Convection-dominated problems

Convection-dominated problems

Scalar transport equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = \nabla \cdot (\epsilon \nabla u)$$

Incompressible Navier-Stokes equations

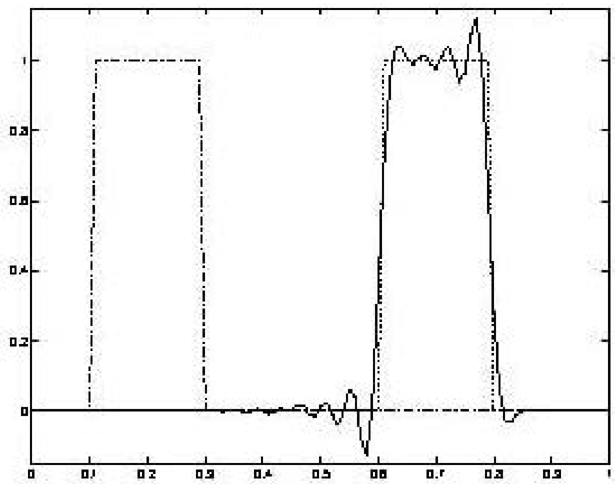
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0$$

Compressible Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0$$

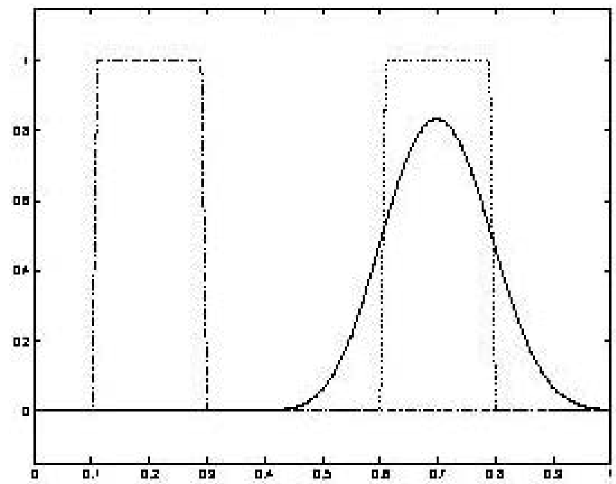
$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho H \mathbf{v}) = 0$$



high-order method: wiggles

Dilemma

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$$



low-order method: smearing

Modern high-resolution schemes

Flux-Corrected Transport (FCT) paradigm — *Boris & Book (1973)*

1. Compute a provisional solution by a monotone method of low order
2. Determine the percentage of artificial diffusion that can be removed without generating new extrema and accentuating already existing ones
3. Add limited *antidiffusion* to recover the high accuracy in smooth regions

State of the art: MUSCL, PPM, TVD, LED, ENO are widely used

- geometric construction
- one-dimensional nature
- cartesian or simplex meshes
- explicit time discretization
- finite differences/volumes

Objective: a general methodology applicable to implicit FEM discretizations

Design of high-resolution schemes

LED criterion (*Jameson, 1993*) A semi-discrete scheme of the form

$$\frac{du_i}{dt} = \sum_j c_{ij} u_j, \qquad \text{where} \quad c_{ii} = -\sum_{j \neq i} c_{ij} \quad \text{and} \quad c_{ij} \geq 0, \quad \forall j \neq i$$

so that $\frac{du_i}{dt} = \sum_{j \neq i} c_{ij}(u_j - u_i)$ proves *local extremum diminishing* since

- maxima do not increase: $u_i = \max_j u_j \Rightarrow u_j - u_i \leq 0 \Rightarrow \frac{du_i}{dt} \leq 0$
- minima do not decrease: $u_i = \min_j u_j \Rightarrow u_j - u_i \geq 0 \Rightarrow \frac{du_i}{dt} \geq 0$

Total variation $TV(u) = 2\left(\sum \max u - \sum \min u\right)$ is non-increasing

Conclusion: the LED criterion represents a handy generalization of Harten's TVD theorem to multidimensional discretizations on unstructured meshes.

Design of high-resolution schemes

Discretization in time by the standard $\theta-$scheme

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\theta\sum_{j\neq i}c_{ij}(u_j^{n+1}-u_i^{n+1})+(1-\theta)\sum_{j\neq i}c_{ij}(u_j^n-u_i^n)$$

Fully discrete LED schemes are *positivity-preserving* i.e. $u^n\geq 0 \;\Rightarrow\; u^{n+1}\geq 0$ under the CFL-like condition $1+\Delta t(1-\theta)\min_i c_{ii}\geq 0$ for $0\leq\theta<1$.

Definition. A nonsingular discrete operator $A\in\mathbb{R}^{N\times N}$ is called an *M-matrix* if $a_{ij}\leq 0$ for $i\neq j$ and all entries of the inverse A^{-1} are nonnegative.

> Positivity criterion A fully discrete scheme is positivity-preserving if it can be cast in the form $Au^{n+1}=Bu^n$ where A is an M-matrix and $B\geq 0$.

Strategy: discretize in space by a standard high-order method (central differences, Galerkin FEM) and modify the matrices so as to satisfy the above criteria.

Roadmap of algebraic manipulations

Scalar conservation law $$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0$$ continuity equation

1. Linear high-order scheme

$$M_C \frac{\mathrm{d}u}{\mathrm{d}t} = Ku \quad \textit{(Galerkin FEM)}$$

such that $\exists\, j \neq i : \; k_{ij} < 0$

2. Linear low-order scheme

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = Lu, \quad L = K + D$$

such that $l_{ij} \geq 0, \; \forall j \neq i$

3. Nonlinear high-resolution scheme

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = K^* u, \quad K^* = L + F$$

such that $\exists\, j \neq i : \; k_{ij}^* < 0$

Equivalent LED representation

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = L^* u, \quad L^* u = K^* u$$

such that $l_{ij}^* \geq 0, \; \forall j \neq i$

The existence of L^* is *sufficient* for $K^* = K + D + F$ to be nonoscillatory

Galerkin FEM: efficient matrix assembly

Variational formulation $$\int_\Omega w \left[\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u)\right] \mathrm{d}\mathbf{x} = 0$$

Group FEM interpolation $$u_h = \sum_j u_j \varphi_j, \quad (\mathbf{v}u)_h = \sum_j (\mathbf{v}_j u_j)\,\varphi_j$$

Semi-discrete scheme $$M_C \frac{\mathrm{d}u}{\mathrm{d}t} = Ku \qquad M_C = \{m_{ij}\}, \quad K = \{k_{ij}\}$$

Matrix coefficients $$m_{ij} = \int_\Omega \varphi_i \varphi_j \,\mathrm{d}\mathbf{x}, \qquad k_{ij} = -\mathbf{v}_j \cdot \mathbf{c}_{ij}$$

Discretized derivatives $$\mathbf{c}_{ij} = \int_\Omega \varphi_i \nabla \varphi_j \,\mathrm{d}\mathbf{x}, \qquad \mathbf{c}_{ii} = -\sum_{j \neq i} \mathbf{c}_{ij}$$

Remark. The coefficients m_{ij} and $\mathbf{c}_{ij}$ remain unchanged as long as the mesh is fixed so that there is no need for costly numerical integration.

