上海交通大学
SHANGHAI JIAO TONG UNIVERSITY

SJTU COMPUTATIONAL MARINE HYDRODYNAMICS LAB
上海交大船舶与海洋工程计算水动力学研究中心

Class-10

NA26018

Finite Element Analysis of Solids and Fluids

万德成

dcwan@sjtu.edu.cn， http://dcwan.sjtu.edu.cn/

上海交通大学

船舶海洋与建筑工程学院
海洋工程国家重点实验室

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XFEM formulation and element integration

The purpose of this section is to introduce the enrichment associated with XFEM such that the structure of the finite element equation set is revealed

The general formulation of the strong form for a body in equilibrium is

$$\tilde{\nabla}\boldsymbol{\sigma} + \mathrm{b} = 0$$

Where σ is the stress tensor and b is the forces in the body, i.e. gravity

The next step is to consider a weak form, or a virtual energy form of the strong formulation of the body's behavior

By multiplying with a arbitrary weight function, v, and integrating over the whole body. The equation then takes the weak form:

$$\int_V (\tilde{\nabla}\mathbf{v})^T \boldsymbol{\sigma}\, dV = \int_S \mathbf{v}^T \mathbf{t}\, dS + \int_V \mathbf{v}^T \mathbf{b}\, dV$$

where t is the surface loading and $\tilde{\nabla}$ is the gradient operator acting on a vector in matrix format

The next step is the choice of the weight function, v

XFEM formulation including enrichment

Weight function v is chosen by the so called Galerkin's method, with the enrichment considered, v and its gradient may be written as

$$\mathbf{v} = \mathbf{N}\mathbf{c} + \mathbf{N}_{enr}\mathbf{q}$$

$$\tilde{\nabla}\mathbf{v} = \mathbf{B}\mathbf{c} + \mathbf{B}_{enr}\mathbf{q}$$

- c is the usual nodal quantities and
- q (additional unknowns) is the part related to the enrichment term
- N and B are the usual shape function matrix and the strain displacement matrix respectively

Insert Eqs which describes the weight function, into weak form. This will produce

$$\mathbf{c}^T[\int_V \mathbf{B}^T\boldsymbol{\sigma}\,dV \int_S \mathbf{N}^T\mathbf{t}\,dS \int_V \mathbf{N}^T\mathbf{b}\,dV]$$
$$+\mathbf{q}^T[\int_V \mathbf{B}_{enr}^T\boldsymbol{\sigma}\,dV \int_S \mathbf{N}_{enr}^T\mathbf{t}\,dS \int_V \mathbf{N}_{enr}^T\mathbf{b}\,dV] = 0$$

Since both c and q are arbitrary, above Eq. can be divided into two parts as

$$\int_V \mathbf{B}^T\boldsymbol{\sigma}\,dV = \int_S \mathbf{N}^T\mathbf{t}\,dS \int_V \mathbf{N}^T\mathbf{b}\,dV$$
$$\int_V \mathbf{B}_{enr}^T\boldsymbol{\sigma}\,dV = \int_S \mathbf{N}_{enr}^T\mathbf{t}\,dS \int_V \mathbf{N}_{enr}^T\mathbf{b}\,dV$$

XFEM in computational solid mechanics

We now have two equations which must both be fulfilled; one with the standard FEM equations and one which involve terms that comes from the enrichment

The next step is to find a constitute relation between stress and strain and to define and approximation for the displacement,

$$\boldsymbol{\sigma} = \mathrm{D}\boldsymbol{\epsilon}$$

$\boldsymbol{\epsilon}$ is the strain matrix defined as

$$\boldsymbol{\epsilon} = \tilde{\nabla}\mathbf{u}$$

and $\mathbf{u} = \mathrm{N}\mathbf{u}_{std} + \mathrm{N}_{enr}\mathbf{u}_{xtra}$

u_{std} are the standard nodal displacements and u_{xtra} are related to the enrichment part

