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SJTU COMPUTATIONAL MARINE HYDRODYNAMICS LAB
上海交大船舶与海洋工程计算水动力学研究中心

Class-10

NA26018

Finite Element Analysis of Solids and Fluids

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Governing equations for incompressible two-phase flow

$$\varrho_i\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)-\nabla\cdot\boldsymbol{\sigma}=\varrho_i\mathbf{f}$$

$$\nabla\cdot\mathbf{u}=0$$

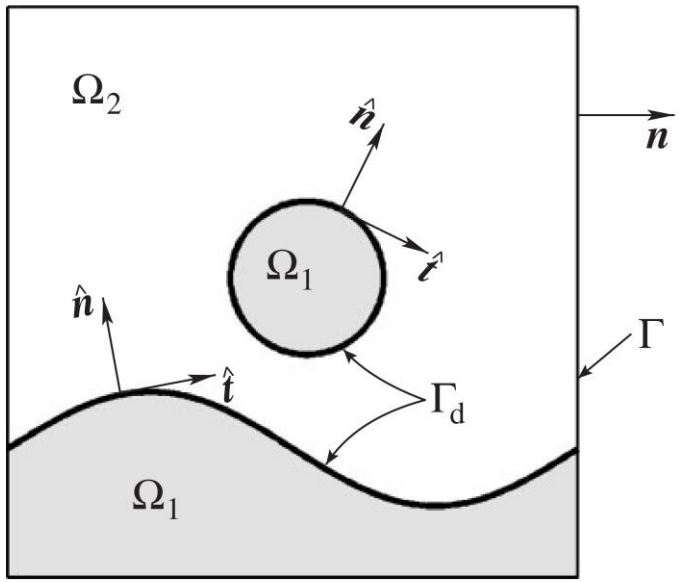
stress tensor

$$\boldsymbol{\sigma}(\mathbf{u},p)=-p\mathbf{I}+2\mu_i\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{with } \boldsymbol{\varepsilon}(\mathbf{u})=\tfrac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}})$$

- **Dirichlet and Neumann boundary conditions on the outer boundary**

$$\mathbf{u}=\hat{\mathbf{u}} \quad \text{on } \Gamma_{\mathbf{u}}\times(0,t_{\text{end}})$$

$$\boldsymbol{\sigma}\cdot\mathbf{n}=\hat{\mathbf{h}} \quad \text{on } \Gamma_{\mathbf{h}}\times(0,t_{\text{end}})$$



- **Conditions apply at the interface:**

$$[\mathbf{u}]_{\Gamma_d}=\mathbf{0} \quad \text{on } \Gamma_d\times(0,t_{\text{end}})$$

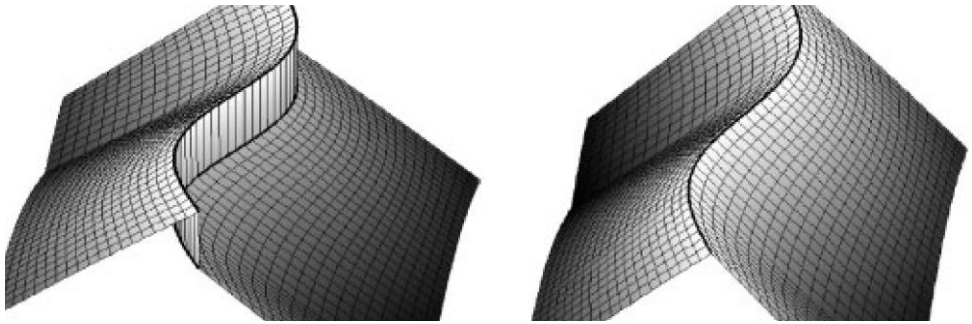
$$-[\boldsymbol{\sigma}]_{\Gamma_d}\cdot\hat{\mathbf{n}}=\gamma\cdot\kappa\cdot\hat{\mathbf{n}} \quad \text{on } \Gamma_d\times(0,t_{\text{end}})$$

Discontinuous state variables at the interface

- The density and viscosity fields

$$\varrho(\mathbf{x},t)=\begin{cases}\varrho_1 & \forall \mathbf{x}\in\Omega_1(t),\\ \varrho_2 & \forall \mathbf{x}\in\Omega_2(t),\end{cases}\qquad \mu(\mathbf{x},t)=\begin{cases}\mu_1 & \forall \mathbf{x}\in\Omega_1(t)\\ \mu_2 & \forall \mathbf{x}\in\Omega_2(t)\end{cases}$$

- Recall that discontinuities may be classified into strong and weak. In the case of strong discontinuities, a **jump** and a change in the gradient are present in the field. For weak discontinuities there is only a **kink** in the field



strong and weak discontinuity

Discontinuous state variables at the interface

$$[\mathbf{u}]_{\Gamma_d} = \mathbf{0} \quad \text{on } \Gamma_d \times (0, t_{\text{end}})$$

$$-[\boldsymbol{\sigma}]_{\Gamma_d} \cdot \hat{\mathbf{n}} = \gamma \cdot \kappa \cdot \hat{\mathbf{n}} \quad \text{on } \Gamma_d \times (0, t_{\text{end}})$$

- Interface condition states that the velocities are continuous across interface, or, in other words, the jump in the velocity field is zero
- The second interface condition, states that the surface tension balances the jump of the normal stress at the interface

Thus

- The velocity fields $\mathbf{u}(\mathbf{x}, t)$ **are weakly discontinuous**, here as the pressure field $p(\mathbf{x}, t)$ **has a strong discontinuity** at the interface

XFEM in computational fluid mechanics

Discontinuous state variables at the interface

- The presence of strong and weak discontinuities when simulating two-fluid flows has been realized by many different approaches: In interface tracking methods, weak discontinuities are accounted for by the standard FEM automatically