Discrete upwinding for finite elements

Lumped-mass Galerkin scheme

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = Ku$$

$$L = K + D$$

Linear LED discretization

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = Lu$$

$$m_i \frac{du_i}{dt} = \sum_{j \neq i} k_{ij}(u_j - u_i) + \delta_i u_i \qquad m_i = \sum_j m_{ij}, \qquad \delta_i = \sum_j k_{ij}$$

Artificial diffusion term: $(Du)_i = \sum_{j \neq i} f_{ij}, \qquad f_{ij} = d_{ij}(u_j - u_i), \quad f_{ji} = -f_{ij}$

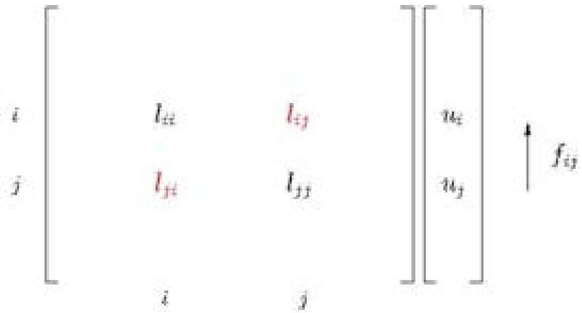
$$d_{ij} = \max\{0, -k_{ij}, -k_{ji}\}$$

Implementation: start with $L := K$

$$l_{ii} := l_{ii} - d_{ij}, \qquad l_{ij} := l_{ij} + d_{ij}$$

$$l_{ji} := l_{ji} + d_{ij}, \qquad l_{jj} := l_{jj} - d_{ij}$$

Graph orientation: let the edge $\vec{ij}$ be directed so that $l_{ji} \geq l_{ij} = \max\{0, k_{ij}\}$



Discrete upwinding in one dimension

Convection equation $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \quad v > 0$ linear finite elements

Element matrices $\hat{M}_C = \frac{\Delta x}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \hat{M}_L = \frac{\Delta x}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\hat{K} = \begin{bmatrix} \frac{v}{2} & -\frac{v}{2} \\ \frac{v}{2} & -\frac{v}{2} \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} -\frac{v}{2} & \frac{v}{2} \\ \frac{v}{2} & -\frac{v}{2} \end{bmatrix} \quad \Rightarrow \quad \hat{L} = \hat{K} + \hat{D} = \begin{bmatrix} 0 & 0 \\ v & -v \end{bmatrix}$$

High-order discretization

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + v\,\frac{u_{i+1} - u_{i-1}}{2\,\Delta x} = 0$$

central difference scheme

Low-order discretization

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + v\,\frac{u_i - u_{i-1}}{\Delta x} = 0$$

upwind difference scheme

Graph orientation: our convention $l_{ji} \geq l_{ij}$ implies that node i is located *upwind*

TVD discretization in one dimension

Antidiffusive correction $\hat{K}^*(u) = \hat{L} - \Phi(r_i)\hat{D} = \hat{K} + [1 - \Phi(r_i)]\hat{D}$

$$r_i = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}$$ *slope ratio* evaluated at the upwind node i

LED representation

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = c_{i-1/2}(u_{i-1} - u_i) + c_{i+1/2}(u_{i+1} - u_i)$$

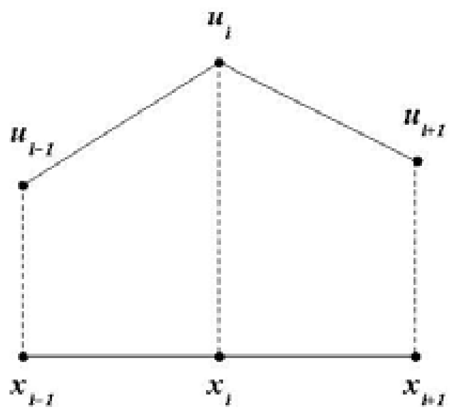
where $$c_{i-1/2} = \frac{v}{2\Delta x}\left[2 + \frac{\Phi(r_i)}{r_i} - \Phi(r_{i-1})\right] \geq 0, \qquad c_{i+1/2} = 0$$

Standard flux limiters $\Phi(r) = \frac{r+|r|}{1+|r|}$ VL

$\Phi(r) = \max\{0, \min\{1, r\}\}$ MM

$\Phi(r) = \max\{0, \min\left\{\frac{1+r}{2}, 2, 2r\right\}\}$ MC

$\Phi(r) = \max\{0, \min\{1, 2r\}, \min\{2, r\}\}$ SB



TVD discretization in one dimension

Standard flux limiters as two-parameter functions

$$\mathcal{S}(a,b) = \frac{\mathrm{sign}(a) + \mathrm{sign}(b)}{2}$$

VL $\quad \mathcal{L}(a,b) = \mathcal{S}(a,b)\,\frac{2|a||b|}{|a|+|b|}$

MM $\quad \mathcal{L}(a,b) = \mathcal{S}(a,b)\,\min\{|a|,|b|\}$

MC $\quad \mathcal{L}(a,b) = \mathcal{S}(a,b)\,\min\{0.5|a+b|, 2|a|, 2|b|\}$

SB $\quad \mathcal{L}(a,b) = \mathcal{S}(a,b)\,\max\{\min\{2|a|,|b|\},\min\{|a|,2|b|\}\}$

$$\Phi(r) = \mathcal{L}(1,r)$$

Properties of limited average operators *Jameson (1993)*

- $\mathcal{L}(a,b) = 0 \quad \text{if} \quad ab \le 0 \quad \Rightarrow \quad \Phi(r) = 0 \quad \text{if} \quad r \le 0$
- $\mathcal{L}(ca,cb) = c\mathcal{L}(a,b) \quad \Rightarrow \quad \Phi(r) = \mathcal{L}(1,r) = r\mathcal{L}(1/r,1)$
- $\mathcal{L}(a,b) = \mathcal{L}(b,a) \quad \Rightarrow \quad \mathcal{L}(1/r,1) = \mathcal{L}(1,1/r) = \Phi(1/r)$

$$\Phi(r_i)(u_{i+1} - u_i) = \mathcal{L}(u_{i+1} - u_i, u_i - u_{i-1}) = \Phi(1/r_i)(u_i - u_{i-1})$$

Generalized FEM-TVD formulation

Modified transport operator $K^*(u) = L + F(u) = K + D + F(u)$

Limited antidiffusive fluxes $(Fu)_i = \sum_{j \neq i} f_{ij}^a$

$$M_L \frac{\mathrm{d}u}{\mathrm{d}t} = K^* u$$

$$f_{ij}^a = \min\{\Phi(r_i) d_{ij}, l_{ji}\}(u_i - u_j), \quad f_{ji}^a := -f_{ij}^a$$

Nontrivial case $d_{ij} > 0, \quad l_{ji} > l_{ij} = 0$ $\qquad a_{ij} := \min\{d_{ij}, l_{ji}/\Phi(r_i)\}$

$$f_{ij}^a = \Phi(r_i) a_{ij}(u_i - u_j) = \Phi(1/r_i) a_{ij} \Delta u_{ij} \qquad \Delta u_{ij} := r_i(u_i - u_j)$$

Sufficient LED condition $\Delta u_{ij} = \sum_{k \neq i} c_{ik}(u_k - u_i), \quad c_{ik} \geq 0, \quad \forall k \neq i$

$$(K^* u)_i \quad \leftarrow \quad k_{ij}^*(u_j - u_i) = \Phi(1/r_i) a_{ij} \Delta u_{ij} \qquad \Phi(1/r_i) a_{ij} \geq 0$$

$$(K^* u)_j \quad \leftarrow \quad k_{ji}^*(u_i - u_j) = [l_{ji} - \Phi(r_i) a_{ij}](u_i - u_j) \qquad l_{ji} \geq \Phi(r_i) a_{ij}$$

Algebraic flux correction of TVD type

Incompressible part of Ku $\quad \sum_{j\neq i} k_{ij}(u_j - u_i) = P_i + Q_i$ $\quad$ *nodal increment*

$P_i = P_i^+ + P_i^-, \qquad P_i^\pm = \sum_{j\neq i} \min\{0, k_{ij}\} \genfrac{}{}{0pt}{}{\min}{\max}\{0, u_j - u_i\}$ $\quad$ *downstream data*

$Q_i = Q_i^+ + Q_i^-, \qquad Q_i^\pm = \sum_{j\neq i} \max\{0, k_{ij}\} \genfrac{}{}{0pt}{}{\max}{\min}\{0, u_j - u_i\}$ $\quad$ *upstream data*

Nodal correction factors $\quad \boxed{R_i^\pm = \Phi(Q_i^\pm / P_i^\pm)}$ $\quad$ for all fluxes f_{ij}^a, $j \neq i$

Smoothness indicator

$$r_i = \begin{cases} Q_i^+/P_i^+ & \text{if } u_i \geq u_j \\ Q_i^-/P_i^- & \text{if } u_i < u_j \end{cases}$$

Limited antidiffusive fluxes

$$f_{ij}^a = \begin{cases} \min\{R_i^+ d_{ij}, l_{ji}\}(u_i - u_j) & \text{if } u_i \geq u_j \\ \min\{R_i^- d_{ij}, l_{ji}\}(u_i - u_j) & \text{if } u_i < u_j \end{cases}$$

LED property $\quad \boxed{\Delta u_{ij} = r_i(u_i - u_j) = \alpha_{ij} Q_i^\pm}$ $\quad \alpha_{ij} = \genfrac{}{}{0pt}{}{\max}{\min}\{0, u_i - u_j\}/P_i^\pm \geq 0$

