations may be needed for…

- applying boundary/interface conditions…
- choosing appropriate time-integration schemes…
- bad condition numbers (of the global Matrix) in some situations…

Introduction of the XFEM

The XFEM: applications

- **Solid mechanics**:

cracks, material interfaces shear bands, solidification dislocations

- **Fluid mechanics**:

two-phase flows, fluid-structure interaction free surface flows

- **Bio-mechanics**:

bone fracture, virtual surgery biofilms

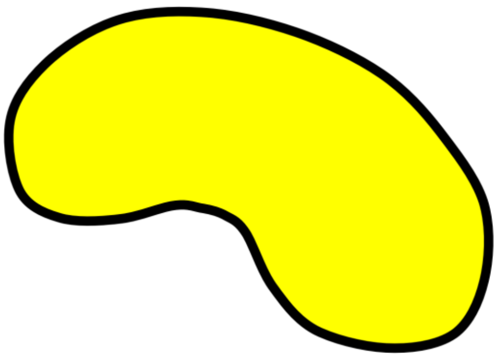
- **Optimization and inverse problems**

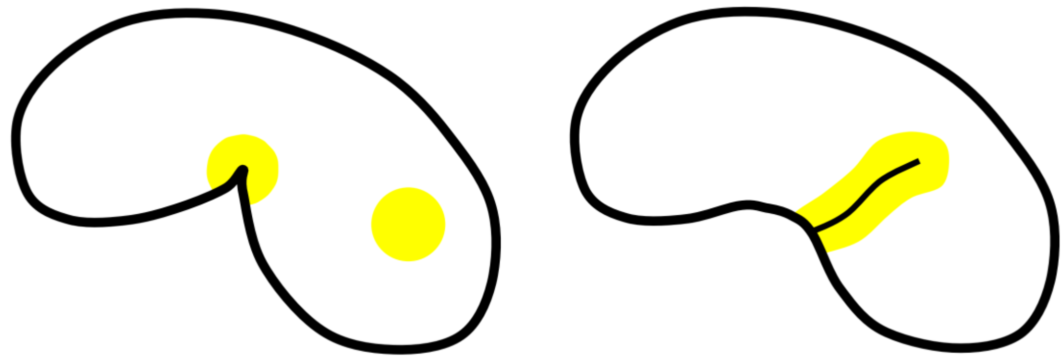
- **Etc.**

Introduction of the XFEM

Basic idea of enriched methods:

- If complex solution characteristics are known a priori, the approximation space can be enriched accordingly
- The enrichment can be done globally or locally



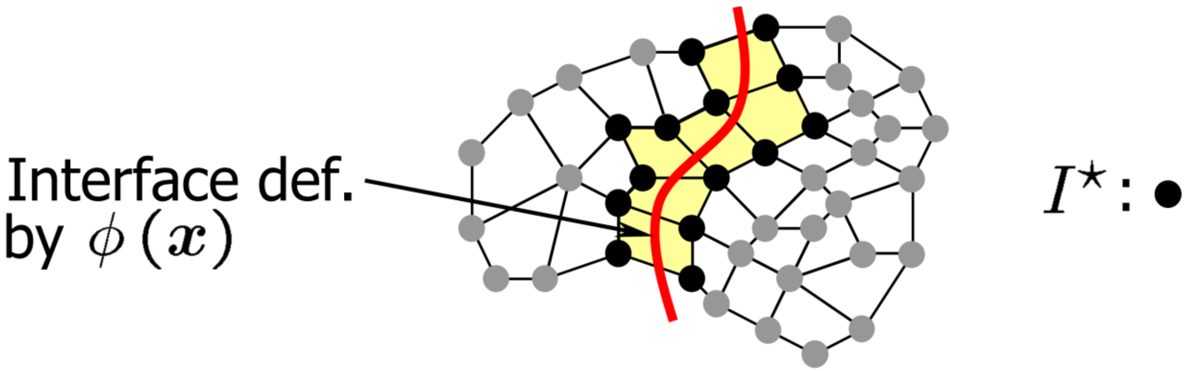


Global: Partion of Unity Method (PUM)

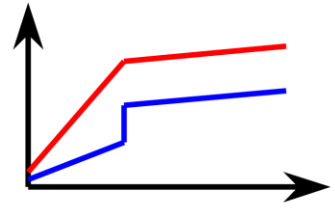
Local: extended finite Element Method (XFEM)