- Strong discontinuity in the pressure field can be realized by using completely decoupled meshes for the two-fluid regions. However, these methods often do not allow topological changes of the two-fluid domains

- In interface-capturing methods, the strong and weak discontinuities may be considered by local h-refinement in the vicinity of the interface. For interface-capturing methods, especially strong discontinuities often pose a serious problem for the accuracy of the overall simulation

- Enriched methods such as the XFEM are ideally suited for two-fluid flow simulations as they consider both, strong and weak discontinuities within elements, by special enrichments of the approximation space

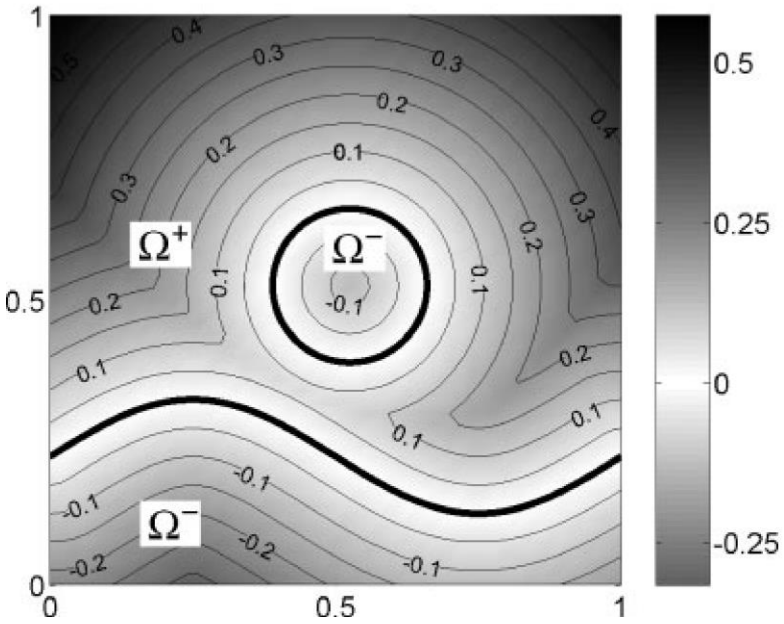
Description of the interfaces by the level-set method

- The signed-distance function is used as a particular level-set function,

$$\phi(\mathbf{x}) = \pm \min_{\mathbf{x}^\star \in \Gamma^\pm} \|\mathbf{x} - \mathbf{x}^\star\| \quad \forall \mathbf{x} \in \Omega$$

- Level-set function is interpolated by standard FE shape function

$$\phi^h(\mathbf{x}) = \sum_{i \in I} N_i^{\mathrm{FEM}}(\mathbf{x}) \phi_i$$



- Standard FE shape function is used for all elements that are not cut by the interface
- In cut elements, we **subdivide** the quadrilateral element into **two triangulars** and employ linear interpolation functions

XFEM in computational fluid mechanics

The intrinsic XFEM for two-fluid flows

intrinsic XFEM falls into three steps

1. **The domain is decomposed into subdomains that overlap in one element layer**

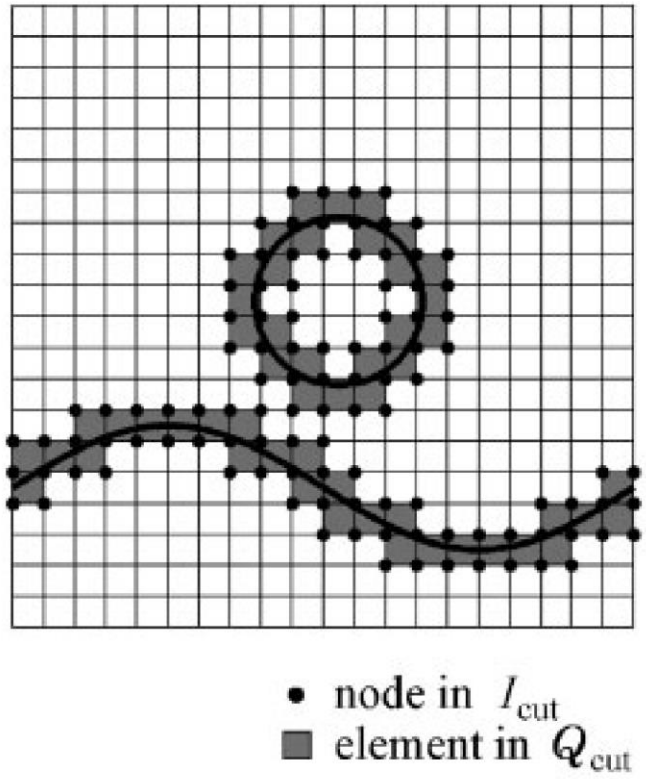
2. **Construction of shape functions for all nodes of each subdomain**

They build partition of unities (PUs) with certain properties. Standard finite element or special enriched MLS functions are employed here