Insert constitute relation and approximation of u into the weak form, we obtain

$$\int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\,dV\,\mathbf{u}_{std} + \int_V \mathbf{B}^T\mathbf{D}\mathbf{B}_{enr}\,dV\,\mathbf{u}_{xtra} = \int_S \mathbf{N}^T\mathbf{t}\,dS \int_V \mathbf{N}^T\mathbf{b}\,dV \qquad \textbf{(a)}$$

$$\int_V \mathbf{B}_{enr}^T\mathbf{D}\mathbf{B}\,dV\,\mathbf{u}_{std} + \int_V \mathbf{B}_{enr}^T\mathbf{D}\mathbf{B}_{enr}\,dV\,\mathbf{u}_{xtra} = \int_S \mathbf{N}_{enr}^T\mathbf{t}\,dS \int_V \mathbf{N}_{enr}^T\mathbf{b}\,dV \qquad \textbf{(b)}$$

Defining the XFEM and FEM matrices

From Eqs (a) and (b), there will be 3 different types of stiffness matrices;

- **one that is the normal stiffness matrix for FEM,**
- **one with only enrichment**
- **2 combinations using both the standard and enriched B-matrix,**

The matrices involving a combination of normal and enriched B-matrices are normally referred to as "blended" stiffness

The equation system can be written as

$$\begin{bmatrix} \mathbf{K}_{std} & \mathbf{K}_{blend} \\ \mathbf{K}_{blend}^T & \mathbf{K}_{xtra} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{std} \\ \mathbf{u}_{xtra} \end{bmatrix} = \mathbf{F}$$

where

$$\mathbf{K}_{std} = \int \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV$$

$$\mathbf{K}_{blend} = \int \mathbf{B}^T \mathbf{D} \mathbf{B}_{enr} \, dV$$

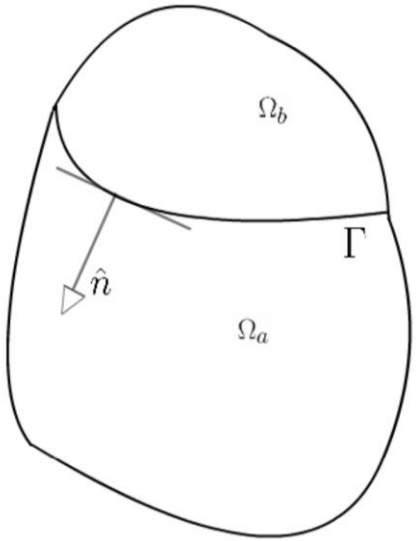
$$\mathbf{K}_{xtra} = \int \mathbf{B}_{enr}^T \mathbf{D} \mathbf{B}_{enr} \, dV$$

Level Sets

As the idea of the XFEM is to capture discontinuities over some boundary without mesh adjustment, it is vital to be able to keep track of this interface

- **The most common way to do this is with a level set function**
- **The level set $\phi(\mathbf{X})$ function has the property of:**

$$\phi(\mathbf{X}) < 0 \; for \; \mathbf{X} \in \Omega_b$$
$$\phi(\mathbf{X}) > 0 \; for \; \mathbf{X} \in \Omega_a$$
$$\phi(\mathbf{X}) = 0 \; for \; \mathbf{X} \in \Gamma$$



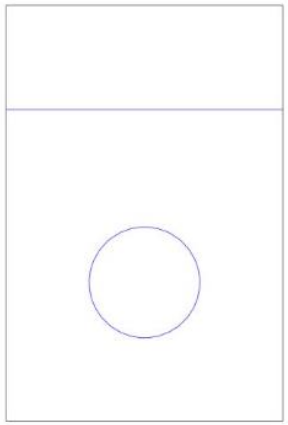
XFEM in computational solid mechanics

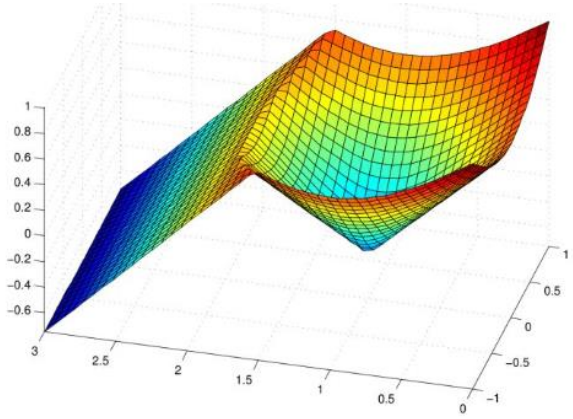
- The sign of the level set function can now be used to reveal what part of the domain certain coordinates X belong to
- Zero value means the point of interest is located on the interface itself
- A convenient choice of level set function is the **signed distance function**