XFEM for discontinuities

- I^* is the set of nodes of all cut elements



- $\psi(\boldsymbol{x})$ depends on the level-set function for the definition of the interface
  - Weak discontinuities: $\psi(\boldsymbol{x}) = \mathrm{abs}(\phi(\boldsymbol{x}))$
  - Strong discontinuities: $\psi(\boldsymbol{x}) = \mathrm{sign}(\phi(\boldsymbol{x}))$



XFEM for discontinuities

The standard XFEM extends the approximation extrinsically by adding more terms

$$u^h(\boldsymbol{x}) = \sum_I N_i(\boldsymbol{x})\, u_i + \sum_{I^\star} M_i(\boldsymbol{x})\, a_i + \ldots,$$

enrichment

additiona unknowns

with

$$M_i(\boldsymbol{x}) = N_i(\boldsymbol{x}) \cdot \psi(\boldsymbol{x})$$

where $\psi(\boldsymbol{x})$ is the enrichment function

(contains discontinuity)

I* and $\psi(\boldsymbol{x})$ define a particular realization of the XFEM

XFEM for discontinuities

The enrichment can be extrinsic or intrinsic

classical FEM

$$u^h(\boldsymbol{x}) = \sum N_i(\boldsymbol{x})\, u_i$$

standard XFEM/PUM

$$u^h(\boldsymbol{x}) = \sum N_i(\boldsymbol{x})\, u_i + \sum M_i(\boldsymbol{x})\, a_i$$

intrinsic XFEM/PUM

$$u^h(\boldsymbol{x}) = \sum N_i^{\star}(\boldsymbol{x})\, u_i$$

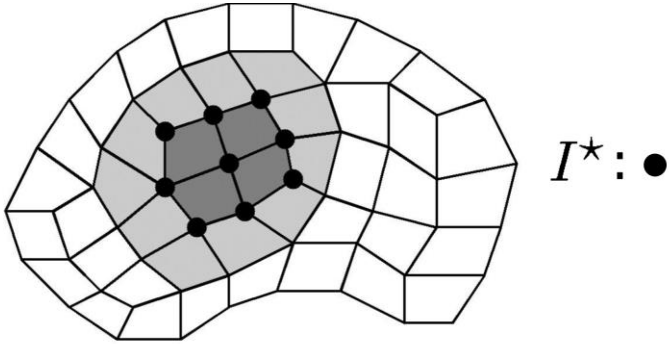
[Fries, Belytschko, IJNME, 2006]
[Fries, Belytschko, Comp. Mech., 2007]

Blending elements

- **Recall standard XFEM-approximation**

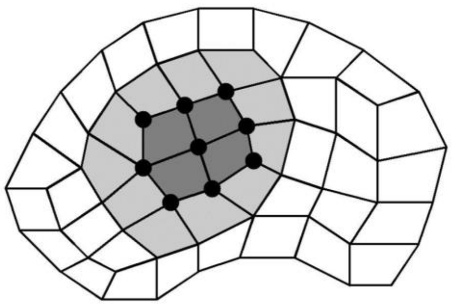
$$u^h(\boldsymbol{x}) = \sum_I N_i(\boldsymbol{x})\, u_i + \sum_{I^\star} N_i(\boldsymbol{x})\, \Psi(\boldsymbol{x})\, a_i$$

- **In the domain one may separate**
  - **Standard finite elements: No enriched nodes**
  - **Reproducing elements: All nodes are enriched**
  - **Blending elements: Some nodes are enriched**

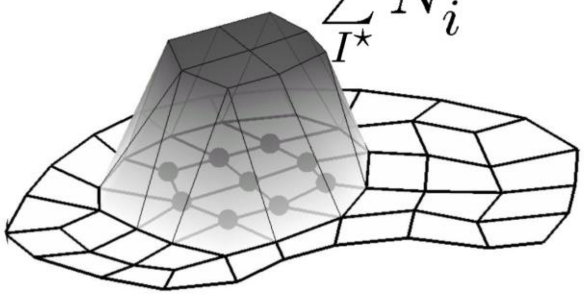


Blending elements

- **Reproducing elements**
  - $\sum_{I^\star} N_i^\star(x) = 1$
  - $\psi(x)$ **can be reproduced exactly**