Iterative defect correction

Discretization in time yields a **nonlinear** algebraic system

$$M_L \frac{u^{n+1} - u^n}{\Delta t} = \theta K^*(u^{n+1})u^{n+1} + (1-\theta)K^*(u^n)u^n$$

standard
$\theta-$scheme

Successive approximations

$$u^{(m+1)} = u^{(m)} + [A(u^{(m)})]^{-1} r^{(m)}$$

$$u^{(0)} = u^n, \qquad m = 0, 1, 2, \ldots$$

Practical implementation

$$A(u^{(m)})\Delta u^{(m+1)} = r^{(m)}$$

$$u^{(m+1)} = u^{(m)} + \Delta u^{(m+1)}$$

'Upwind' preconditioner $\quad A(u^{(m)}) = M_L - \theta \Delta t L(u^{(m)}), \qquad 0 < \theta \leq 1$

Defect vector and right-hand side

$$r^{(m)} = b^n - A(u^{(m)})u^{(m)} + \theta \Delta t f^{(m)}$$

$$b^n = M_L u^n + (1-\theta)\Delta t[L(u^n)u^n + f^n]$$

Limited antidiffusive fluxes

$$f_i = \sum_{j \neq i} f_{ij}^a, \quad f_{ji}^a = -f_{ij}^a$$

constructed edge-by-edge

Summary of the FEM-TVD algorithm

In a loop over edges:

1. Retrieve the entries k_{ij} and k_{ji} of the high-order transport operator
2. Increment the sums of upstream and downstream edge contributions
3. Determine the artificial diffusion coefficient d_{ij} for discrete upwinding
4. Modify the entries a_{ii}, a_{ij}, a_{ji}, a_{jj} of the low-order preconditioner A
5. Adopt the 'upwind-downwind' edge orientation $\vec{ij}$ such that $l_{ji} \geq l_{ji}$

In a loop over nodes:

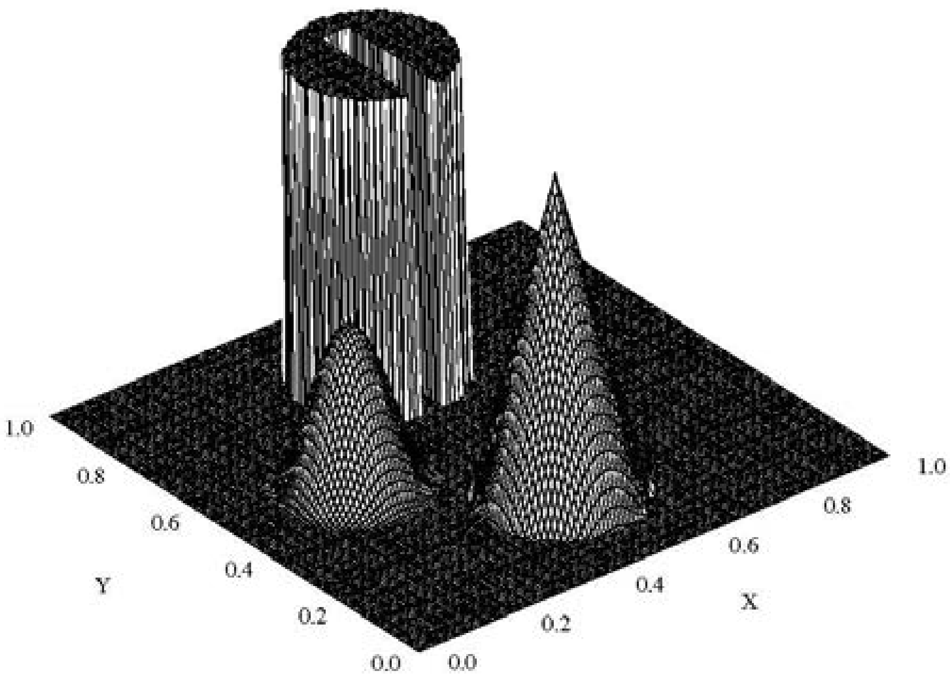
6. Evaluate the ratio $Q_i^{\pm}/P_i^{\pm}$ and apply a TVD limiter Φ to obtain $R_i^{\pm}$

In a loop over edges:

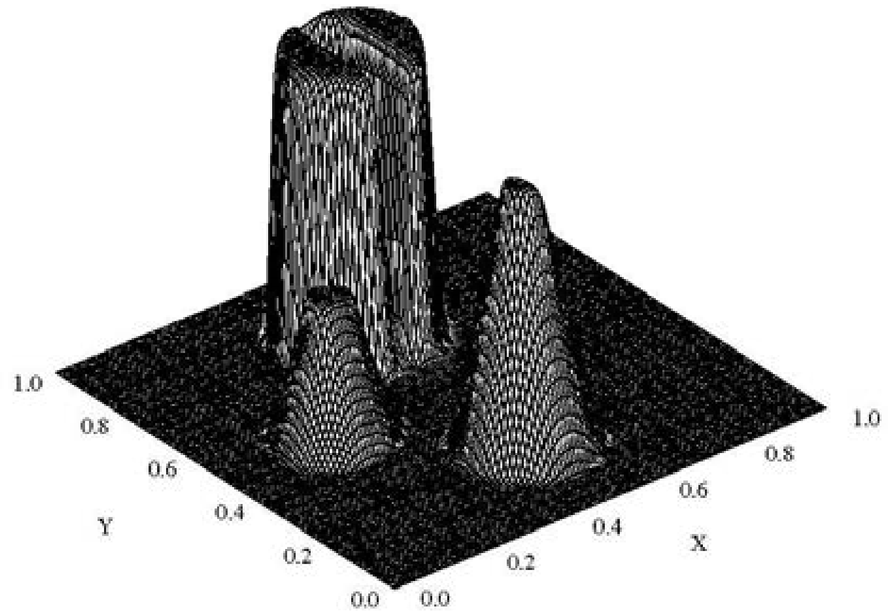
7. Compute the diffusive flux $f_{ij}^d = d_{ij}(u_j - u_i)$ due to discrete upwinding
8. Check the sign of $u_j - u_i$ and compute the limited antidiffusive flux f_{ij}^a
9. Insert the net artificial diffusion $f_{ij} = f_{ij}^d + f_{ij}^a$ into the defect vector r