Let us define a vector $\mathbf{d}$ as the minimum distance from a point of interest $\mathbf{X}$ to the boundary $\mathbf{X}_\Gamma$. Using the outwards normal $\hat{n}$ to the interface the signed distance function is defined as

$$\phi(\mathbf{X}) = \mathbf{d} \cdot \hat{n}$$

e.g. two-phase flow (bubble under free surface)





Zero distance curve and distance function field for the test case

Crack Specific Treatment:

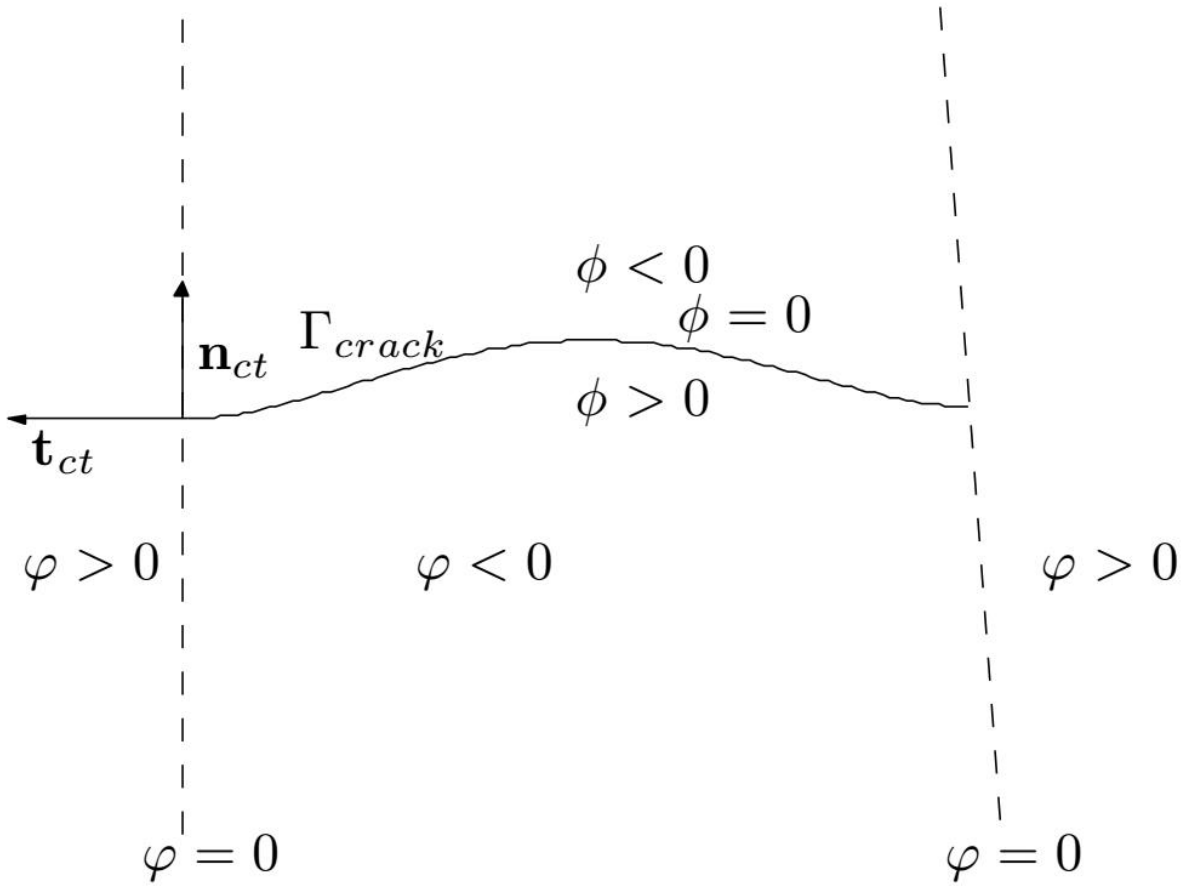
As a crack can not split a body into two well defined, separate regions on its own, it is not enough to define the discontinuity with the signed distance function normal to the interface

