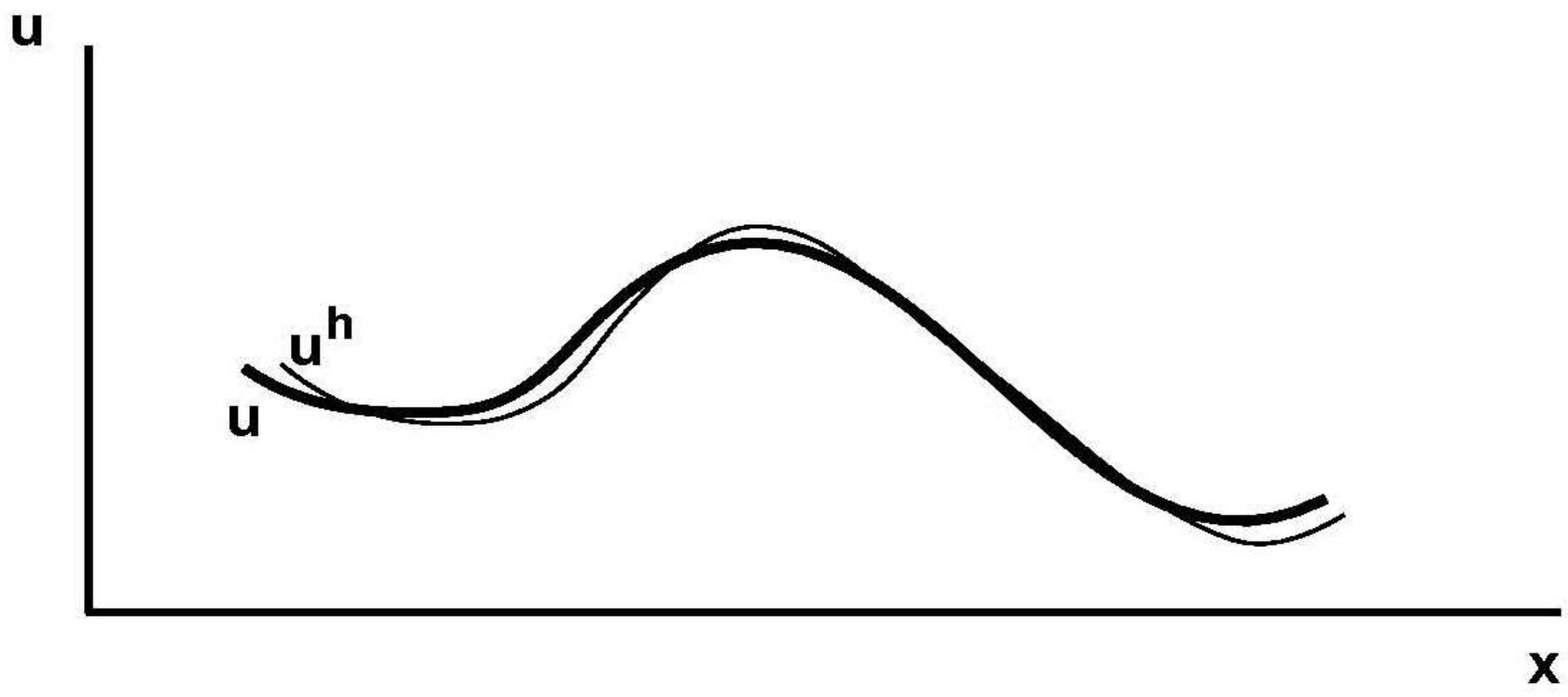
Approximation Theory

To do: given $u(x)$ in Ω, approximate by known functions

$$u(x) \approx u^h(x) = f^i(x)a_i = N^i(x)\hat{u}_i$$



Approximation of Functions

Least Squares Problem

$$I_{ls} = \int_\Omega (\epsilon^h)^2 d\Omega = \int_\Omega (u^h - u)^2 d\Omega$$
$$= \int_\Omega (N^k a_k - u)^2 d\Omega \rightarrow min$$

$\Rightarrow$

$$\delta I_{ls} = \delta a_k \int_\Omega N^k (N^l a_l - u) d\Omega = 0$$

$$\int_\Omega N^k N^l d\Omega \; a_l = \int_\Omega N^k u d\Omega$$

$\Rightarrow$ Equivalent to Galerkin WRM

$\Rightarrow$ Choice of W^i from same set as N^i optimal

2. Weighted Residual Methods(WRM):

Define: $\epsilon^h = u - u^h$ - the error or residual

Require: $\epsilon^h \to 0$ in Ω

Introduce a set of weighting functions W^i ; $i = 1, 2, \ldots M$
Require that:

$$\int_\Omega W^i \epsilon^h d\Omega = 0 \quad , \quad i = 1, 2, \ldots M$$

Then, as $M \to \infty$, $\epsilon^h \to 0$ at all points in Ω

FROM APPROXIMATION TO OPERATORS

Before: given u, approximate: $||u - u^h|| \rightarrow min$

Now: given $L(u) = 0$ approximate: $||L(u) - L(u^h)|| \rightarrow min$

$\Rightarrow$ $||L(u^h)|| \rightarrow min$

Minimize error:

$$\epsilon_L^h = L(u^h) = L(N^i \hat{u}_i)$$

using WRM

$$\int_\Omega W^i \epsilon_L^h d\Omega = 0 \quad , \quad i = 1, M$$

From Approximation to Operators

Choice of N^i, W^i defines the method:

- N^i polynomial, $W^i = \delta(x_i)$: FDM
- N^i polynomial, $W^i = 1$ if x$\subset \Omega_{el}$, 0 otherwise : FVM
- N^i polynomial, $W^i = N^i$: GFEM
- N^i polynomial, $W^i \neq N^i$: Petrov-GFEM
- N^i spectral, $W^i = \delta(x_i)$: SEM

Description of Governing Equations

CFD Notations

PDE of p-th order $\quad f\left(u, \mathbf{x}, t, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^p u}{\partial t^p}\right) = 0$

scalar unknowns $\quad u = u(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \ t \in \mathbb{R}, \quad n = 1, 2, 3$

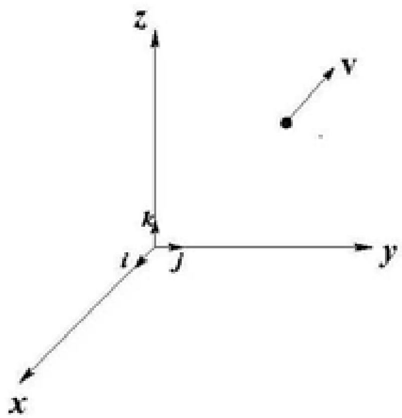
vector unknowns $\quad \mathbf{v} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{v} \in \mathbb{R}^m, \quad m = 1, 2, \dots$

Nabla operator $\quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \qquad \mathbf{x} = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z)$

$$\nabla u = \mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z} = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]^T$$ gradient

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$ divergence

$$\nabla \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix}$$ curl

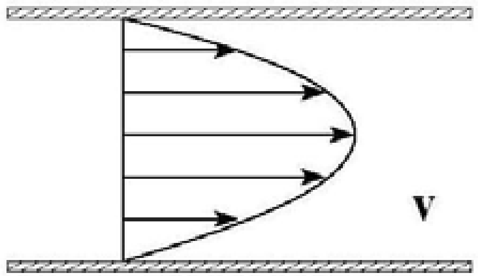


$$\Delta u = \nabla \cdot (\nabla u) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$ Laplacian

CFD Notations

Velocity gradient

$$\nabla \mathbf{v} = [\nabla v_x, \nabla v_y, \nabla v_z] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$



Remark. The trace (sum of diagonal elements) of $\nabla \mathbf{v}$ equals $\nabla \cdot \mathbf{v}$.

Deformation rate tensor (symmetric part of $\nabla \mathbf{v}$)

$$\mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Spin tensor $\quad \mathcal{S}(\mathbf{v}) = \nabla \mathbf{v} - \mathcal{D}(\mathbf{v}) \quad$ (skew-symmetric part of $\nabla \mathbf{v}$)

CFD Notations

Scalar product of two vectors

$$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \qquad \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = [a_1 \;\; a_2 \;\; a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \in \mathbb{R}$$

Example. $\mathbf{v} \cdot \nabla u = v_x \frac{\partial u}{\partial x} + v_y \frac{\partial u}{\partial y} + v_z \frac{\partial u}{\partial z}$ *convective derivative*

Dyadic product of two vectors

$$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \qquad \mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \;\; b_2 \;\; b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

CFD Notations

1. $\alpha\mathcal{T} = \{\alpha t_{ij}\}, \qquad \mathcal{T} = \{t_{ij}\} \in \mathbb{R}^{3\times 3},\ \alpha \in \mathbb{R}$

2. $\mathcal{T}^1 + \mathcal{T}^2 = \{t^1_{ij} + t^2_{ij}\}, \qquad \mathcal{T}^1, \mathcal{T}^2 \in \mathbb{R}^{3\times 3}, \quad \mathbf{a} \in \mathbb{R}^3$

3. $$\mathbf{a}\cdot\mathcal{T} = [a_1, a_2, a_3]\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} = \sum_{i=1}^{3} a_i \underbrace{[t_{i1}, t_{i2}, t_{i3}]}_{i\text{-th row}}$$

4. $$\mathcal{T}\cdot\mathbf{a} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \sum_{j=1}^{3}\begin{bmatrix} t_{1j} \\ t_{2j} \\ t_{3j} \end{bmatrix} a_j \quad (j\text{-th column})$$

5. $$\mathcal{T}^1\cdot\mathcal{T}^2 = \begin{bmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ t^1_{21} & t^1_{22} & t^1_{23} \\ t^1_{31} & t^1_{32} & t^1_{33} \end{bmatrix}\begin{bmatrix} t^2_{11} & t^2_{12} & t^2_{13} \\ t^2_{21} & t^2_{22} & t^2_{23} \\ t^2_{31} & t^2_{32} & t^2_{33} \end{bmatrix} = \left\{\sum_{k=1}^{3} t^1_{ik} t^2_{kj}\right\}$$

6. $$\mathcal{T}^1 : \mathcal{T}^2 = tr(\mathcal{T}^1\cdot(\mathcal{T}^2)^T) = \sum_{i=1}^{3}\sum_{k=1}^{3} t^1_{ik} t^2_{ik}$$

Conservation Laws

Quantities Conserved:

- Mass
- Momentum (Newton' Law)
- Energy (First Law of Thermodynamics)