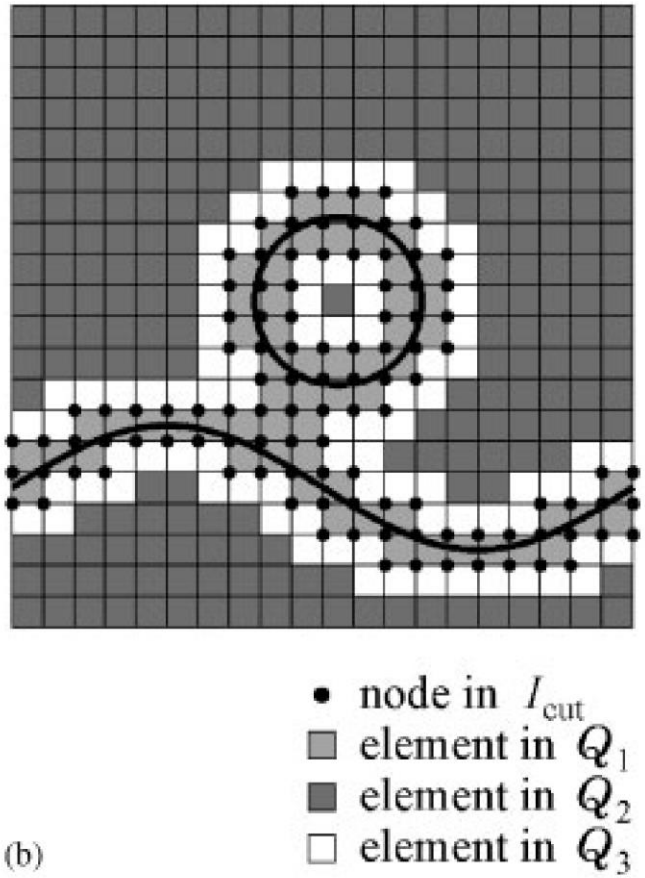
3. **The shape functions in the overlapping element layers are coupled such that only one shape function per node results**

Decomposition of the domain

- The set of nodes associated with elements that are cut by the interface : I_{cut}



(a)



(b)

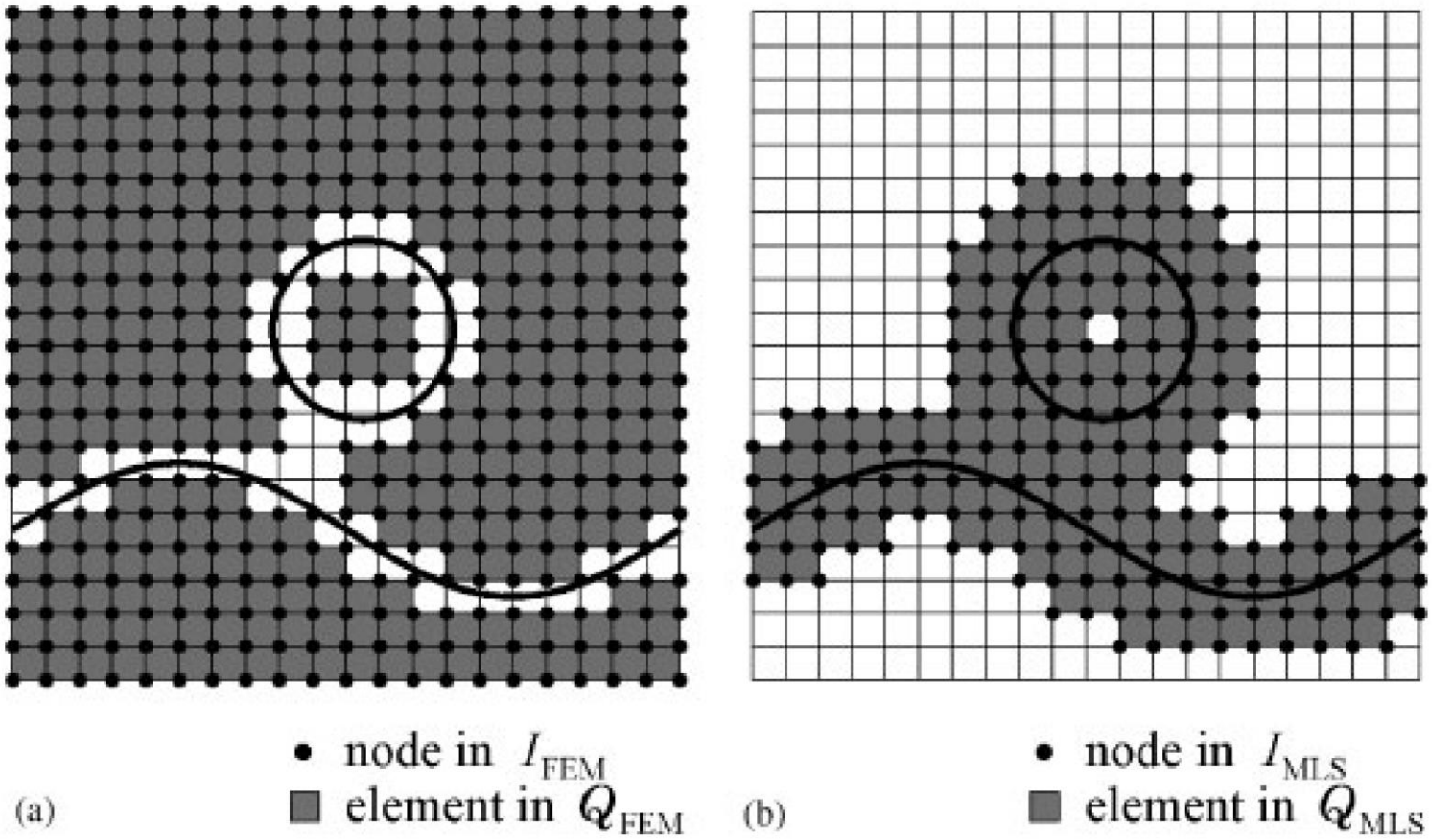
Q_1 : cut element

Q_2 : standard element

Q_1 : blending element

Decomposition of the domain

- Subdomain Q_{FEM} and Q_{MLS} (overlapped)



XFEM in computational fluid mechanics

Construction of PUs in each subdomain

- In each subdomain Q_{FEM} and Q_{MLS}, a set of shape functions is constructed for each of the nodes in I_{FEM} and I_{MLS}, respectively

- Standard bi-linear finite element shape functions are used in Q_{FEM}. It is noted that these shape functions are not constructed in cut elements, because they are not suited for capturing the strong or weak discontinuity across the interface

- In Q_{MLS}, which is in the vicinity of the interface, special enriched MLS functions are constructed. The enrichment enables the shape functions to represent jumps and kinks in the solution of a field variable