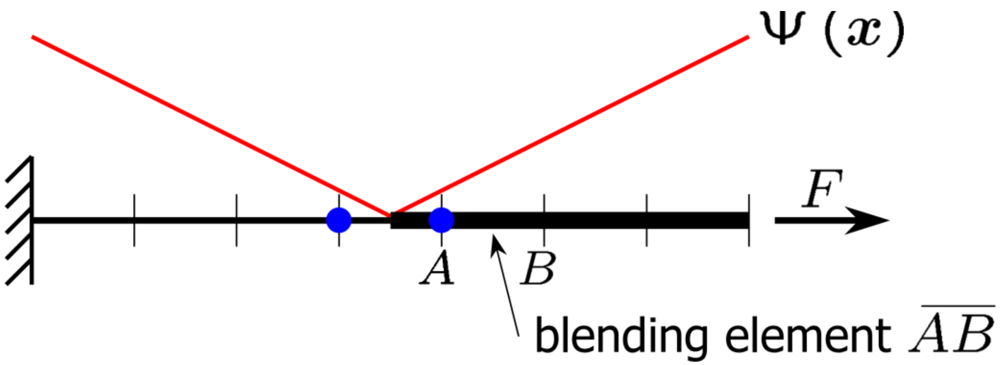
- **Blending elements**
  - $\sum_{I^\star} N_i^\star(x) \neq 1$
  - $\psi(x)$ **can not be reproduced**



- **Unwanted terms are introduced**
- **These terms considerably decrease the accuracy**
- **Exception: Enrichment function is constant in the blending elements, e.g. sign-enrichment**

Blending elements

- **1D example: Bi-material bar**



$$u^h_{AB}(x) = \underbrace{N_A u_A}_{\text{linear}} + \underbrace{N_B u_B}_{\text{linear}} + \underbrace{N^\star_A \cdot \Psi}_{\text{quadratic}} \cdot a_A$$

For $a_A \neq 0$ quad term cannot be compensated by strd FE part
For $a_A = 0$ enrichment is deactivated

Corrected XFEM

- The corrected XFEM has no problems in blending elements

- The approximation is

$$u^h(\boldsymbol{x}) = \sum_I N_i(\boldsymbol{x}) u_i + \sum_{J^\star} N_i^\star(\boldsymbol{x}) \psi^{\mathsf{mod}}(\boldsymbol{x}) a_i$$

- Two differences:

  - a modified enrichment function is used
  - a different set of nodes is enriched

Corrected XFEM

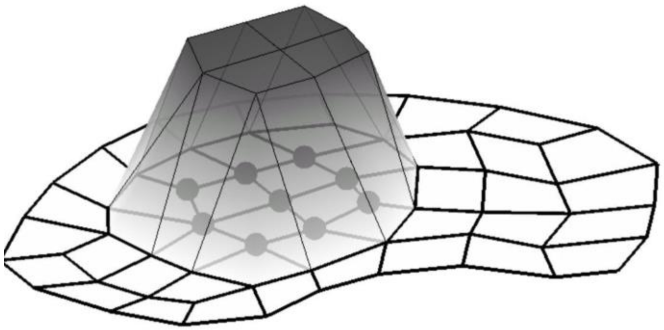
Step 1 : Modification of the enrichment function

- **The enrichment function is modified**

$$\psi^{\text{mod}}(\boldsymbol{x}) = \psi(\boldsymbol{x}) \cdot R(\boldsymbol{x})$$

with the ramp function

$$R(\boldsymbol{x}) = \sum_{I^{\star}} N_i(\boldsymbol{x})$$



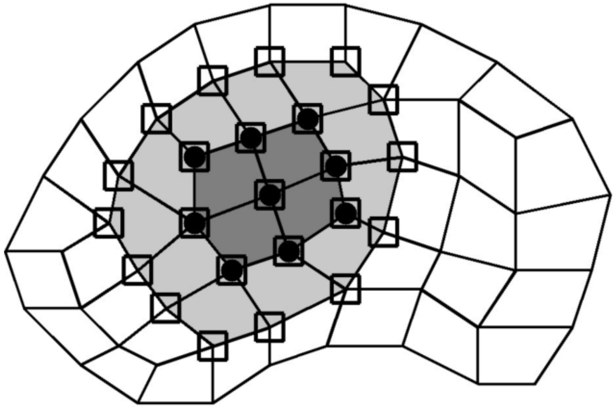
- $\psi^{\text{mod}}(\boldsymbol{x})$ **is unchanged in reproducing elements**
- $\psi^{\text{mod}}(\boldsymbol{x}) = 0$ **in the standard finite elements**
- $\psi^{\text{mod}}(\boldsymbol{x})$ **varies continuously inbetween**