General Form:

$$Change = Production(Destruction)$$

In Eulerian Frame:

$$Change|_{fixed\mathbf{x}} + Transport = Diffusion + Production$$

Or:

$$\Phi_{,t} + \nabla \cdot \mathbf{v}\Phi = \nabla \cdot q + S$$

Conservation Laws

Physical principles

1. Mass is conserved
2. Newton's second law
3. Energy is conserved

→

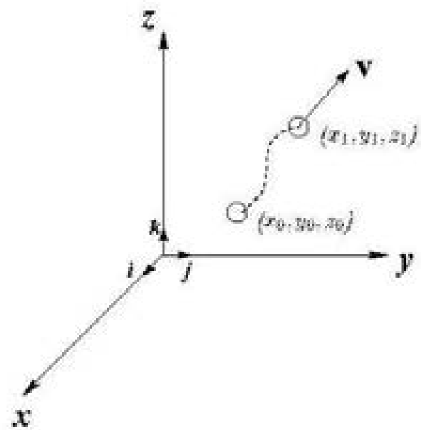
Mathematical equations

- continuity equation
- momentum equations
- energy equation

It is important to understand the meaning and significance of each equation in order to develop a good numerical method and properly interpret the results

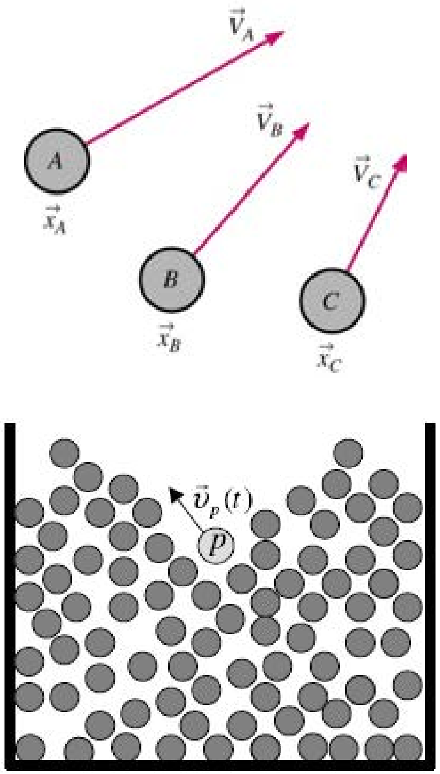
Description of fluid motion

Eulerian	monitor the flow characteristics in a fixed control volume
Lagrangian	track individual fluid particles as they move through the flow field