Moving least-squares (MLS) method

- **MLS shape functions is defined as**

$$N_i^{\mathrm{MLS}}(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})[\mathbf{M}(\mathbf{x})]^{-1} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)$$

$$\text{with} \quad \mathbf{M}(\mathbf{x}) = \sum_{i \in I_{\mathrm{MLS}}} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)$$

$w_i(\mathbf{x})$: **MLS weight functions**
$\mathbf{p}(\mathbf{x})$: **the intrinsic basis consisting of *k* components**

- **The set of MLS functions builds a PU over the domain Ω_{MLS}, i.e. for any function in the basis $\mathbf{p}(\mathbf{x})$:**

$$\sum_{i \in I_{\mathrm{MLS}}} N_i^{\mathrm{MLS}}(\mathbf{x})\mathbf{p}(\mathbf{x}_i) = \mathbf{p}(\mathbf{x}), \quad \mathbf{x} \in \Omega_{\mathrm{MLS}}$$

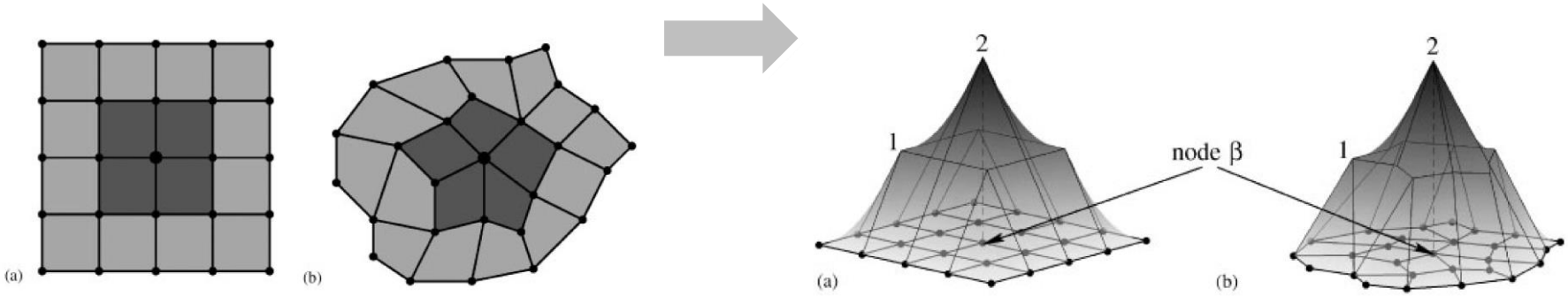
XFEM in computational fluid mechanics

Special weight functions $w_i(\mathbf{x})$

The weight functions must be constructed so as to ensure a sufficient overlap such that the MLS moment matrices are invertible

$$w_\beta(\mathbf{x}) = 2 \cdot N_\beta^{\mathrm{FEM}}(\mathbf{x}) + \sum_{i \in I_\beta^\star} N_i^{\mathrm{FEM}}(\mathbf{x})$$

The weight function corresponding to the center node has a support which includes the neighboring elements of that node (dark-gray area) and the next-neighboring elements (light-gray area)



Basis functions for discontinuities

- For smooth solutions, the basis vector $\mathbf{p}(\mathbf{x})$ of the MLS often consists of monomials depending on the desired order of accuracy

For example, $\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = [1, x, x^2]$ serves as a quadratic basis in one dimension, $\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = [1, x, y]$ as a linear basis in two dimensions

- In contrast to finite element shape functions, the basis in the MLS method can be easily enriched by any desirable terms which enable the resulting MLS shape functions to improve their approximation properties in the presence of non-smooth solutions

For weak discontinuities we define

$$\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = [1, x, y, \mathrm{abs}(\phi(\mathbf{x}))]$$

For strong discontinuities we define

$$\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = [1, x, y, \mathrm{abs}(\phi(\mathbf{x})), \mathrm{sign}(\phi(\mathbf{x}))]$$

Coupling the PUs

Different sets of shape functions are individually defined over the overlapping subdomains Ω_{FEM} and Ω_{MLS} with respect to the nodal sets $\mathrm{I}_{\mathrm{FEM}}$ and $\mathrm{I}_{\mathrm{MLS}}$, respectively. The subdomains Ω_{FEM} and Ω_{MLS}overlap in the transition area Ω_{TRANS}

- In elements that are in $\Omega_{\mathrm{FEM}} \backslash \Omega_{\mathrm{TRANS}}$, only finite element shape functions are evaluated

$$N_i(\mathbf{x}) = N_i^{\mathrm{FEM}}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{\mathrm{FEM}} \setminus \Omega_{\mathrm{trans}} \quad \forall i \in I_{\mathrm{FEM}}$$

- For elements in $\Omega_{\mathrm{MLS}} \backslash \Omega_{\mathrm{TRANS}}$, only MLS shape functions are present

$$N_i(\mathbf{x}) = N_i^{\mathrm{MLS}}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{\mathrm{MLS}} \setminus \Omega_{\mathrm{trans}} \quad \forall i \in I_{\mathrm{MLS}}$$

- In the transition area Ω_{TRANS}, for each node in $\mathrm{I}_{\mathrm{TRANS}}$, a coupling is required