Example: solid body rotation

Exact solution / initial data



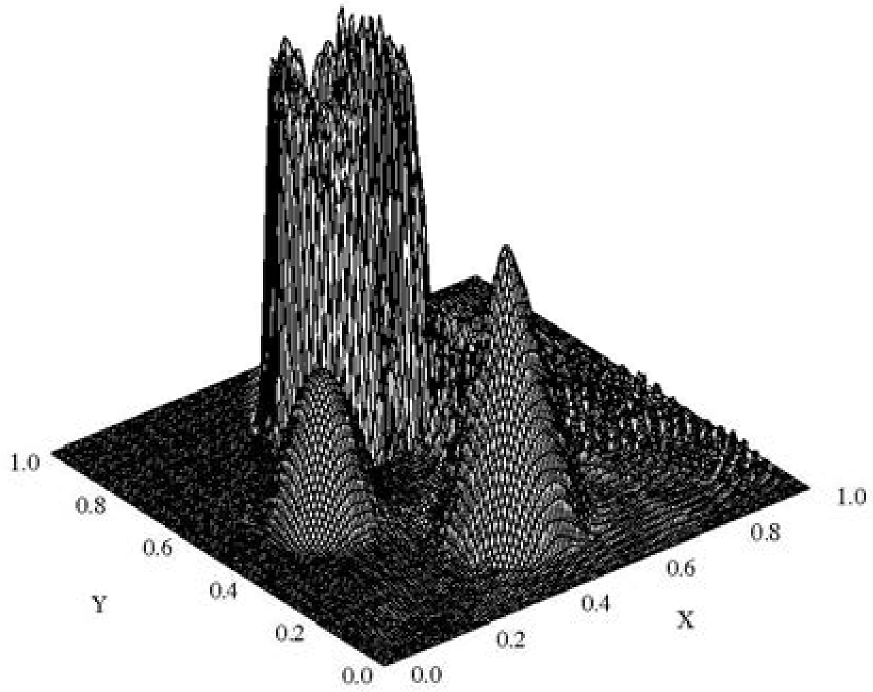
FEM-TVD / superbee limiter



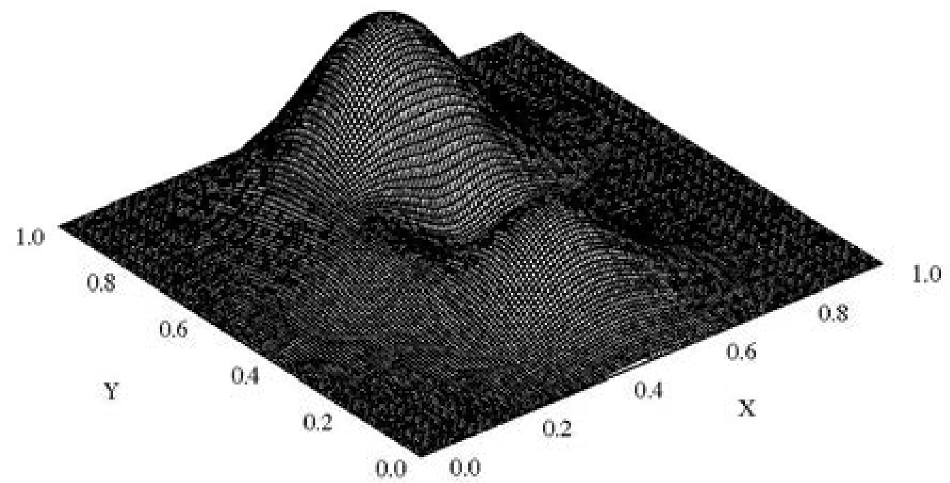
Crank-Nicolson time-stepping $\Delta t = 10^{-3}$, 128×128 bilinear elements

Example: solid body rotation

High-order solution



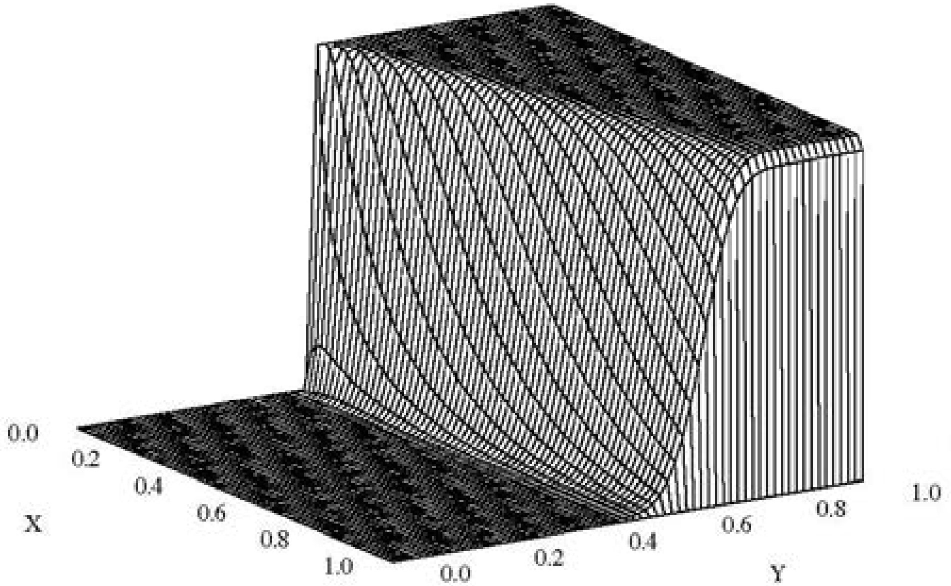
Low-order solution



Crank-Nicolson time-stepping $\Delta t = 10^{-3}$, 128×128 bilinear elements

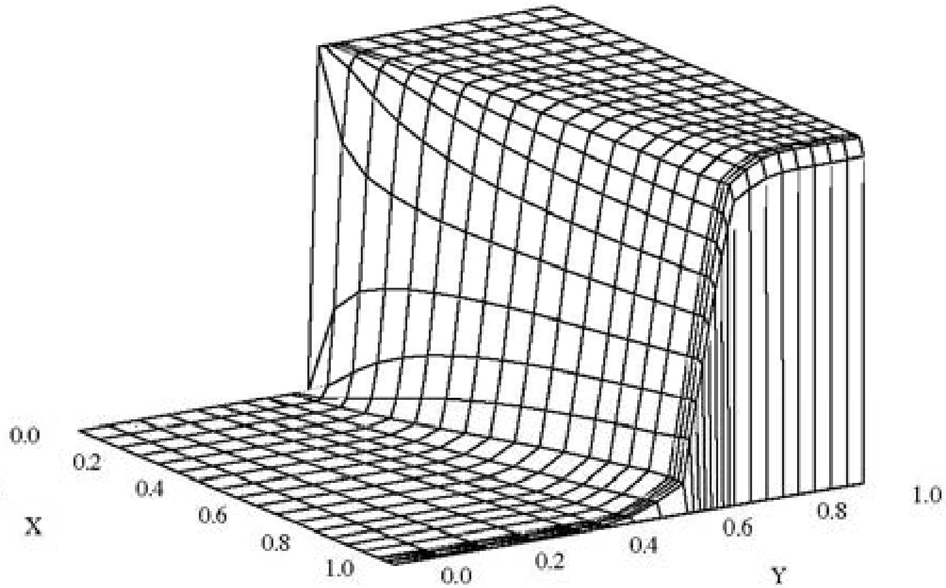
Example: steady-state convection-diffusion

Uniform structured mesh



64×64 bilinear elements

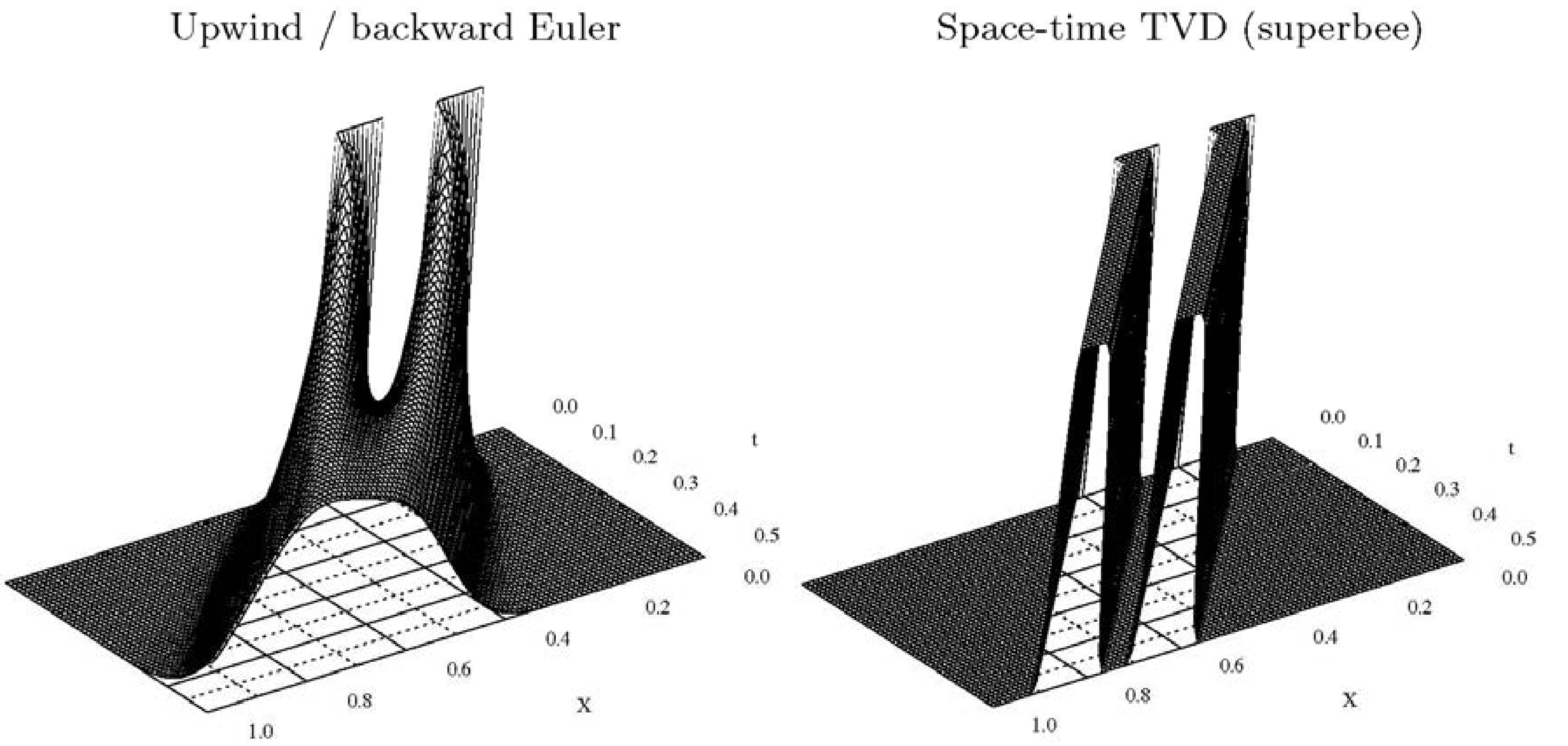
Adaptive structured mesh



20×24 bilinear elements

Backward Euler time-stepping $\mathbf{v} = (\cos 10^o, \sin 10^o),\ \epsilon = 10^{-3},\ \Delta t = 1.0$