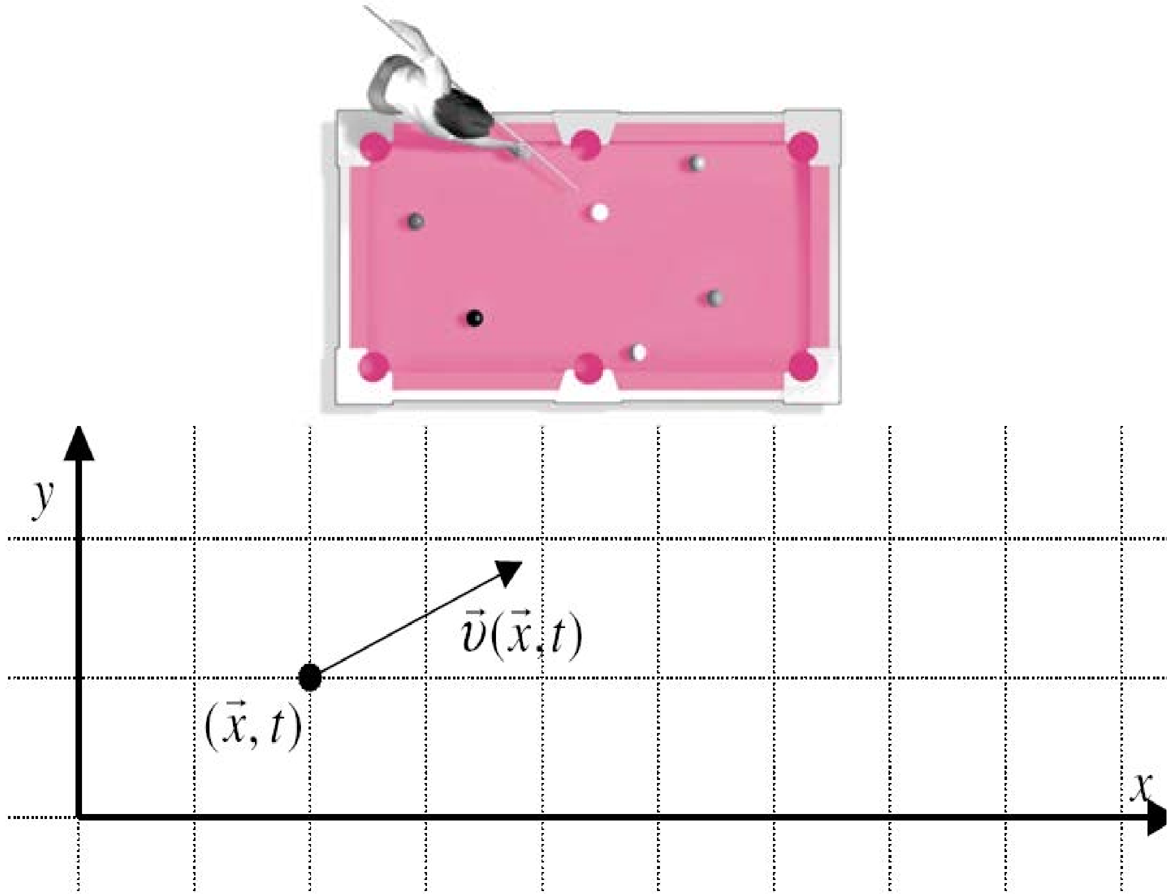


Two ways of describing a fluid flow

- **Lagrangian description, Eulerian description**

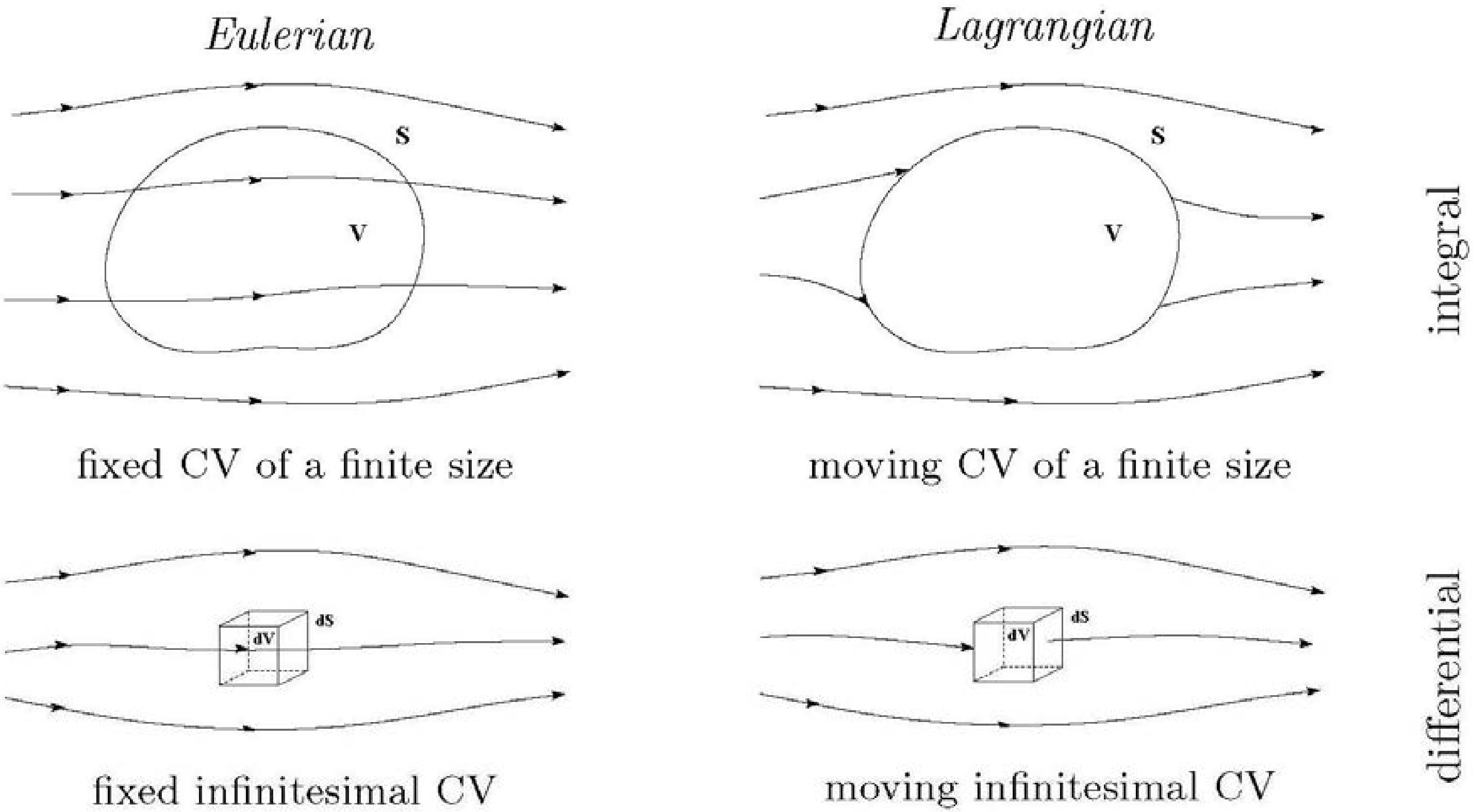


Lagrangian description; snapshot



Eulerian description; Cartesian grid

Flow models and reference frames



Good news: all flow models lead to the same equations

Eulerian vs. Lagrangian

Definition. *Substantial time derivative* $\frac{d}{dt}$ *is the rate of change for a moving fluid particle. Local time derivative* $\frac{\partial}{\partial t}$ *is the rate of change at a fixed point.*

Let $u = u(\mathbf{x}, t)$, where $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$. The chain rule yields