- This is preferably solved by adding an extra level set function φ, this time represented as the signed distance tangential to the closest crack tip from the query point
- φ can be defined with two help functions, one for each tip

$$\varphi(\mathbf{X}) = \max(\varphi_1, \varphi_2)$$
$$\varphi_i = (\mathbf{X} - \mathbf{X}_{cTip\,i}) \cdot \mathbf{t}_i$$

Where t_i is the tangent to the ith crack tip (as an imagined extension of the crack). Identifying the crack as all points containing the level sets $\{\phi = 0, \varphi \leq 0\}$

XFEM in computational solid mechanics



The two types of level set functions for cracks

Enrichment

The level set function provides knowledge of the location of the geometry. The remaining part is to define the enrichment function in the XFEM approximation

- The idea is to add new degrees of freedom to the system and superimpose these on top of the standard FEM DOFs with some weight function

- The task of the enrichment function is to supply this weight in a fashion that captures the behaviour of the discontinuity

As the behaviour of the discontinuity strongly relates to the shape of the interface, it is common to choose an enrichment function formulated with the level set function

Partition of Unity

As the discontinuities are an uniquely local event some restriction must be applied. A partition of unity is a set of functions which sum is one over a specified domain Ω^{PU}

$$\sum_i f_i(\mathbf{X}) = 1$$

$$\forall\, \mathbf{X} \in \Omega^{PU}$$

The partition of unity method allows for the introduction of enrichment function $\psi(\mathbf{X})$

A FEM approximation using a partition of unity function f, over a domain consisting of M nodes

$$u^{aprox}(\mathbf{X}) = \sum_{I=1}^{M} f_I u_I^{std}$$

Adding the XFEM part to this, the domain has L enriched nodes starting with node number L_1

$$u^{aprox}(\mathbf{X}) = \sum_{I=1}^{M} f_I u_I^{std} + \sum_{I=1}^{M} f_I \sum_{J=L_1}^{L} \psi(\mathbf{X}) u_J^{xtra}$$

When introducing the shape functions, i.e.

$$f_I = N_I$$

this approximation becomes exclusively local

$$u^{aprox}(\mathbf{X}_I) = u_I^{std} + \psi(\mathbf{X}_I) u_I^{xtra}$$

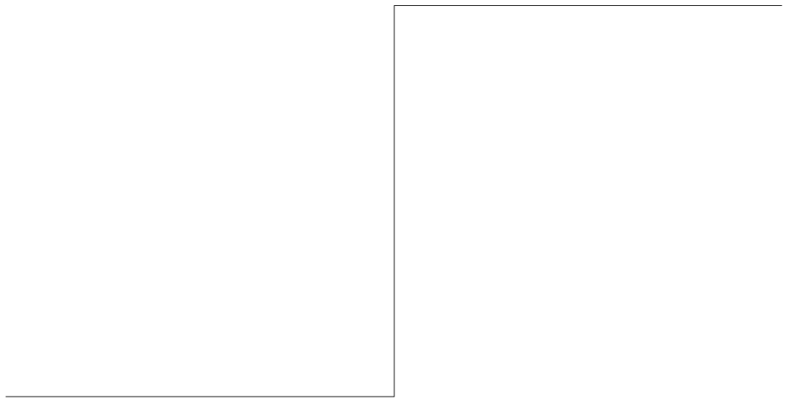
Strong vs. Weak Discontinuities

The single most important criteria for choosing enrichment function is weather the discontinuity is strong or weak

For Strong discontinuity:

A strong discontinuity has a jump in some field variable. E.g., holes and cracks.