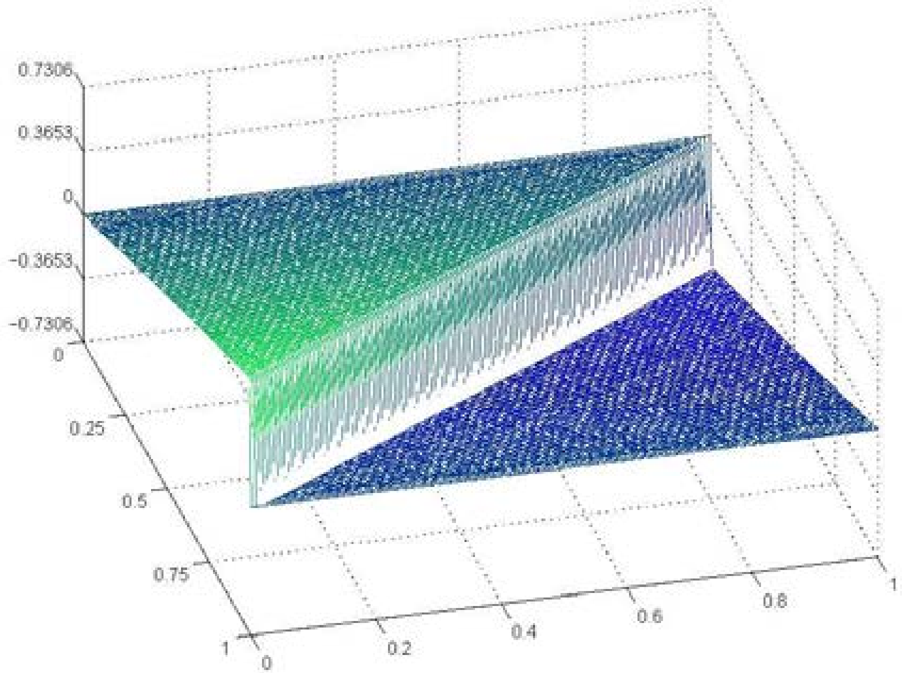
Example: scalar convection in space-time



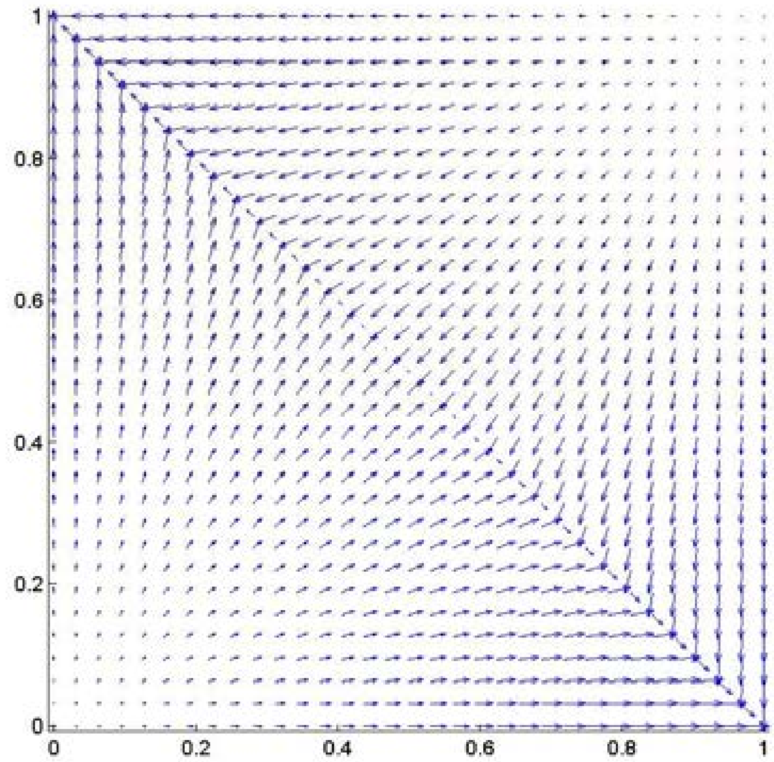
Uniform Cartesian mesh $\Delta x = \Delta t = 10^{-2}$, $\Omega = (0,1) \times (0,0.5)$

Example: two-dimensional Burgers equation

Initial data: $u(x,y,0) = \sin(\pi x)\cos(\pi y), \quad v(x,y,0) = \cos(\pi x)\sin(\pi y)$



FEM-TVD / MC limiter, $u(x,y,1)$



Backward Euler time-stepping $\quad \Delta t = 10^{-2}$, 128×128 bilinear elements

Compressible Euler equations

Divergence form $$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = 0$$ where $$\nabla \cdot \mathbf{F} = \sum_{d=1}^{3} \frac{\partial F^d}{\partial x_d}$$

Conservative variables and fluxes

$$U = (\rho, \rho\mathbf{v}, \rho E)^T$$

$$\mathbf{F} = (F^1, F^2, F^3)$$

$$\mathbf{F} = \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v} \otimes \mathbf{v} + p\mathcal{I} \\ \rho H \mathbf{v} \end{bmatrix}$$

$$H = E + \frac{p}{\rho}$$

$$\gamma = c_p / c_v$$

Equation of state $$p = (\gamma - 1)\rho\left(E - |\mathbf{v}|^2/2\right)$$ for a polytropic gas

Quasi-linear form $$\frac{\partial U}{\partial t} + \mathbf{A} \cdot \nabla U = 0$$ where $$\mathbf{A} \cdot \nabla U = \sum_{d=1}^{3} A^d \frac{\partial U}{\partial x_d}$$

Jacobian matrices $$\mathbf{A} = (A^1, A^2, A^3)$$

$$F^d = A^d U, \qquad A^d = \frac{\partial F^d}{\partial U}, \qquad d = 1, 2, 3$$

Galerkin FEM for the Euler equations

Group FEM formulation

$$M_C \frac{\mathrm{d}\mathrm{U}}{\mathrm{d}t} = K\mathrm{U}$$

$$(K\mathrm{U})_i = -\sum_{j \neq i} \mathbf{c}_{ij} \cdot (\mathbf{F}_j - \mathbf{F}_i)$$

since the basis functions satisfy $\sum_j \varphi_j \equiv 1$ and thus $\mathbf{c}_{ii} = -\sum_{j \neq i} \mathbf{c}_{ij}$

Roe averaging $\mathbf{F}_j - \mathbf{F}_i = \hat{\mathbf{A}}_{ij}(\mathrm{U}_j - \mathrm{U}_i)$, where $\hat{\mathbf{A}}_{ij} = \mathbf{A}(\hat{\rho}_{ij}, \hat{\mathbf{v}}_{ij}, \hat{H}_{ij})$

$$\hat{\rho}_{ij} = \sqrt{\rho_i \rho_j}, \qquad \hat{\mathbf{v}}_{ij} = \frac{\sqrt{\rho_i}\mathbf{v}_i + \sqrt{\rho_j}\mathbf{v}_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}, \qquad \hat{H}_{ij} = \frac{\sqrt{\rho_i}H_i + \sqrt{\rho_j}H_j}{\sqrt{\rho_i} + \sqrt{\rho_j}}$$