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = \frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u$$

substantial derivative = *local derivative* + *convective derivative*

Reynolds transport theorem

$$\frac{d}{dt}\int_{V_t} u(\mathbf{x}, t)\, dV = \int_{V \equiv V_t} \frac{\partial u(\mathbf{x}, t)}{\partial t}\, dV + \int_{S \equiv S_t} u(\mathbf{x}, t)\mathbf{v} \cdot \mathbf{n}\, dS$$

rate of change in a moving volume = *rate of change in a fixed volume* + *convective transfer through the surface*

$$\underbrace{\frac{D}{Dt}}_{\text{Lagrangian}} \equiv \underbrace{\frac{\partial}{\partial t} + \overrightarrow{v_p} \cdot \nabla}_{\text{Eulerian}}$$

$$\underbrace{\frac{D\vec{G}}{Dt}}_{\substack{\text{Lagragian} \\ \text{rate of change}}} = \underbrace{\frac{\partial \vec{G}}{\partial t}}_{\substack{\text{Eulerian} \\ \text{rate of change}}} + \underbrace{\vec{v} \cdot \nabla \vec{G}}_{\substack{\text{Convective} \\ \text{rate of change}}}$$

$$\underbrace{\frac{D\vec{v}}{Dt}}_{\substack{\text{Lagragian} \\ \text{acceleration}}} = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\substack{\text{Eulerian} \\ \text{acceleration}}} + \underbrace{\vec{v} \cdot \nabla \vec{v}}_{\substack{\text{Convective} \\ \text{acceleration}}}$$

Continuity Equation

Physical principle: conservation of mass

$$\frac{dm}{dt} = \frac{d}{dt}\int_{V_t} \rho\, dV = \int_{V\equiv V_t} \frac{\partial \rho}{\partial t}\, dV + \int_{S\equiv S_t} \rho\mathbf{v}\cdot\mathbf{n}\, dS = 0$$

accumulation of mass inside CV = net influx through the surface

Divergence theorem yields

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV = 0 \quad \Rightarrow$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

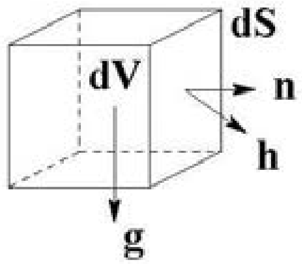
Lagrangian representation

$$\nabla\cdot(\rho\mathbf{v}) = \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} \quad \Rightarrow \quad \frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0$$

Incompressible flows: $\frac{d\rho}{dt} = \nabla\cdot\mathbf{v} = 0$ (constant density)

Momentum Equation

Physical principle: $\mathbf{f} = m\mathbf{a}$ (Newton's second law)



total force	$\mathbf{f} = \rho\mathbf{g}\, dV + \mathbf{h}\, dS$,	where $\mathbf{h} = \sigma \cdot \mathbf{n}$
body forces	$\mathbf{g}$	gravitational, electromagnetic,...
surface forces	$\mathbf{h}$	pressure + viscous stress

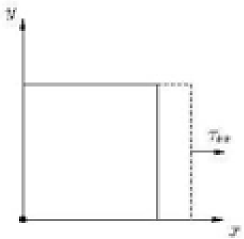
Stress tensor $\sigma = -p\mathcal{I} + \tau$ momentum flux

For a *newtonian fluid* viscous stress is proportional to velocity gradients:

$$\tau = (\lambda \nabla \cdot \mathbf{v})\mathcal{I} + 2\mu\mathcal{D}(\mathbf{v}), \quad \text{where} \quad \mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T), \quad \lambda \approx -\frac{2}{3}\mu$$

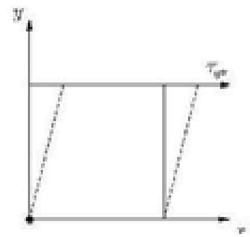
Normal stress: *stretching*

$$\tau_{xx} = \lambda\nabla \cdot \mathbf{v} + 2\mu\frac{\partial v_x}{\partial x}$$
$$\tau_{yy} = \lambda\nabla \cdot \mathbf{v} + 2\mu\frac{\partial v_y}{\partial y}$$
$$\tau_{zz} = \lambda\nabla \cdot \mathbf{v} + 2\mu\frac{\partial v_z}{\partial z}$$



Shear stress: *deformation*

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)$$
$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$
$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$



Momentum Equation

Newton's law for a moving volume

$$\frac{d}{dt}\int_{V_t} \rho\mathbf{v}\,dV = \int_{V\equiv V_t} \frac{\partial(\rho\mathbf{v})}{\partial t}\,dV + \int_{S\equiv S_t} (\rho\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{n}\,dS$$

$$= \int_{V\equiv V_t} \rho\mathbf{g}\,dV + \int_{S\equiv S_t} \sigma\cdot\mathbf{n}\,dS$$

Transformation of surface integrals

$$\int_V \left[\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})\right] dV = \int_V \left[\nabla\cdot\sigma + \rho\mathbf{g}\right]\,dV, \qquad \sigma = -p\mathcal{I} + \tau$$

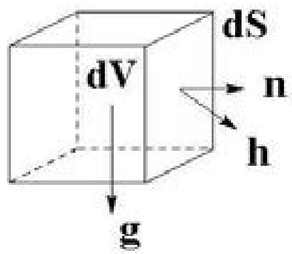
Momentum equations

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = -\nabla p + \nabla\cdot\tau + \rho\mathbf{g}$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = \rho\underbrace{\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right]}_{\text{substantial derivative}} + \mathbf{v}\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right]}_{\text{continuity equation}} = \rho\frac{d\mathbf{v}}{dt}$$

Energy Equation

Physical principle: $\delta e = s + w$ (first law of thermodynamics)



δe accumulation of internal energy

s heat transmitted to the fluid particle

w rate of work done by external forces

Heating: $s = \rho q \, dV - f_q \, dS$

q internal heat sources

f_q diffusive heat transfer

T absolute temperature

κ thermal conductivity

Fourier's law of heat conduction

$$f_q = -\kappa \nabla T$$

the heat flux is proportional to the local temperature gradient

Work done per unit time = total force × velocity

$$w = \mathbf{f} \cdot \mathbf{v} = \rho \mathbf{g} \cdot \mathbf{v} \, dV + \mathbf{v} \cdot (\sigma \cdot \mathbf{n}) \, dS, \qquad \sigma = -p\mathcal{I} + \tau$$