Step type enrichment function

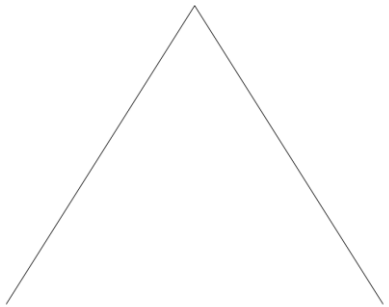


- **Depending on the specific type of discontinuity, the chosen enrichment function are often some sort of binary on/off type of function. Examples of this are the Heaviside and step- functions**

For Weak discontinuity:

A weak discontinuity has a kink, rather than jump in some field variable.
E.g., inclusion of another material

- In these cases the field (displacement) is actually continuous and the discontinuity only appears in the derivative (strain) of the primary field
  Examples of this are the ramp functions



Ramp type of enrichment function

Enrichment function for Voids/Holes

A common choice of enrichment function for voids is the Heaviside function

$$H(\mathbf{X}) = \begin{cases} 1 \; if \; \phi(\mathbf{X}) > 0 \\ 0 \; if \; \phi(\mathbf{X}) < 0 \end{cases}$$

Note:

- A binary distinction between material and no material is achieved (negative level set value indicates inside the void)

- In practice, it is common to remove all degrees of freedom associated with elements located completely inside the void and only consider contributions from elements only containing material and the elements whose nodal support is cut of by the void

Enrichment function for Inclusions

Inclusions fall under the category of weak discontinuities. As mentioned earlier, it is not the primary field that is discontinuous but its derivative. An obvious contender is the following type of ramp function

$$\psi(\mathbf{X}) = |\,\phi(\mathbf{X})\,| = \left|\sum_I N_I \phi_I\right|$$

- A continuous shape can be acquired by a kink of undefined derivative where ϕ equals zero
- It is sufficient for most purposes but might introduce convergence problems in blended elements

Alternatively,

$$\psi(\mathbf{X}) = \sum_I |\phi_I| N_I - \left| \sum_I \phi_I N_I \right|$$

- It has the advantage of being zero at all points outside the intersected elements
- The blended elements will not contribute to the extended parts of the stiffness matrix

Enrichment function for Cracks

Besides having a discontinuous primary field (displacement) on opposite sides of the crack, special consideration must be given to the crack tip

- **Because elements containing a crack tip can not be fully divided by the crack, a step function can not be used as enrichment function here, for this purpose the extra level set function comes in handy**

First, define following help variables using the level set functions ϕ and φ

$$\theta = \arctan \frac{\varphi}{\phi}$$

$$r = \sqrt{\phi^2 + \varphi^2}$$

Without diving too deep into the realm of fracture mechanics, it can be stated that for linear elastic fracture mechanics, the displacement field near the crack tip, can be written as

$$u_x = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right) + \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 + 2\cos^2\frac{\theta}{2}\right)$$

$$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right) - \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 - 2\sin^2\frac{\theta}{2}\right)$$

Where K_I and K_{II} are the mode 1 and 2 stress intensity factors. The Koslov constant is defined as

$$\kappa = 3 - 4\nu \qquad (plane\ stress)$$

$$\kappa = \frac{3-\nu}{1+\nu} \qquad (plane\ strain)$$

It can be shown that the crack tip displacement field is contained by four functions

$$\gamma(r,\theta) = \left[\begin{array}{c} \sqrt{r}\cos\frac{\theta}{2} \\ \sqrt{r}\sin\frac{\theta}{2} \\ \sqrt{r}\sin\frac{\theta}{2}\sin\theta \\ \sqrt{r}\cos\frac{\theta}{2}\sin\theta \end{array} \right]$$

- **It is possible to combine these functions for the crack tip enrichment**
- **This enrichment requires that one new degree of freedom per node is introduced for each of these four functions**

XFEM in computational solid mechanics

- **If we consider a domain containing crack tip enrichment is denoted Ω_{ct} and the part of with normal crack enrichment Ω_H, the XFEM displacement approximation for such a case would read as**

$$u^{aprox}(\mathbf{X}_I) = u_I^{std} + H(\mathbf{X}_I)u_I^{xtra}$$
$$for I \in \Omega_H$$
$$u^{aprox}(\mathbf{X}_I) = u_I^{std} + \gamma_1(\mathbf{X}_I)u_I^{xtra1} + \gamma_2(\mathbf{X}_I)u_I^{xtra2} + \gamma_3(\mathbf{X}_I)u_I^{xtra3} + \gamma_4(\mathbf{X}_I)u_I^{xtra4}$$
$$for I \in \Omega_{ct}$$