Quasi-linear Galerkin discretization

$$(K\mathrm{U})_i = -\sum_{j \neq i} \mathbf{c}_{ij} \cdot \hat{\mathbf{A}}_{ij}(\mathrm{U}_j - \mathrm{U}_i) = -\sum_{j \neq i} (\mathrm{A}_{ij} + \mathrm{B}_{ij})(\mathrm{U}_j - \mathrm{U}_i)$$

Cumulative Roe matrices

$$\mathrm{A}_{ij} = \mathbf{a}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{a}_{ij} = \frac{\mathbf{c}_{ij} - \mathbf{c}_{ji}}{2}$$

$$\mathrm{B}_{ij} = \mathbf{b}_{ij} \cdot \hat{\mathbf{A}}_{ij}, \qquad \mathbf{b}_{ij} = \frac{\mathbf{c}_{ij} + \mathbf{c}_{ji}}{2}$$

Contribution of the edge $\vec{ij}$

$$(\mathrm{A}_{ij} + \mathrm{B}_{ij})(\mathrm{U}_i - \mathrm{U}_j) \longrightarrow (K\mathrm{U})_i$$

$$(\mathrm{A}_{ij} - \mathrm{B}_{ij})(\mathrm{U}_i - \mathrm{U}_j) \longrightarrow (K\mathrm{U})_j$$

Matrix assembly for the Euler equations

Edge contribution to the operator K

$$\mathrm{K}_{ii} = \mathrm{A}_{ij} + \mathrm{B}_{ij} \qquad \mathrm{K}_{ij} = -\mathrm{A}_{ij} - \mathrm{B}_{ij}$$

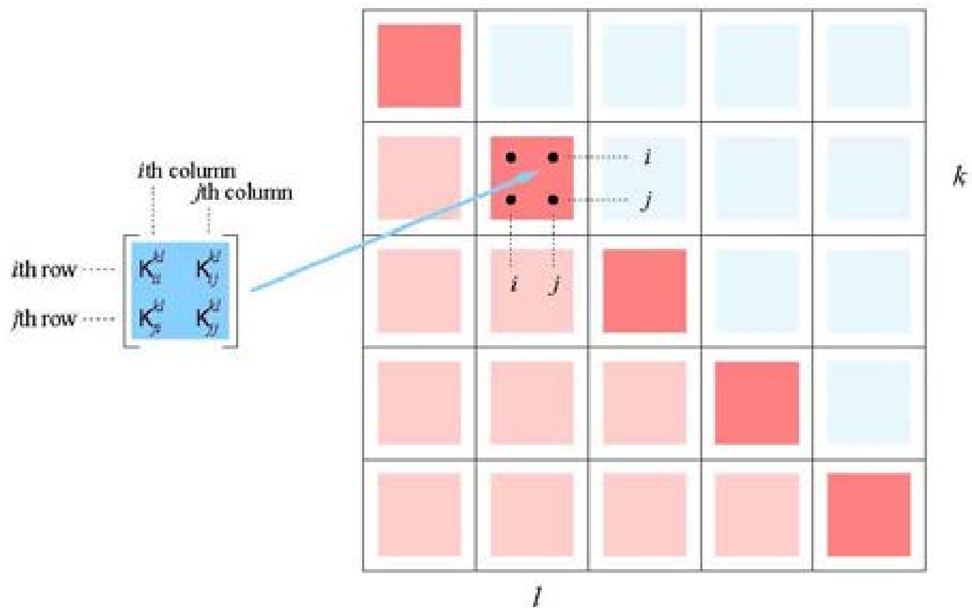
$$\mathrm{K}_{ji} = \mathrm{A}_{ij} - \mathrm{B}_{ij} \qquad \mathrm{K}_{jj} = -\mathrm{A}_{ij} + \mathrm{B}_{ij}$$

Edge contribution to the operator L

$$\mathrm{L}_{ii} = \mathrm{A}_{ij} - \mathrm{D}_{ij} \qquad \mathrm{L}_{ij} = -\mathrm{A}_{ij} + \mathrm{D}_{ij}$$

$$\mathrm{L}_{ji} = \mathrm{A}_{ij} + \mathrm{D}_{ij} \qquad \mathrm{L}_{jj} = -\mathrm{A}_{ij} - \mathrm{D}_{ij}$$

Structure of the global matrix



Raw antidiffusive flux for the edge $\vec{ij}$

$$\mathrm{F}_{ij} = -\left(\mathrm{M}_{ij}\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{D}_{ij} + \mathrm{B}_{ij}\right)(\mathrm{U}_j - \mathrm{U}_i), \qquad \mathrm{F}_{ji} = -\mathrm{F}_{ij} \qquad \text{(semi-discrete)}$$

where $\mathrm{M}_{ij} = m_{ij}\mathrm{I}$ and D_{ij} is a local tensor of artificial diffusion (to be defined)

Construction of artificial viscosities

Generalized LED principle for systems

Off-diagonal blocks of the global matrix should be positive semi-definite

Characteristic decomposition $\quad \mathrm{A}_{ij} = \mathrm{R}_{ij}\Lambda_{ij}\mathrm{R}_{ij}^{-1} \qquad |\mathbf{a}_{ij}| = \sqrt{\mathbf{a}_{ij}\cdot\mathbf{a}_{ij}}$

where $\quad \Lambda_{ij} = |\mathbf{a}_{ij}|\mathrm{diag}\{\lambda_1,\dots,\lambda_5\}$ and R_{ij} is the matrix of eigenvectors

Eigenvalues $\quad \lambda_1 = \hat{v}_{ij} - \hat{c}_{ij}, \qquad \lambda_2 = \lambda_3 = \lambda_4 = \hat{v}_{ij}, \qquad \lambda_5 = \hat{v}_{ij} + \hat{c}_{ij}$

Characteristic velocities $\quad \hat{v}_{ij} = \frac{\mathbf{a}_{ij}\cdot\hat{\mathbf{v}}_{ij}}{|\mathbf{a}_{ij}|}, \quad \hat{c}_{ij} = \sqrt{(\gamma-1)\left(\hat{H}_{ij} - \frac{|\hat{\mathbf{v}}_{ij}|^2}{2}\right)}$

Generalization of Roe's Riemann solver

$$\mathrm{D}_{ij} = |\mathrm{A}_{ij}| = \mathrm{R}_{ij}|\Lambda_{ij}|\mathrm{R}_{ij}^{-1} \qquad \text{or} \qquad \mathrm{D}_{ij} = \sum_{d=1}^{3} |\mathrm{A}_{ij}^d| \qquad \text{(coordinate splitting)}$$

Remark. Scalar artificial viscosity $\mathrm{D}_{ij} = |\mathbf{a}_{ij}|\max_i|\lambda_i|\mathrm{I}$ is a cheaper alternative

Decoupling of the Euler equations

Iterative defect correction

$$\mathrm{U}^{(m+1)} = \mathrm{U}^{(m)} + [A(\mathrm{U}^{(m)})]^{-1}\mathrm{R}^{(m)}$$

$$A(\mathrm{U}^{(m)})\Delta\mathrm{U}^{(m+1)} = \mathrm{R}^{(m)}$$

$$\mathrm{U}^{(m+1)} = \mathrm{U}^{(m)} + \Delta\mathrm{U}^{(m+1)}$$

Linearized global system for the $m-$th iteration

$$\begin{bmatrix} A_{11}^{(m)} & A_{12}^{(m)} & A_{13}^{(m)} & A_{14}^{(m)} & A_{15}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} & A_{23}^{(m)} & A_{24}^{(m)} & A_{25}^{(m)} \\ A_{31}^{(m)} & A_{32}^{(m)} & A_{33}^{(m)} & A_{34}^{(m)} & A_{35}^{(m)} \\ A_{41}^{(m)} & A_{42}^{(m)} & A_{43}^{(m)} & A_{44}^{(m)} & A_{45}^{(m)} \\ A_{51}^{(m)} & A_{52}^{(m)} & A_{53}^{(m)} & A_{54}^{(m)} & A_{55}^{(m)} \end{bmatrix} \begin{bmatrix} \Delta\mathrm{U}_1^{(m+1)} \\ \Delta\mathrm{U}_2^{(m+1)} \\ \Delta\mathrm{U}_3^{(m+1)} \\ \Delta\mathrm{U}_4^{(m+1)} \\ \Delta\mathrm{U}_5^{(m+1)} \end{bmatrix} = \begin{bmatrix} \mathrm{R}_1^{(m)} \\ \mathrm{R}_2^{(m)} \\ \mathrm{R}_3^{(m)} \\ \mathrm{R}_4^{(m)} \\ \mathrm{R}_5^{(m)} \end{bmatrix}$$