Energy Equation

Total energy per unit mass: $E = e + \frac{|\mathbf{v}|^2}{2}$

e specific internal energy due to random molecular motion

$\frac{|\mathbf{v}|^2}{2}$ specific kinetic energy due to translational motion

Integral conservation law for a moving volume

$$\begin{aligned}
\frac{d}{dt}\int_{V_t} \rho E \, dV &= \int_{V\equiv V_t} \frac{\partial(\rho E)}{\partial t} dV + \int_{S\equiv S_t} \rho E \, \mathbf{v}\cdot\mathbf{n} \, dS && \text{accumulation} \\
&= \int_{V\equiv V_t} \rho q \, dV + \int_{S\equiv S_t} \kappa \nabla T \cdot \mathbf{n} \, dS && \text{heating} \\
&+ \int_{V\equiv V_t} \rho \mathbf{g}\cdot\mathbf{v} \, dV + \int_{S\equiv S_t} \mathbf{v}\cdot(\sigma\cdot\mathbf{n}) \, dS && \text{work done}
\end{aligned}$$

Transformation of surface integrals

$$\int_V \left[\frac{\partial(\rho E)}{\partial t} + \nabla\cdot(\rho E\mathbf{v})\right] dV = \int_V \left[\nabla\cdot(\kappa\nabla T) + \rho q + \nabla\cdot(\sigma\cdot\mathbf{v}) + \rho\mathbf{g}\cdot\mathbf{v}\right] dV,$$

where $\nabla\cdot(\sigma\cdot\mathbf{v}) = -\nabla\cdot(p\mathbf{v}) + \nabla\cdot(\tau\cdot\mathbf{v}) = -\nabla\cdot(p\mathbf{v}) + \mathbf{v}\cdot(\nabla\cdot\tau) + \nabla\mathbf{v} : \tau$

Energy Equation

Total energy equation

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - \nabla \cdot (p\mathbf{v}) + \mathbf{v} \cdot (\nabla \cdot \tau) + \nabla \mathbf{v} : \tau + \rho \mathbf{g} \cdot \mathbf{v}$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{v}) = \underbrace{\rho \left[\frac{\partial E}{\partial t} + \mathbf{v} \cdot \nabla E \right]}_{\text{substantial derivative}} + \underbrace{E \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right]}_{\text{continuity equation}} = \rho \frac{dE}{dt}$$

Momentum equations $\quad \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g} \quad$ (Lagrangian form)

$$\rho \frac{dE}{dt} = \rho \frac{de}{dt} + \mathbf{v} \cdot \rho \frac{d\mathbf{v}}{dt} = \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) + \mathbf{v} \cdot [-\nabla p + \nabla \cdot \tau + \rho \mathbf{g}]$$

Internal energy equation

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - p \nabla \cdot \mathbf{v} + \nabla \mathbf{v} : \tau$$

Summary of GE

1. Continuity equation / conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

2. Momentum equations / Newton's second law

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g}$$

3. Energy equation / first law of thermodynamics

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - \nabla \cdot (p \mathbf{v}) + \mathbf{v} \cdot (\nabla \cdot \tau) + \nabla \mathbf{v} : \tau + \rho \mathbf{g} \cdot \mathbf{v}$$

$$E = e + \frac{|\mathbf{v}|^2}{2}, \qquad \frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - p \nabla \cdot \mathbf{v} + \nabla \mathbf{v} : \tau$$

This PDE system is referred to as the *compressible Navier-Stokes equations*

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Conservation form of GE

Generic conservation law for a scalar quantity

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f} = q, \qquad \text{where} \quad \mathbf{f} = \mathbf{f}(u, \mathbf{x}, t) \quad \text{is the flux function}$$

Conservative variables, fluxes and sources

$$U = \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathcal{I} - \tau \\ (\rho E + p)\mathbf{v} - \kappa \nabla T - \tau \cdot \mathbf{v} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 \\ \rho \mathbf{g} \\ \rho(q + \mathbf{g} \cdot \mathbf{v}) \end{bmatrix}$$

Navier-Stokes equations in divergence form

$$\boxed{\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = Q} \qquad U \in \mathbb{R}^5, \quad \mathbf{F} \in \mathbb{R}^{3\times 5}, \quad Q \in \mathbb{R}^5$$

- representing all equations in the same generic form simplifies the programming
- it suffices to develop discretization techniques for the generic conservation law

Constitive relations

Variables: ρ, $\mathbf{v}$, e, p, τ, T *Equations:* continuity, momentum, energy

⚠ *The number of unknowns exceeds the number of equations.*

1. Newtonian stress tensor

$$\tau = (\lambda \nabla \cdot \mathbf{v})\mathcal{I} + 2\mu\mathcal{D}(\mathbf{v}), \qquad \mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \quad \lambda \approx -\frac{2}{3}\mu$$

2. Thermodynamic relations, e.g.

$p = \rho R T$	ideal gas law	R	specific gas constant
$e = c_v T$	caloric equation of state	c_v	specific heat at constant volume

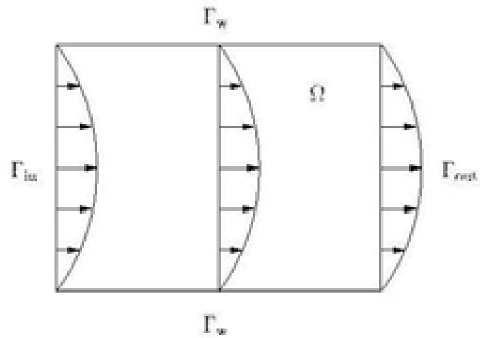
Now the system is closed: it contains five PDEs for five independent variables ρ, $\mathbf{v}$, e and algebraic formulae for the computation of p, τ and T. It remains to specify appropriate initial and boundary conditions.