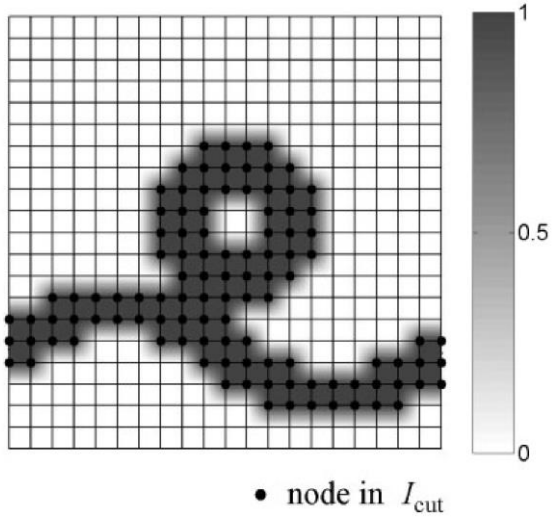
Coupling the PUs

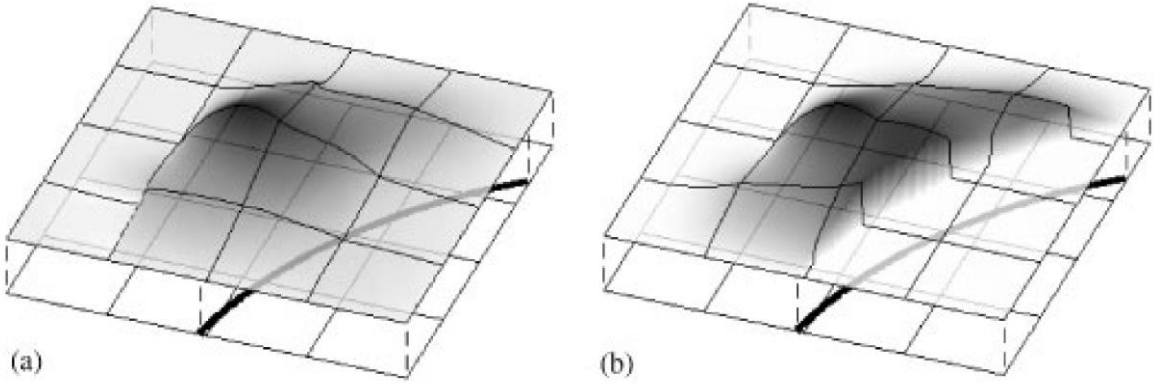
$$R(\mathbf{x}) = \sum_{i \in I_{\text{cut}}} N_i^{\text{FEM}}(\mathbf{x})$$

- By the definition of ramp function $R(\mathbf{x})$, the resulting shape function is

$$N_i(\mathbf{x}) = N_i^{\text{FEM}}(\mathbf{x}) \cdot [1 - R(\mathbf{x})] + N_i^{\text{MLS}}(\mathbf{x}) \cdot R(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{\text{trans}} \quad \forall i \in I_{\text{trans}}$$

ramp function $R(\mathbf{x})$





Shape functions of a particular node for the case of a (a) weak discontinuity and (b) strong discontinuity

Governing equations in weak form

- The streamline-upwind Petrov–Galerkin (SUPG) and pressure-stabilizing Petrov–Galerkin (PSPG) formulation of the weak form is considered here, recall that

$$\int_\Omega \mathbf{w}^h \cdot \varrho_i \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \nabla \mathbf{u}^h \right) \mathrm{d}\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(\mathbf{u}^h, p^h)\, \mathrm{d}\Omega$$

$$- \int_{\Gamma_\mathbf{h}} \mathbf{w}^h \cdot \hat{\mathbf{h}}\, \mathrm{d}\Gamma + \int_\Omega q^h \nabla \cdot \mathbf{u}^h\, \mathrm{d}\Omega + \sum_{j \in \mathcal{Q}} \int_{\Omega_j^{\mathrm{el}}} \tau_j \left(\mathbf{u}^h \cdot \nabla \mathbf{w}^h + \frac{1}{\varrho_i} \nabla q^h \right)$$

$$\cdot \left[\varrho_i \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \nabla \mathbf{u}^h \right) - \nabla \cdot \boldsymbol{\sigma} \left(\mathbf{u}^h, p^h \right) \right] \mathrm{d}\Omega$$

$$= \int_\Omega \varrho_i \mathbf{w}^h \cdot \mathbf{f}\, \mathrm{d}\Omega + \int_{\Gamma^\pm} \gamma \kappa \mathbf{w}^h \cdot \hat{\mathbf{n}}\, \mathrm{d}\Gamma$$

- The discretization of weak form will not be presented here, since it has no more differences as we discussed in last course, except the enrichment terms

Numerical Integration

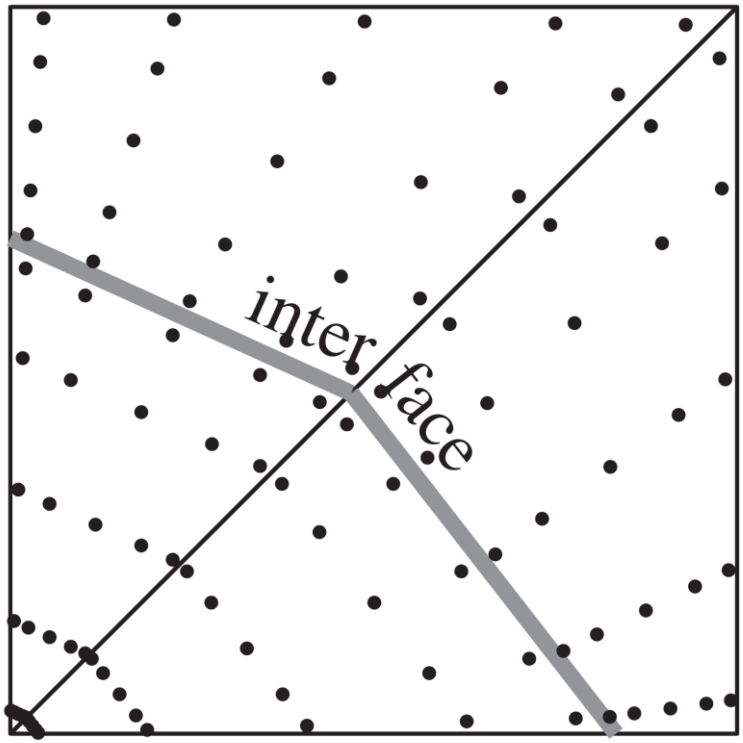
- In elements not cut by the interface, the proposed shape functions of the intrinsic XFEM are sufficiently smooth such that standard Gauss integration is suitable