Block-diagonal preconditioner $A_{kk}^{(m)} = M_{kk} - \theta\Delta t L_{kk}^{(m)}, \quad A_{kl}^{(m)} = 0, \quad \forall l \neq k$

is employed to save memory; equations can be solved separately or in parallel

Segregated FEM-TVD algorithm

Sequence of scalar subproblems

$$A_{kk}^{(m)} \Delta \mathrm{U}_k^{(m+1)} = \mathrm{R}_k^{(m)}, \qquad k = 1, \ldots, 5$$

$$\mathrm{U}_k^{(m+1)} = \mathrm{U}_k^{(m)} + \Delta \mathrm{U}_k^{(m+1)}, \quad \mathrm{U}_k^{(0)} = \mathrm{U}_k^n$$

Characteristic TVD limiter

$$\mathrm{F}_{ij}^a = \mathrm{R}_{ij} |\Lambda_{ij}| \Delta \hat{\mathrm{W}}_{ij}$$

$$\Delta \hat{\mathrm{W}}_{ij} = \Phi_{ij} \mathrm{R}_{ij}^{-1} (\mathrm{U}_i - \mathrm{U}_j)$$

Implementation of characteristic boundary conditions

Algebraic manipulations for $\mathbf{x}_i \in \Gamma$

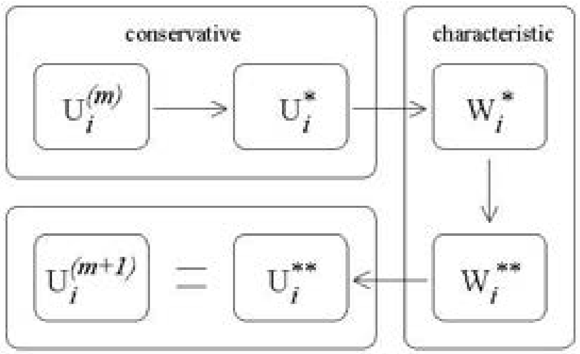
1. Prediction of $\mathrm{U}_i = [u_{1,i}, \ldots, u_{5,i}]^T$

$$a_{ij}^{kk} := 0 \qquad u_{k,i}^* = u_{k,i}^{(m)} + r_{k,i}^{(m)} / a_{ii}^{kk} \qquad r_{k,i}^{(m)} := 0$$

2. Correction of $\mathrm{W}_i = [w_{1,i}, \ldots, w_{5,i}]^T$

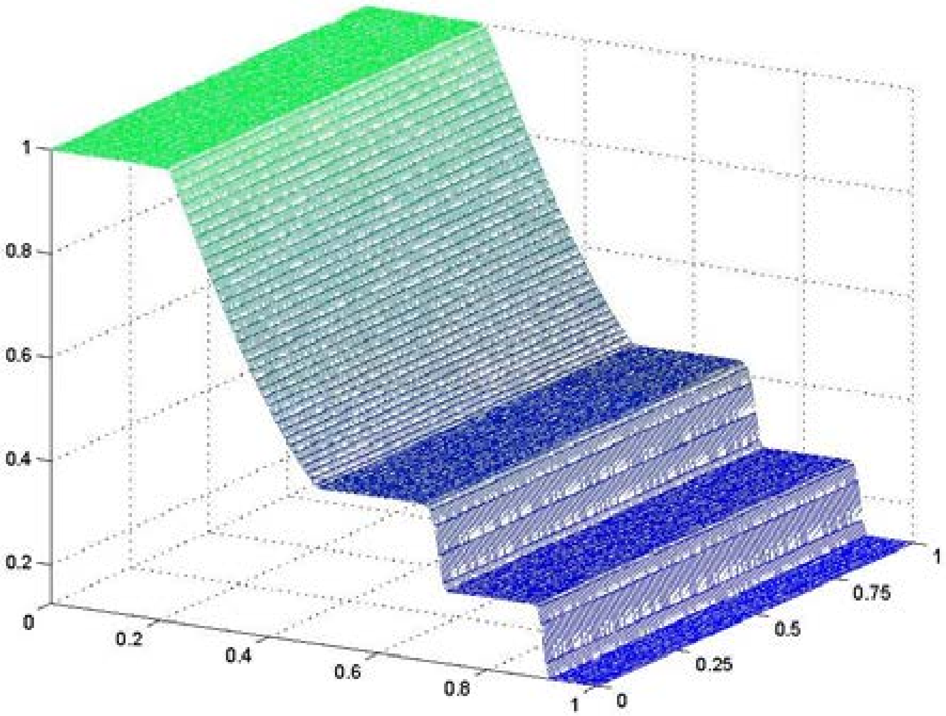
- transform U_i^* into W_i^* and prescribe the incoming Riemann invariants
- convert the resulting vector W_i^{**} back to the conservative variables U_i^{**}

Variable transformations

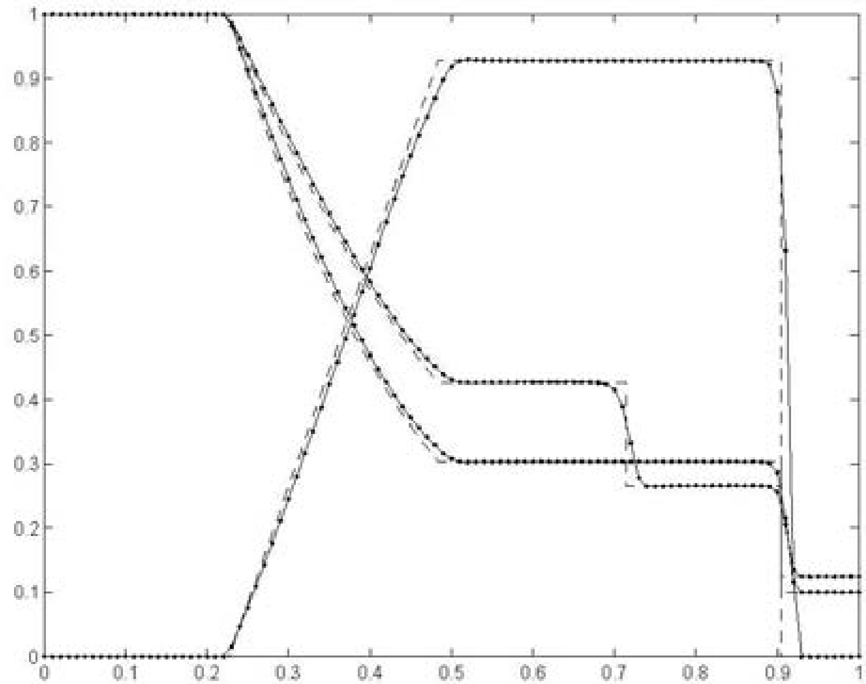


Example: Sod's shock tube problem

Density distribution (2D)