Note: no node can have with both heaviside and crack tip enrichment

Implementation for Straight Crack

1. Enrichment Function

Two types of enrichments are needed, Heaviside and cracktip

- **Heaviside enrichment accounts for the loss of connection between nodes in an element cut by the crack,**
- **Cracktip enrichment is used for capturing special crack tip behavior**
- **In the crack case the role of this enrichment is to break connection between the two sides of the crack**
- **Either side still contains material, however there is no contribution to the stiffness to the other side**

In practice, for Heaviside enrichment, for an arbitrary Gauss point dealing with element node i:

$$H(\mathbf{X_i}) = \frac{1 + sign(\phi(\mathbf{X}_{gp}))sign(\phi(\mathbf{X}_i))}{2}$$

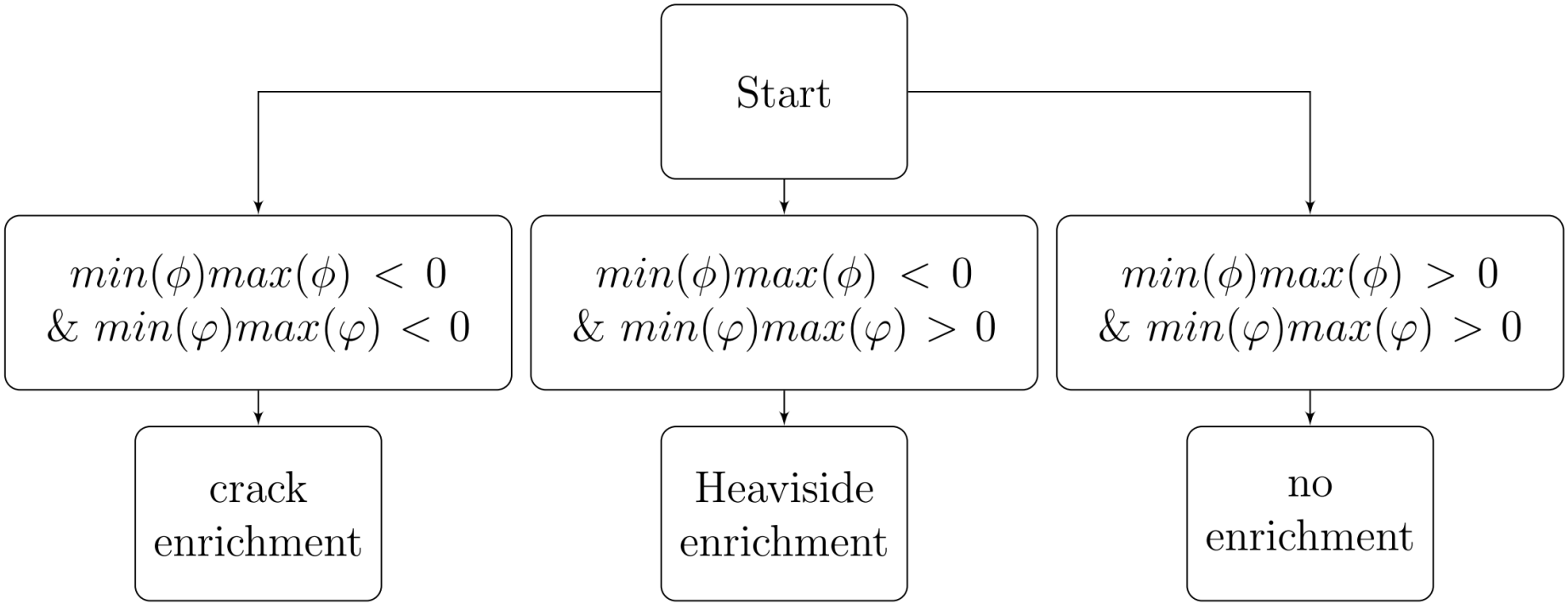
Cracktip enrichment needs to be expressed in crack tip coordinates θ and r, the entire order of operations are as follows