Initial and Boundary conditions

Initial conditions $\rho|_{t=0} = \rho_0(\mathbf{x}), \quad \mathbf{v}|_{t=0} = \mathbf{v}_0(\mathbf{x}), \quad e|_{t=0} = e_0(\mathbf{x}) \qquad \text{in } \Omega$

Boundary conditions

Let $\Gamma = \Gamma_{\text{in}} \cup \Gamma_{\text{w}} \cup \Gamma_{\text{out}}$



Inlet $\Gamma_{\text{in}} = \{\mathbf{x} \in \Gamma : \mathbf{v} \cdot \mathbf{n} < 0\}$

$$\rho = \rho_{in}, \quad \mathbf{v} = \mathbf{v}_{in}, \quad e = e_{in}$$

prescribed density, energy and velocity

Solid wall $\Gamma_{\text{w}} = \{\mathbf{x} \in \Gamma : \mathbf{v} \cdot \mathbf{n} = 0\}$

$\mathbf{v} = 0$ no-slip condition

$T = T_w$ given temperature or

$\left(\frac{\partial T}{\partial n}\right) = -\frac{f_q}{\kappa}$ prescribed heat flux

Outlet $\Gamma_{\text{out}} = \{\mathbf{x} \in \Gamma : \mathbf{v} \cdot \mathbf{n} > 0\}$

$\mathbf{v} \cdot \mathbf{n} = v_n$ or $-p + \mathbf{n} \cdot \tau \cdot \mathbf{n} = 0$

$\mathbf{v} \cdot \mathbf{s} = v_s$ or $\mathbf{s} \cdot \tau \cdot \mathbf{n} = 0$

prescribed velocity vanishing stress

The problem is well-posed if the solution exists, is unique and depends continuously on IC and BC. Insufficient or incorrect IC/BC may lead to wrong results (if any).

Dimensionless form of GE

Underlying principle: dynamic similarity of flows

Purpose: equations are normalized in order to

- facilitate the scale-up of obtained results to real flow conditions
- avoid round-off due to manipulations with large/small numbers
- assess the relative importance of terms in the model equations

Dimensionless variables

$$t^* = \frac{t}{t_0}, \qquad \mathbf{x}^* = \frac{\mathbf{x}}{L_0}, \qquad \mathbf{v}^* = \frac{\mathbf{v}}{v_0}, \qquad p^* = \frac{p}{\rho v_0^2}, \qquad T^* = \frac{T - T_0}{T_1 - T_0}$$

Dimensionless numbers

Reynolds number	$Re = \frac{\rho v_0 L_0}{\mu}$	$\frac{\text{inertia}}{\text{viscosity}}$	Mach number	$M = \frac{\lvert\mathbf{v}\rvert}{c}$
Froude number	$Fr = \frac{v_0}{\sqrt{L_0 g}}$	$\frac{\text{inertia}}{\text{gravity}}$	Strouhal number	$St = \frac{L_0}{v_0 t_0}$
Peclet number	$Pe = \frac{v_0 L_0}{\kappa}$	$\frac{\text{convection}}{\text{diffusion}}$	Prandtl number	$Pr = \frac{\mu}{\rho \kappa}$

Dimensionless form of GE

Typically:

L	:	Characteristic Length
$\lvert\mathbf{v}_\infty\rvert$	:	Inflow/Free-Stream Velocity
ρ_∞	:	Inflow/Free-Stream Density
T_∞	:	Inflow/Free-Stream Temperature
μ_∞	:	Inflow/Free-Stream Viscosity
k_∞	:	Inflow/Free-Stream Conductivity

Dimensionless form of GE

Define Non-Dimensional Quantities:

$$t^\star = \frac{t\,|\mathbf{v}_\infty|}{L} \quad , \quad x_i^\star = \frac{x_i}{L} \quad , \quad v_i^\star = \frac{v_i}{|\mathbf{v}_\infty|}$$

$$\rho^\star = \frac{\rho}{\rho_\infty} \quad , \quad T^\star = \frac{T}{T_\infty} \quad , \quad p^\star = \frac{p}{\rho_\infty |\mathbf{v}_\infty|^2} \quad , \quad e^\star = \frac{e}{|\mathbf{v}_\infty|^2}$$

$$\mu^\star = \frac{1}{Re_{\infty,L}} \cdot \frac{\mu}{\mu_\infty} \quad , \quad k^\star = \frac{1}{(\gamma - 1) M_\infty^2 Pr Re_{\infty,L}} \cdot \frac{k}{k_\infty}$$

Dimensionless form of GE

Characteristic Numbers:

$$Re_{\infty,L} = \frac{\rho_\infty |\mathbf{v}_\infty| L}{\mu_\infty}:$$ Reynolds-Number;