- However, in cut elements, the shape functions have jumps or kinks along the interface depending on the enrichment. This must be considered adequately which is achieved by dividing the elements into integration cells

The partitioning depends on the interpolation of the level-set function $\phi(\mathbf{x})$

Numerical Integration

- E.g., here each cut quadrilateral element is subdivided into two triangulars. In each triangular, linear interpolation functions are employed
- In each integration cell, standard Gauss points are placed



Moving interfaces in the level-set method

- The level-set method is used for the implicit description of the moving interfaces between the two fluids

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$

- The SUPG-stabilized and discretized weak form may be stated as

$$\int_{\Omega} (\psi^h + \tau \mathbf{u}^h \cdot \nabla \psi^h) \cdot \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \nabla \phi^h \right) \mathrm{d}\Omega = 0$$

- Standard bi-linear shape functions can be used. This is justified due to the fact that the level-set function is smooth everywhere in the computational domain

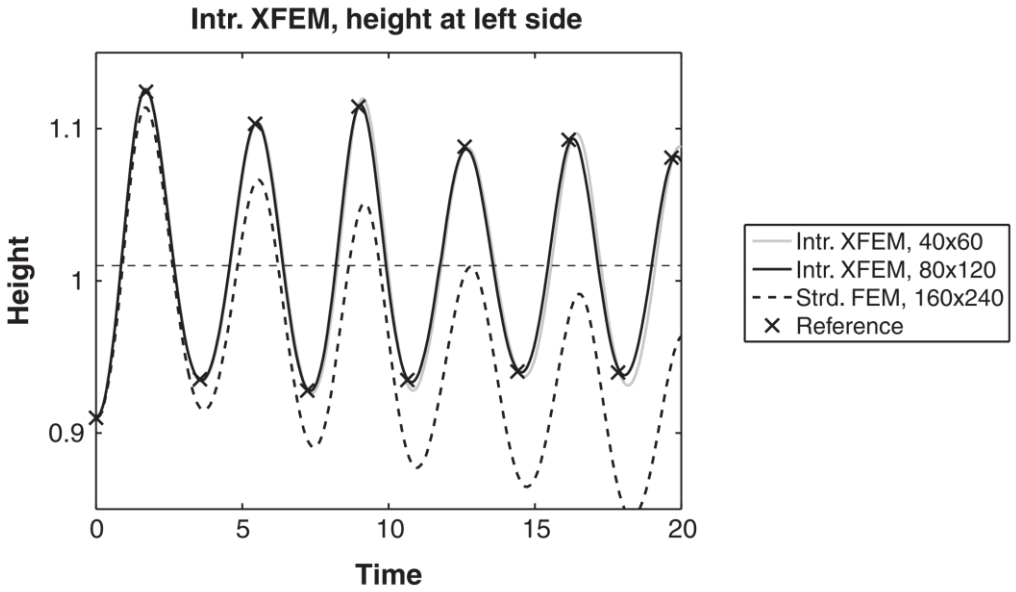
Numerical results

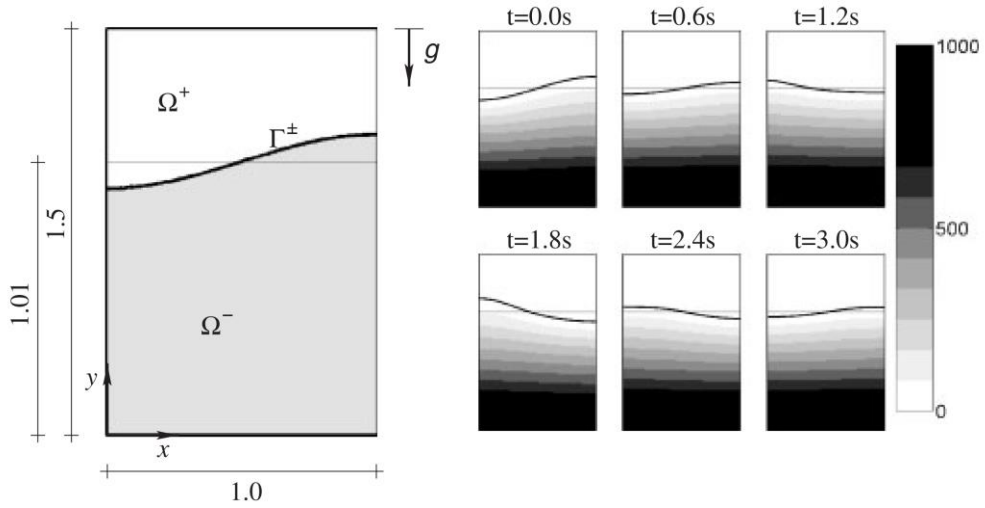
- **Sloshing tank**

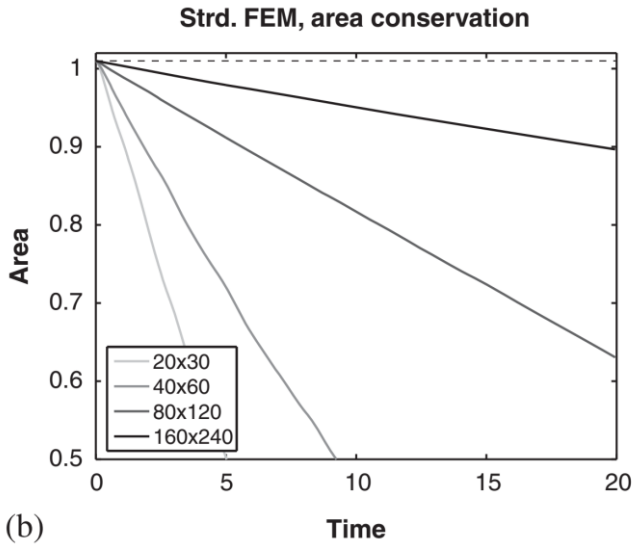
Initially, the two fluids are separated by a sinusoidal interface

$$\Gamma^{\pm}=\{(x,y): y=1.01+0.1\cdot\sin((x-0.5)\cdot\pi), 0\leqslant x\leqslant 1\}$$