1. Calculate crack angle $\omega_1 = \arctan\left(\frac{y_{ct2}-y_{ct1}}{x_{ct2}-x_{ct1}}\right)$
2. Rotate coordinate system $\mathbf{X}_{roti} = \begin{bmatrix} cos(\omega_i) & sin(\omega_i) \\ -sin(\omega_i) & cos(\omega_i) \end{bmatrix} (\mathbf{X} - \mathbf{X_{cti}})$
3. Calculate help variables r and θ
   (a) $r_i = \sqrt{x_{roti}^2 + y_{roti}^2}$
   $r = min(r_1, r_2)$
   (b) $\theta = atan2(y_{rot}, x_{rot})$
4. use help variables to extract the four crack tip enrichment functions.
   (a) $f_1 = \sqrt{r}\ \cos(\frac{\theta}{2})$
   (b) $f_2 = \sqrt{r}\ \sin(\frac{\theta}{2})$
   (c) $f_3 = \sqrt{r}\ \sin(\theta)\ \sin(\frac{\theta}{2})$
   (d) $f_4 = \sqrt{r}\ \sin(\theta)\ \cos(\frac{\theta}{2})$

Implementation for Straight Crack

2. Stiffness Matrix

First the code determines for each element if it is enriched or not and what kind of enrichment should be used



XFEM in computational solid mechanics

- Note that node can not have both Heaviside and crack tip enrichment.
- If two neighboring elements sharing one or more nodes are found to have different types of enrichment, crack tip enrichment takes precedents over Heaviside enrichment
- It is important to recall that crack tip have four types of enrichment and will add eight new DOFs to the node (four in the x-direction and four in the y direction)

The B-matrix

$$B_{tot} = [B_{std}\ B_H\ B_{f1}\ B_{f2}\ B_{f3}\ B_{f4}]$$

If node i have been found to have Heaviside enrichment, the B-matrix are extended according to

$$B_{Hi} = H_{shifti} \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

Similarly, in case node j needs crack tip extra DOFs, the code extends the B-matrix as follows

$$B_{f1j} = \begin{bmatrix} \frac{\partial N_j}{\partial x} f_{1j} + \frac{\partial f_{1j}}{\partial x} N_j & 0 \\ 0 & \frac{\partial N_j}{\partial y} f_{1j} + \frac{\partial f_{1j}}{\partial y} N_j \\ \frac{\partial N_j}{\partial y} f_{1j} + \frac{\partial f_{1j}}{\partial y} N_j & \frac{\partial N_j}{\partial x} f_{1j} + \frac{\partial f_{1j}}{\partial x} N_j \end{bmatrix}$$

The derivative of enrichment function is given in crack tip polar coordinates

XFEM in computational solid mechanics

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f_1}{\partial \theta}\frac{\partial \theta}{\partial x}$$

A comprehensive list of all crack tip enrichment derivatives with respect to global coordinates expressed in available crack tip coordinates can be obtained

$$\frac{\partial f_1}{\partial x} = \frac{1}{2\sqrt{r}}\left(\cos\left(\frac{\theta}{2}\right)\cos\left(\omega\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\omega\right)\right)$$

$$\frac{\partial f_2}{\partial x} = \frac{1}{2\sqrt{r}}\left(-\sin\left(\frac{\theta}{2}\right)\cos\left(\omega\right) - \cos\left(\frac{\theta}{2}\right)\sin\left(\omega\right)\right)$$

$$\frac{\partial f_3}{\partial x} = \frac{1}{2\sqrt{r}}\left(-\sin\left(\frac{3\theta}{2}\right)\sin(\theta)\cos(\omega) - \left(\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\cos(\theta)\right)\sin(\omega)\right)$$

$$\frac{\partial f_4}{\partial x} = \frac{1}{2\sqrt{r}}\left(-\cos\left(\frac{3\theta}{2}\right)\sin(\theta)\cos(\omega) - \left(\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{3\theta}{2}\right)\cos(\theta)\right)\sin(\omega)\right)$$

and

$$\frac{\partial f_1}{\partial y} = \frac{1}{2\sqrt{r}}\left(\cos\left(\frac{\theta}{2}\right)\sin(\omega) + \sin\left(\frac{\theta}{2}\right)\cos(\omega)\right)$$

$$\frac{\partial f_2}{\partial y} = \frac{1}{2\sqrt{r}}\left(-\sin\left(\frac{\theta}{2}\right)\sin(\omega) + \cos\left(\frac{\theta}{2}\right)\cos(\omega)\right)$$

$$\frac{\partial f_3}{\partial y} = \frac{1}{2\sqrt{r}}\left(-\sin\left(\frac{3\theta}{2}\right)\sin(\theta)\sin(\omega) + \left(\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2}\right)\cos(\theta)\right)\cos(\omega)\right)$$

$$\frac{\partial f_4}{\partial y} = \frac{1}{2\sqrt{r}}\left(-\cos\left(\frac{3\theta}{2}\right)\sin(\theta)\sin(\omega) + \left(\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{3\theta}{2}\right)\cos(\theta)\right)\cos(\omega)\right)$$