Corrected XFEM

Step 2 : Choice of the enriched nodes

- **In the modified XFEM, all nodes of the reproducing and blending elements are enriched**

$$J^\star = \text{Element nodes of elements that share nodes with } I^\star$$



$\psi^{\text{mod}} = 0$

$\psi^{\text{mod}} = \psi \cdot R$

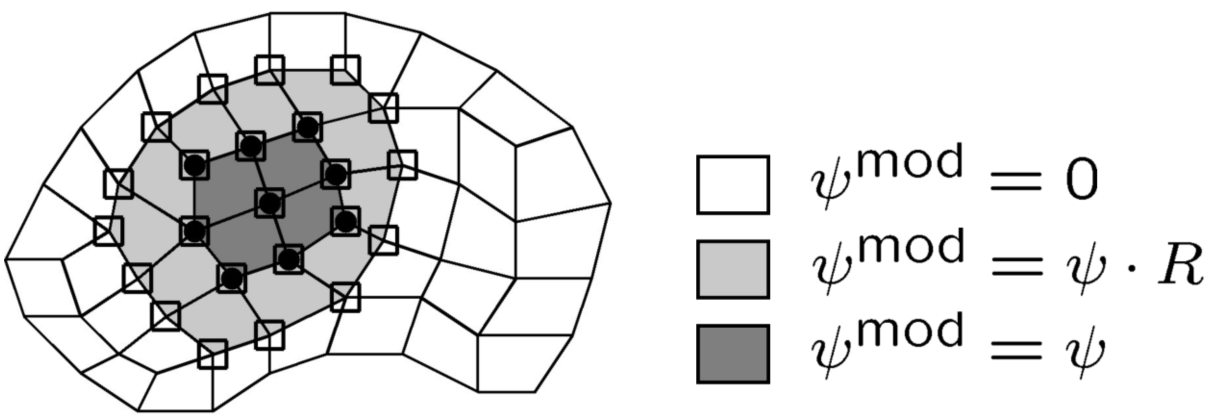
$\psi^{\text{mod}} = \psi$

Corrected XFEM

Properties of the modified approximation

- In elements with only some of their nodes in J^*, ψ^{mod} is zero => no unwanted terms

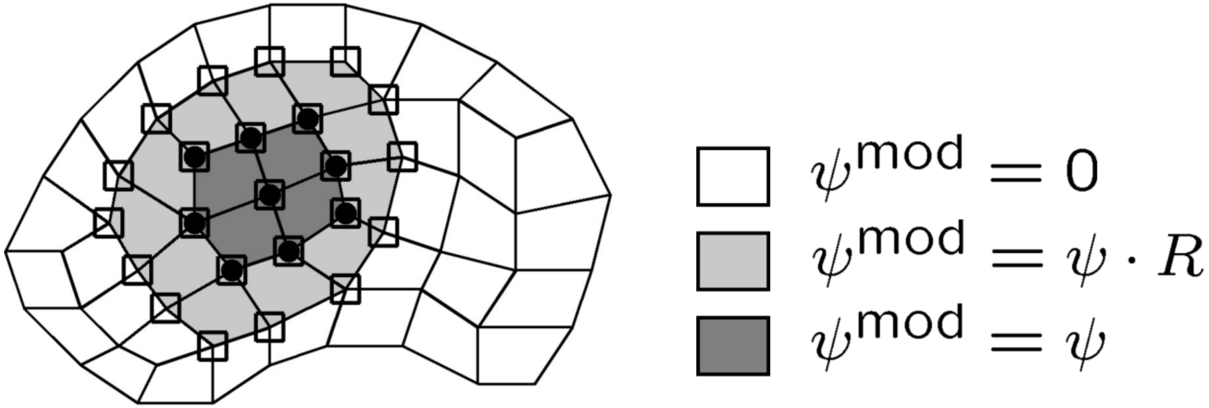
- In strd. XFEM: In elements with only some of their nodes in I^*, ψ^{mod}, inn-zero => unwanted terms



Corrected XFEM

Properties of the modified approximation

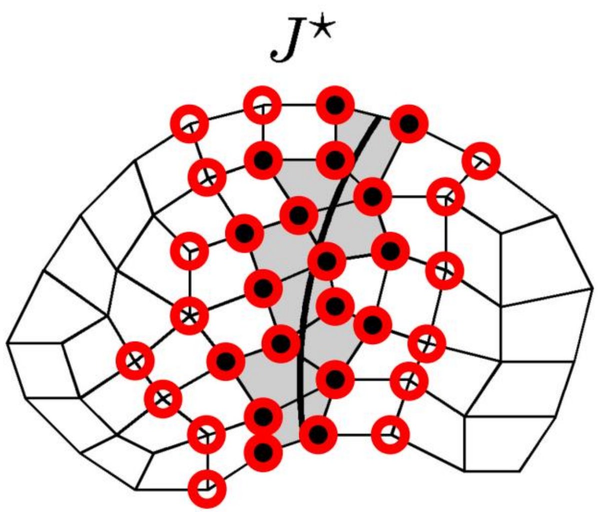
- The functions N_I^*, build a partition of unity in the blending and reproducing elements. =>ψ^{mod} can be reproduced exactly everywhere in the domain