The properties of two fluids are set up as water and gas, no surface tension is considered here



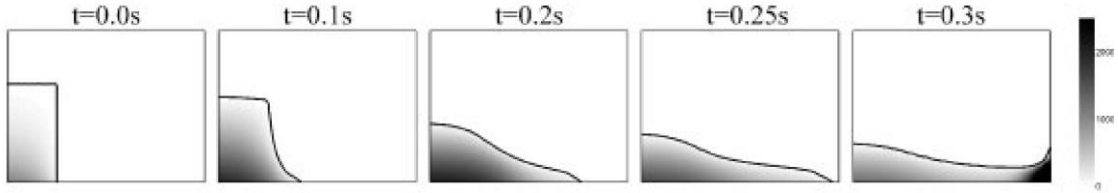


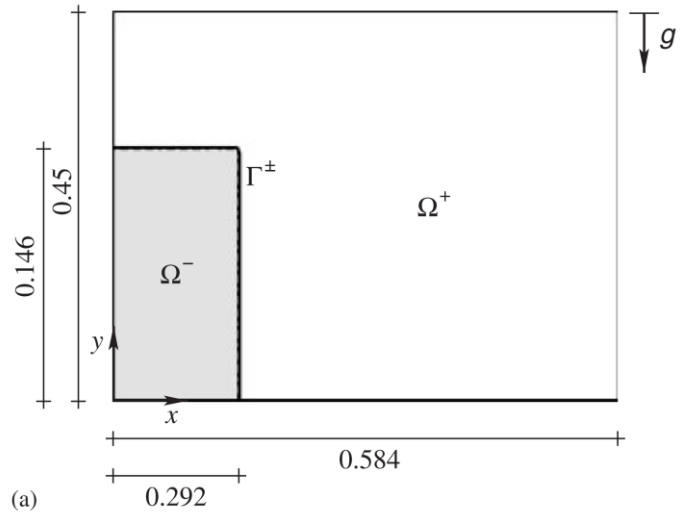


Numerical results

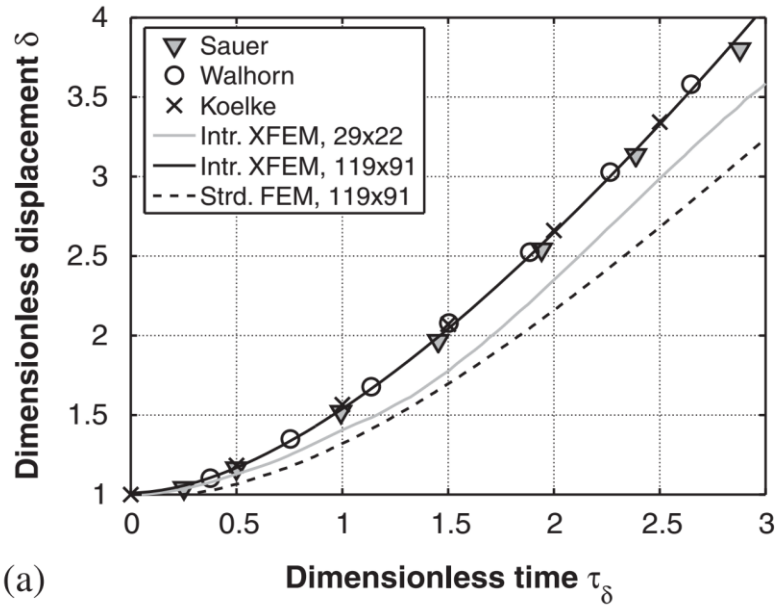
- **Collapsing water column**

  This test case considers a collapsing water column in a domain 0.584m×0.45m

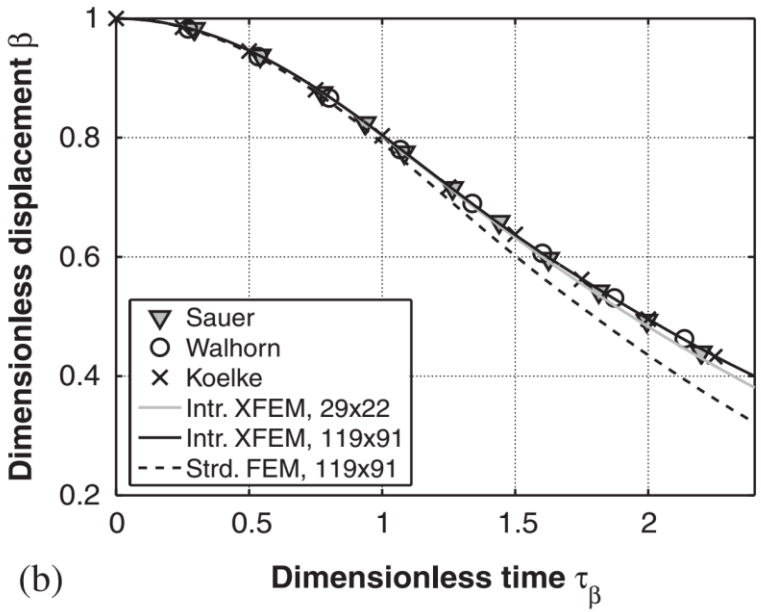




Water column, width



Water column, height



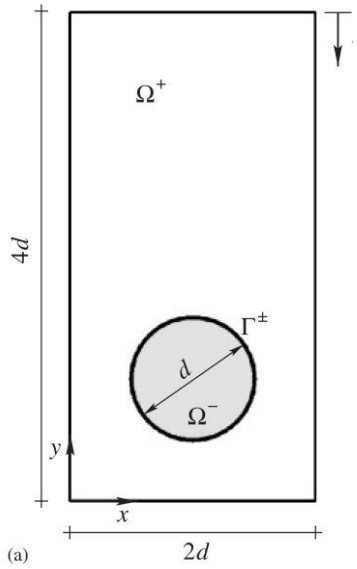
Numerical results

- **Rising bubble**

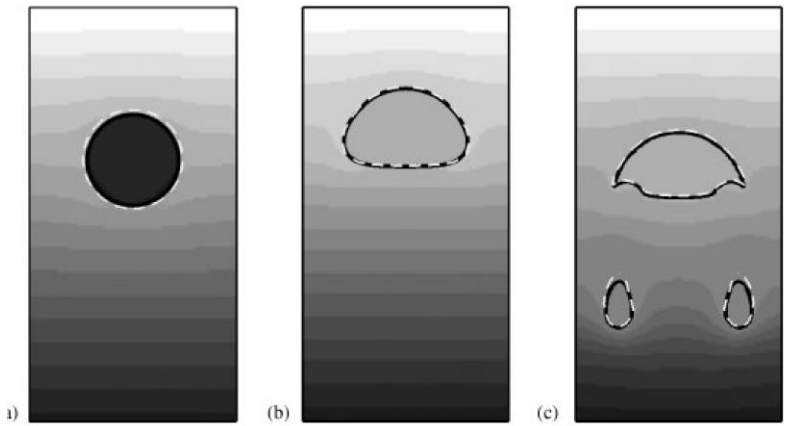
Bubble flows at different Eotvos (Eo) numbers are considered

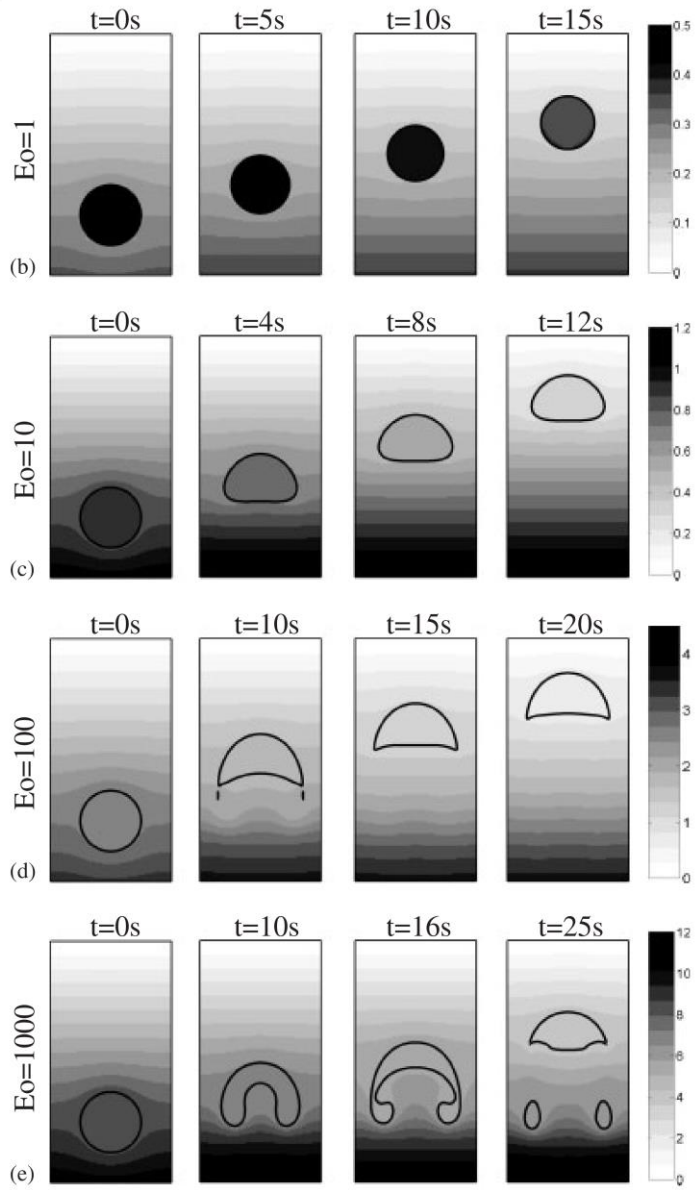
$$Eo = \frac{g \cdot (\rho^+ - \rho^-) \cdot d^2}{\gamma}$$

Surface tension $\gamma = 10^{-3}\,\mathrm{kg/s^2}$



Comparison of the interface positions obtained by the intrinsic XFEM and standard XFEM (dashed white lines)





Summary

- The approximation space used in the intrinsic XFEM is able to represent inner-element jumps and kinks exactly. This approximation space is built by standard finite element shape functions in the majority of the domain and special enriched moving least-squares functions in the vicinity of the interface
- Inbetween the FE and MLS regions, a coupling of the two classes of shape functions is realized
- Most importantly, the resulting approximation in the intrinsic XFEM has the same number of unknowns as a classical FE approximation
- The computational work for the evaluation of the MLS shape functions is increased, however, these functions are only needed locally near the interface

谢 谢！

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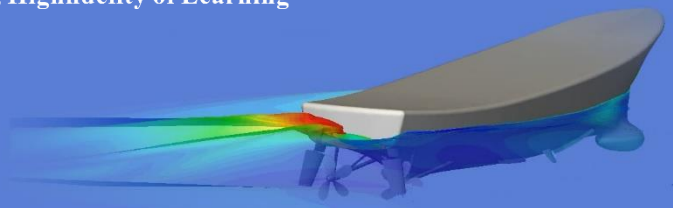

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