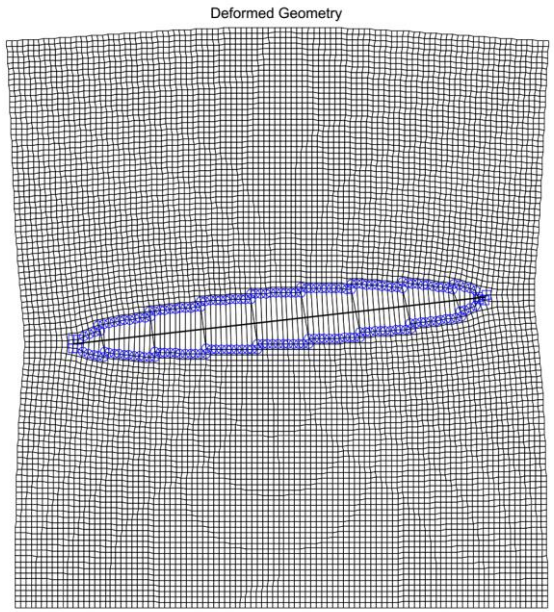
Once the extended B-matrix is sorted out the element stiffness matrix is calculated in the same manner as a previously, i.e.

$$\mathbf{K_{elm}} = \mathbf{B_{ext}}^T \mathbf{D} \mathbf{B_{ext}}$$

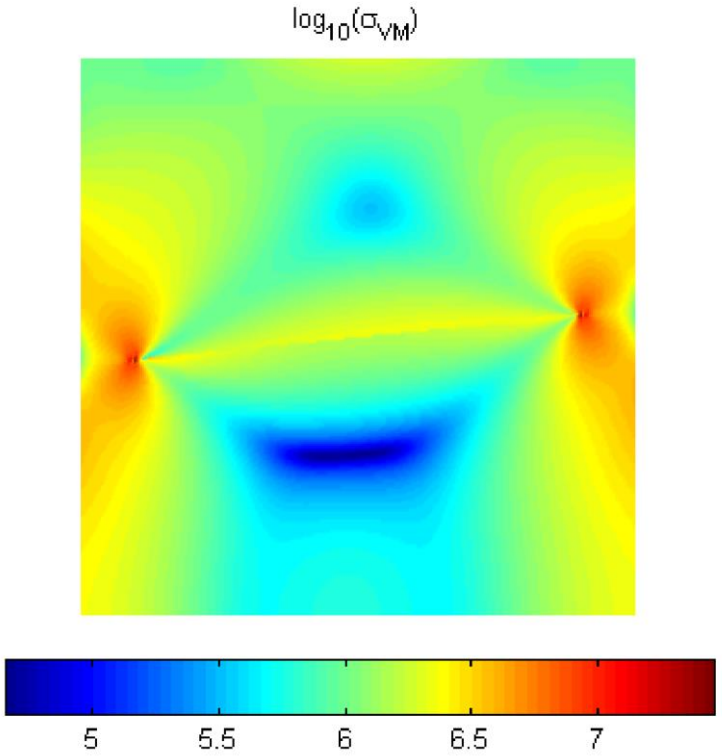
Validation

After implementing XFEM, the straight crack problem mentioned at the beginning can be simulated,

Displacement field for a crack



von Mises stresses for a crack



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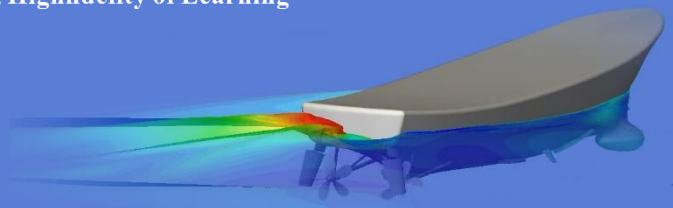

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