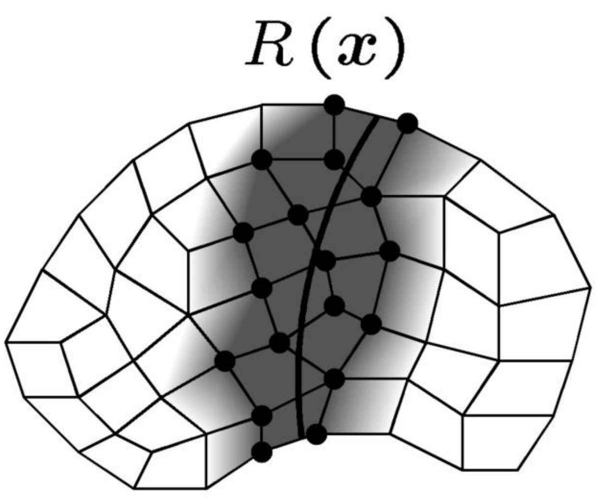


Corrected XFEM

Properties of the modified approximation

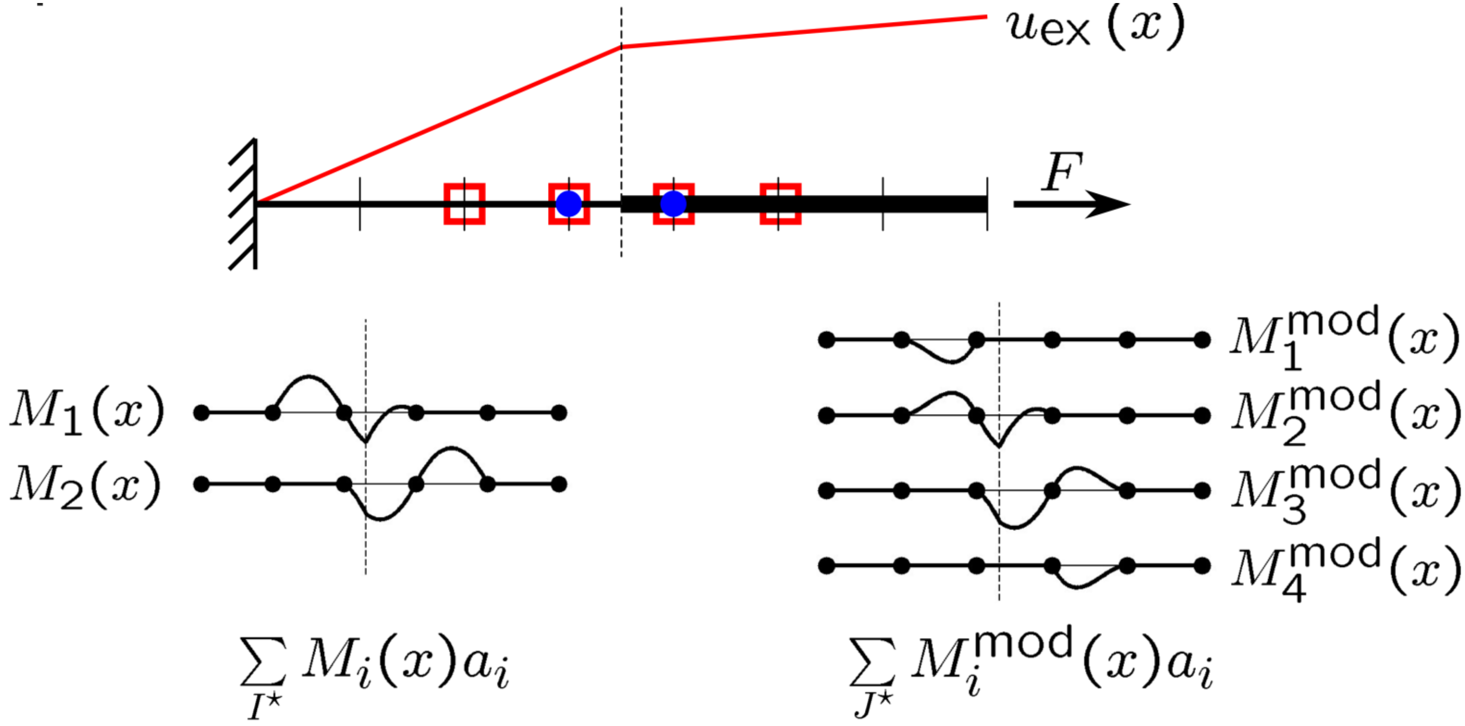
- I^* are the element nodes of cut elements
- $\psi = abs(\phi(x))$ for weak discontinuities
- $\psi = sign(\phi(x))$ for strong discontinuities
- $J^*$and $R(x)$ follow immediately