Ratio of: Intertial Forces:Viscous Forces

$$M_\infty = \frac{|\mathbf{v}_\infty|}{c_\infty}:$$ Mach-Number;

Ratio of: Fluid Velocity:Speed of Sound

$$Pr_\infty = \frac{c_p \mu_\infty}{k_\infty}:$$ Prandtl-Number;

Ratio of: Viscosity:Conductivity

$$\gamma = \frac{c_p}{c_v}:$$

Ratio of: Specific Heat at Constant Pressure:
Specific Heat at Constant Volume

Dimensionless form of GE

By Going to Dimensionless Form, Dropping the *:

$$\mathbf{u}_{,t} + \nabla \cdot \mathbf{F}^a = \nabla \cdot \mathbf{F}^v + \mathbf{S}$$

$$\mathbf{u} = \left\{ \begin{array}{r} \rho \\ \rho v_i \\ \rho e \end{array} \right\} , \ \mathbf{F}_j^a = \left\{ \begin{array}{r} \rho v_j \\ \rho v_i v_j + p\delta_{ij} \\ v_j(\rho e + p) \end{array} \right\} , \ \mathbf{F}_j^v = \left\{ \begin{array}{r} 0 \\ \sigma_{ij} \\ v_l \sigma_{lj} - q_j \end{array} \right\}$$

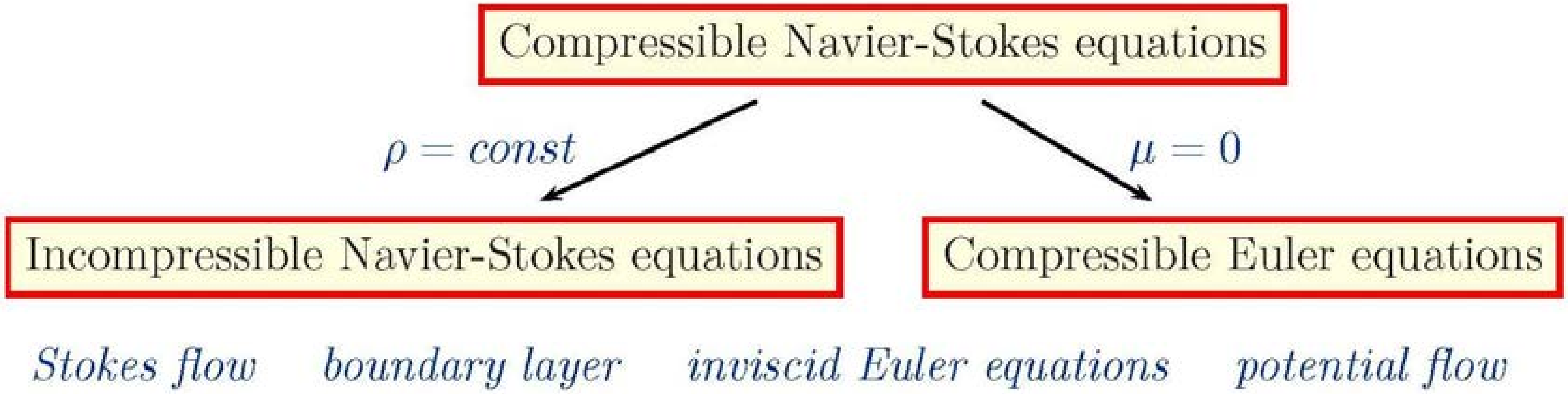
$$p = (\gamma - 1)\rho[e - \frac{1}{2} v_j v_j]$$

$$\sigma_{ij} = \frac{1}{Re}\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij} \right)$$

$$\mathbf{q} = -\frac{1}{PrRe} k \nabla T$$

Model Simplification

Purpose: to reduce the computational cost



Derivation of a simplified model

1. determine the type of flow to be simulated
2. separate important and unimportant effects
3. leave irrelevant features out of consideration
4. omit redundant terms/equations from the model
5. prescribe suitable initial/boundary conditions

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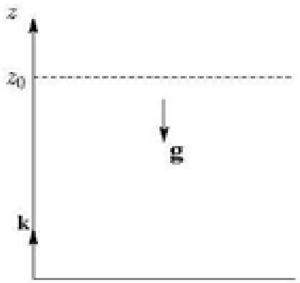
Incompressible flows

Simplification: $\rho = const, \quad \mu = const$

continuity equation $\quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \longrightarrow \quad \nabla \cdot \mathbf{v} = 0$

inertial term $\quad \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = \rho \frac{d\mathbf{v}}{dt}$

stress tensor $\quad \nabla \cdot \tau = \mu \nabla \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T) = \mu (\nabla \cdot \nabla \mathbf{v} + \nabla \nabla \cdot \mathbf{v}) = \mu \Delta \mathbf{v}$



Let $\quad \rho \mathbf{g} = -\rho g \mathbf{k} = -\nabla (\rho g z) = \nabla p_0$

$p_0 = \rho g (z_0 - z)$ hydrostatic pressure

$\tilde{p} = \frac{p - p_h}{\rho} = \frac{p}{\rho} + g(z_0 - z)$ reduced pressure

$\nu = \frac{\mu}{\rho}$ kinematic viscosity

Incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \tilde{p} + \nu \Delta \mathbf{v} - \underbrace{\beta \mathbf{g} (T - T_0)}_{\text{Boussinesq}} \qquad \text{momentum equations}$$

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{continuity equation}$$