FEM-TVD solution at $t = 0.231$



Crank-Nicolson time-stepping $\Delta t = 10^{-3}$, 128×128 bilinear elements

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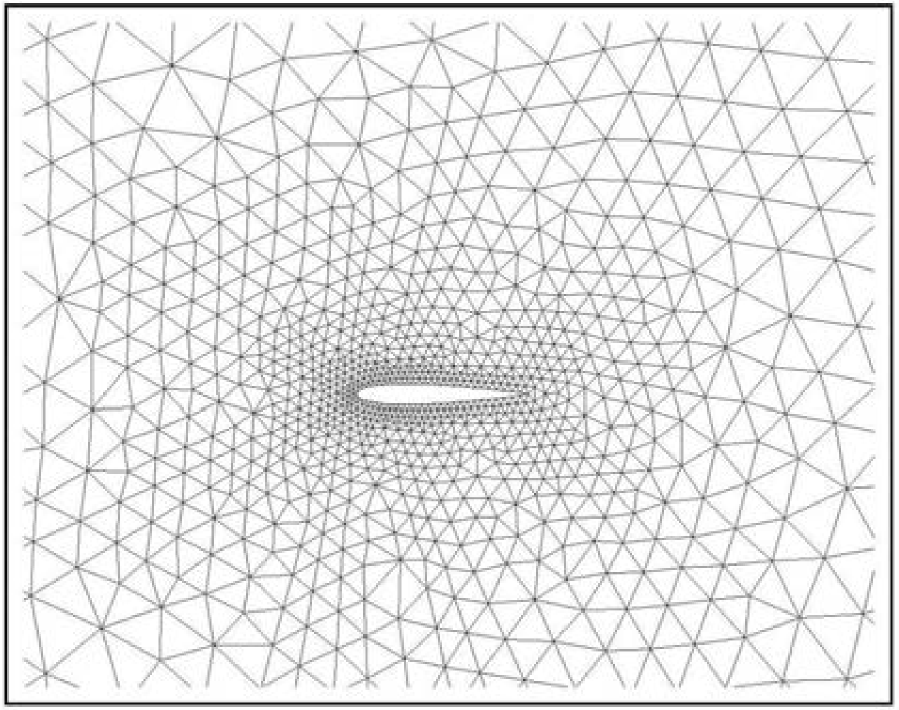
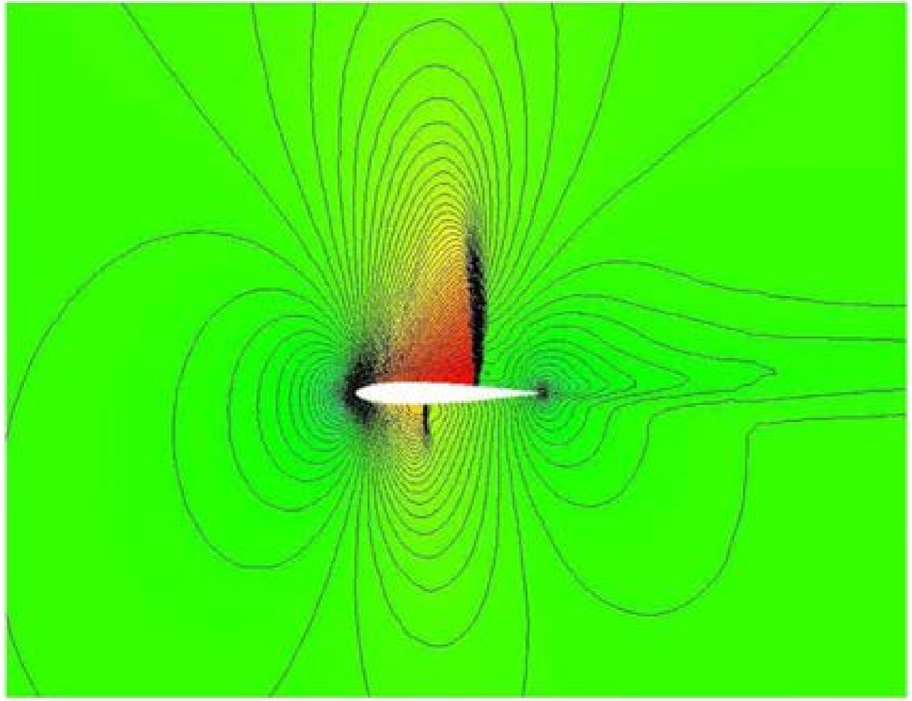
Example: Transonic flow past a NACA 0012 airfoil

Triangular coarse mesh



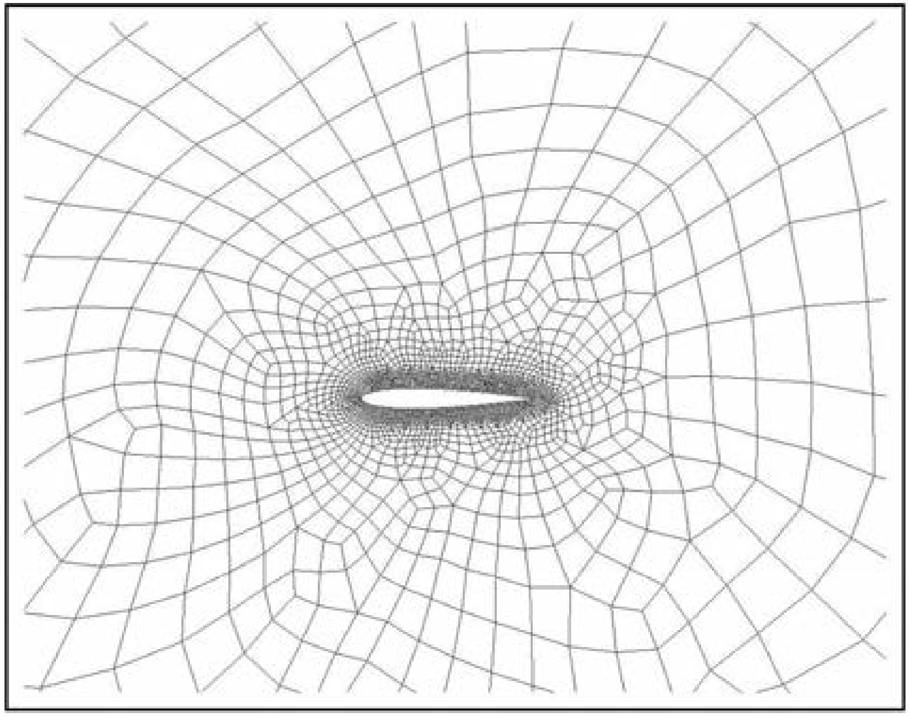
Mach number isolines



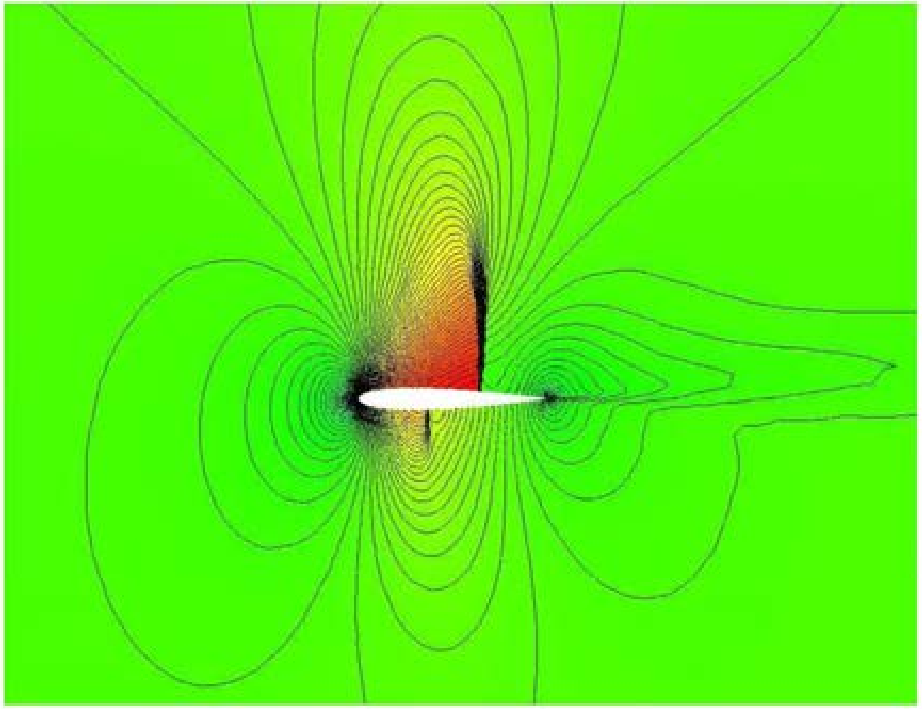
Characteristic FEM-TVD method, MC limiter, $M_\infty = 0.8$, $\alpha = 1.25°$

Example: Transonic flow past a NACA 0012 airfoil

Quadrilateral coarse mesh



Mach number isolines



Characteristic FEM-TVD method, MC limiter, $M_\infty = 0.8, \;\; \alpha = 1.25°$

The algebraic approach to the design of high-resolution schemes

- deals with matrices and their sparsity pattern
- is applicable to arbitrary discrete operators
  - finite elements/differences/volumes
  - nonuniform and unstructured meshes
  - explicit and implicit time-stepping
  - coupled space-time discretizations
- leads to a node-oriented flux limiter of TVD type which is readily portable to multidimensions
- reduces to Harten's TVD schemes in the 1D case
- is very simple to implement and to incorporate into existing CFD codes as a 'black-box' module