Corrected XFEM

Numerical results 1d Bi-material bar



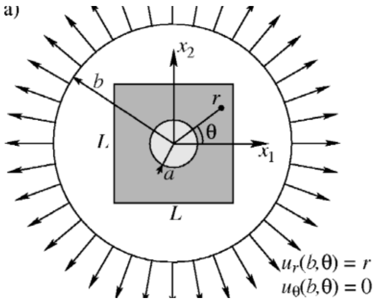
The exact solution can be found with the modified XFEM

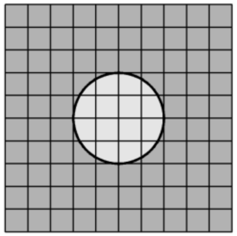
Introduction of the XFEM

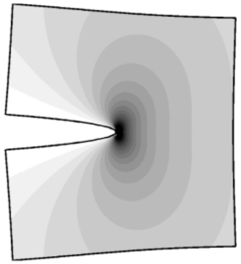
Corrected XFEM

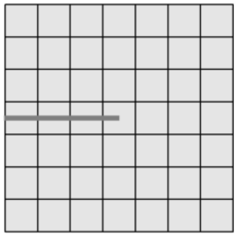
- **Optimal convergence rates have been achieved with the corrected XFEM in the following applications**

- **Solid mechanics**

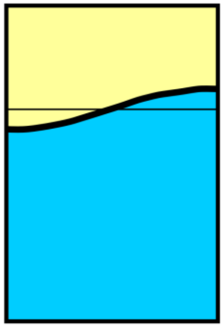


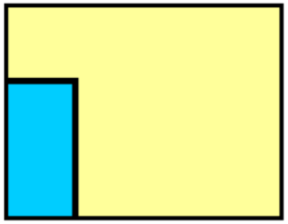


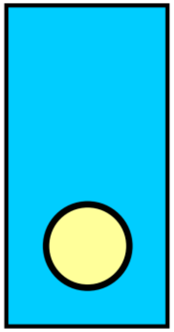




- **Fluid mechanics**







[Fries, Belytschko, IJNME, 2006]
[Fries, Belytschko, Comp. Mech., 2007]

Introduction of the XFEM

Summary

- The XFEM manipulates the approximation space but not the mesh. => No mesh refinement in regions with high gradients, no mesh alignment with discontinuities
- This is achieved by a local, extrinsic enrichment through the partition of unity concept
- A particular realization of the XFEM is defined by the choice of the enriched nodes and the enrichment functions
- The XFEM can be used successfully for the simulation of floating bodies

Introduction of the XFEM

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Introduction of the XFEM

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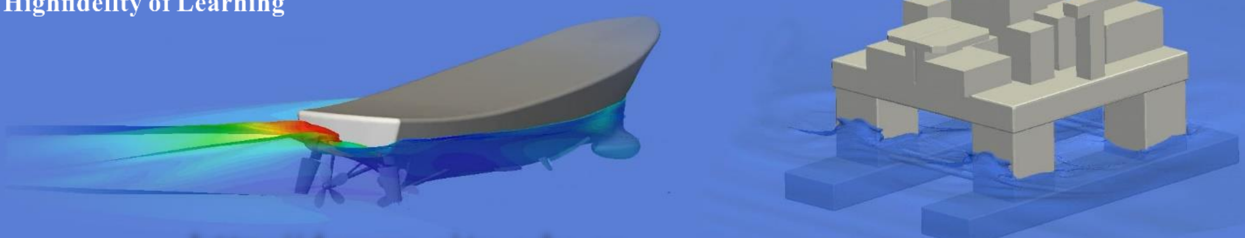

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